Phenomenological Quantum Electrodynamics. Part III. Dispersion

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The quantum theory of a radiation field in an infinite dielectric medium is extended to include dispersion. The Čerenkov radiation is discussed in terms of the theory both for the case that the medium is at rest and the electron moving and for the case that the electron is at rest and the medium moving. In each case the results agree closely with those of corresponding classical treatment.

A. THE CLASSICAL FIELD EQUATIONS

IN the first two parts[1] we have discussed the electromagnetic theory of dielectric media and its quantization. It was noted that this theory was not satisfactory for a treatment of many physical problems. In particular, the radiation emitted per unit time by a charge moving with velocity $u>c/n$ in a refractive medium at rest is infinite. The reason for this is that we did not take into account the phenomenon of dispersion, which is a characteristic of physical media. In the present discussion we extend our results to include dispersion.

In the coordinate system for which the medium is at rest the dielectric constant ϵ and the magnetic permeability μ will now be functions of the frequency k^0 of the light wave that is passing through the medium, or

$$\epsilon=\epsilon(k^0),\quad \mu=\mu(k^0). \tag{1}$$

The field vectors will then be related by the equations

$$\mathbf{D}=(2\pi)^{-\frac{3}{2}}\int d^3k\mathbf{E}(k)\epsilon(k^0)\exp[i\mathbf{k}\cdot\mathbf{x}],$$

$$\mathbf{B}=(2\pi)^{-\frac{3}{2}}\int d^3k\mathbf{H}(k)\mu(k^0)\exp[i\mathbf{k}\cdot\mathbf{x}], \tag{2}$$

where $k^0=k/n$ ($n=(\epsilon\mu)^{\frac{1}{2}}$ is the refractive index of the medium) and $\mathbf{E}(k)$ and $\mathbf{H}(k)$ are the Fourier components of $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{H}(\mathbf{x},t)$, respectively. Here and in the following the notation is the same as that of Parts I and II.[1]

In the case of a general coordinate system in which the medium is moving with the four-velocity $(\mathbf{v},v^0)$, the above equations must be generalized. We have

$$\epsilon=\epsilon(v^0k^0-\mathbf{v}\cdot\mathbf{k}),\quad \mu=\mu(v^0k^0-\mathbf{v}\cdot\mathbf{k}), \tag{3}$$

since ϵ and μ are invariants. Here $(\mathbf{k},k^0)$ represents the wave number-frequency four-vector for a Fourier component of the electromagnetic wave.

The field variables $F_{\lambda\mu}$ and $G_{\lambda\mu}$ satisfy Maxwell's equations (Eqs. (6) and (7) of Part I). Thus $F_{\lambda\mu}$ can be expressed in terms of a vector potential ϕ_σ, as given by Eq. (8,I):[1]

$$F_{\lambda\mu}=\partial_\lambda\phi_\mu-\partial_\mu\phi_\lambda. \tag{4}$$

However, to obtain the definition of $G_{\lambda\mu}$ in accordance with Eq. (9,I) we perform a Fourier decomposition of the field variables, setting

$$\phi_\sigma(x)=(2\pi)^{-2}\int d^4kQ_\sigma(k)\exp[i(\mathbf{k}\cdot\mathbf{x}-k^0x^0)]. \tag{5}$$

Equation (9,I) is modified by putting $\kappa=\epsilon\mu-1$ under the integral sign in the Fourier representation. This equation may also be written as

$$G_{\lambda\mu}=F_{\lambda\mu}+\int d^4x'q(x-x')\times[F_{\mu\sigma}(x')v_\lambda-F_{\lambda\sigma}(x')v_\mu]v^\sigma, \tag{6}$$

where

$$q(z)=(2\pi)^{-4}\int d^4k\kappa(v^0k^0-\mathbf{v}\cdot\mathbf{k})\exp[ik_\sigma z^\sigma]. \tag{7}$$

As it is not possible with dispersive media to introduce μ as a multiplicative factor, we will find it convenient to introduce a new set of variables $H_{\lambda\mu}$ derived from the $G_{\lambda\mu}$ by the relation

$$H_{\lambda\mu}(x)=(2\pi)^{-2}\int d^4k(1/\mu)G_{\lambda\mu}(k)\exp[ik_\sigma x^\sigma], \tag{8}$$

where $G_{\lambda\mu}(k)$ is the Fourier transform of $G_{\lambda\mu}(x)$. $H_{\lambda\mu}$ will also satisfy Maxwell's equation (7,I).

It is apparent that the other equations of Part I defining the field variables can be expressed in a similar way by replacing functions of κ, say $f(\kappa)$, by the integral operator

$$\int d^4x'h(x-x'), \tag{9}$$

[1] J. M. Jauch and K. M. Watson, Phys. Rev. **74**, 950 (1948); **74**, 1485 (1948). The notation in the present paper follows that of Parts I and II. Reference to equations in the first two parts will be given as "Eq. (8,I)" to mean Eq. (8) of Part I, etc. Greek indices run from 0 to 3, while Latin indices may have values from 1 to 3 inclusive.

where

$$h(z)=(2\pi)^{-4}\int d^4kf(\kappa)\exp[ik_\sigma z^\sigma].$$

Thus Eqs. (10), (12), and (13) of Part I become

$$\chi=\partial^\rho\phi_\rho-v^\sigma\partial_\sigma v^\rho\int q(x-x')\phi_\rho(x')d^4x', \quad (10)$$

$$\Psi^\lambda=\phi^\lambda-v_\sigma v^\lambda\int q(x-x')\phi^\sigma(x')d^4x', \quad (11)$$

$$\phi^\lambda=\Psi^\lambda+v_\sigma v^\lambda\int r(x-x')\Psi^\sigma(x')d^4x', \quad (12)$$

where

$$r(z)=(2\pi)^{-4}\int d^4k[\kappa/(1+\kappa)]\exp[ik_\sigma z^\sigma].$$

For the subsequent quantization of the field it will be convenient to generalize the canonical variables π_μ of Eq. (22,I):

$$\pi_\mu(x)=G_\mu{}^0(x)-g_\mu{}^0\chi(x)+v_\mu v^0\int d^4x'q(x-x')\chi(x')$$

and

$$\pi_\mu'(x)=(2\pi)^{-2}\int d^4k(1/\mu)\pi_\mu(k)\exp[ik_\sigma x^\sigma], \quad (13)$$

where $\pi_\mu(k)$ is the Fourier transform of $\pi_\mu(x)$. It is to be noted that π_μ' is related to π_μ as was $H_{\lambda\mu}$ related to $G_{\lambda\mu}$.

In order that these four-dimensional integrals may have meaning, we must suppose that they converge. It might be too restrictive to insist that our field variables vanish for large time intervals, so we shall restrict ourselves to κ's of such a form that all the integrals will converge. This restriction would seem to be physically reasonable in that one would hardly expect the state of the field at sufficiently great space-time distances to affect a particular region of interest. We thus assume our κ's to be so restricted; then the differential and integral operators commute. For instance,

$$\partial_\rho\int q(x-x')\phi_\sigma(x')d^4x'=\int q(x-x')\partial_\rho'\phi_\sigma(x')d^4x'$$

where $\partial_\rho'\equiv\partial/\partial x'^\rho$.

Our field equation then is,[2] analogous to Eq. (11,I)

$$\int d^4x'[\delta(x-x')\partial_\mu\partial^\mu - q(x-x')\partial_\mu'v^\mu\partial_\sigma'v^\sigma]\Psi^\lambda(x')=0. \quad (14)$$

[2] This equation can be derived by variation of the integral of the Lagrangean density which is obtained from that of Eq. (16,I) by replacing κ by the integral operator of Eq. (7).

It is also easily shown that ϕ_σ and χ (Eq. (10)) satisfy Eq. (14) when Ψ^λ does.

Transforming Eq. (14) into k space by a Fourier transform.ation, we obtain the following condition on the k's.

$$k_\lambda k^\lambda-\kappa v_\lambda v_\nu k^\lambda k^\nu=0, \quad (15)$$

which is formally equivalent to Eq. (41,I). Formal solutions for k^0 are

$$k_0'(\mathbf{k},k^0)=\frac{\kappa v^0\mathbf{v}\cdot\mathbf{k}+[(1+\kappa v_0{}^2)k^2-\kappa(\mathbf{v}\cdot\mathbf{k})^2]^{\frac{1}{2}}}{1+\kappa v_0{}^2},$$

$$k_0''(\mathbf{k},k^0)=\frac{\kappa v^0\mathbf{v}\cdot\mathbf{k}-[(1+\kappa v_0{}^2)k^2-\kappa(\mathbf{v}\cdot\mathbf{k})^2]^{\frac{1}{2}}}{1+\kappa v_0{}^2}, \quad (16)$$

where

$$\kappa=\kappa(v^0k^0-\mathbf{v}\cdot\mathbf{k}). \quad (17)$$

k_0' and k_0'' are functions of $\mathbf{k}$ and k^0 and define roots k' and k'' of Eq. (15) by the conditions

$$k_0'(\mathbf{k},k')=k', \quad k_0''(\mathbf{k},k'')=k''. \quad (18)$$

In order to further develop the theory it seems necessary to make some restrictive assumptions regarding the functional form of κ. We shall not endeavor to find the most general functional forms of κ under which our present form of the theory can be developed. Instead, we shall impose on κ the following conditions which seem to be physically meaningful and which allow considerable simplifications.

(a) The three functions κ, ϵ, and μ shall be real valued functions satisfying the inequality $1+\kappa>0$ on the real axis.

(b) $\kappa(z)$ and $\mu(z)$ are analytic in a neighborhood of $z=\infty$.

(c) $\lim_{z\to\infty}\kappa(z)\to 0$, $\lim_{z\to\infty}\mu(z)\to 1$.

(d) $\kappa(z)=\kappa(-z)$.

(e) $[1+\kappa(z)]z^2$ is monotone increasing for increasing real positive values of z.

The physical meaning of condition (a) is that the medium is non-absorbing. Condition (d) is necessary to insure the reality of the field variables. Conditions (d) and (e) imply that for each value of $\mathbf{k}$ there exist two real and distinct solutions of Eqs. (18). This can be seen immediately in the rest system of the medium where Eq. (15) becomes

$$k^2=[1+\kappa(k^0)]k^{0^2}. \quad (19)$$

Since the right-hand side is monotone increasing with k^0, there exist two real roots k', $k''\neq 0$ for $k^2\neq 0$ which satisfy the condition $k'+k''=0$. In a general coordinate-system the roots are obtained from those of Eq. (19) by a Lorentz transformation. Since a Lorentz transformation is a continuous, real transformation, the new roots are again distinct and real.

If condition (e) were not satisfied, we could have photons of different energy for a given momentum. This would not seem to involve any fundamental difficulty; however, we have excluded this case explicitly in the present paper for reasons of simplicity even though it might be of considerable physical interest.

A further consequence of condition (d) is the functional identity

$$k'(-\mathbf{k})=-k''(\mathbf{k}). \tag{20}$$

This equation is necessary in order that we may have real solutions of Eq. (14). That is, the solutions will be of the form

$$\begin{aligned}\phi(x)&=\int d^3k[A(\mathbf{k})\exp[i(\mathbf{k}\cdot\mathbf{x}-k'x^0)]\\&\qquad+A^*(-\mathbf{k})\exp[i(\mathbf{k}\cdot\mathbf{x}-k''x^0)]]\\&=\int d^3k[A(\mathbf{k})\exp[i(\mathbf{k}\cdot\mathbf{x}-k'x^0)]\\&\qquad+A^*(\mathbf{k})\exp[-i(\mathbf{k}\cdot\mathbf{x}-k'x^0)]].\end{aligned}$$

The fact that we have only two (real) solutions of Eq. (15) implies that the vanishing of χ in Eq. (10) (the subsidiary condition) and its time derivative at all space points and at a single instant is sufficient to make it vanish for all time. This may be readily verified by using a Fourier representation such as that used above. The conditions stated imply that the Fourier coefficients vanish.

For later use we desire a D function analogous to that of Part I. We define

$$\begin{aligned}D(y)&=\pm(2\pi)^{-4}\int_{C_{\mp}'}d^4k\frac{\exp[ik_\rho y^\rho]}{k_\rho k^\rho-\kappa v^\rho v^\sigma k_\rho k_\sigma}\\&=\pm(2\pi)^{-4}\int d^3k\,\exp[i\mathbf{k}\cdot\mathbf{y}]\\&\quad\times\int_{C'_{\mp}}\frac{dk^0\exp[-ik^0y^0]}{(k^0-k_0')(k^0-k_0'')(1+\kappa v_0^2)},\end{aligned} \tag{21}$$

where k_0' and k_0'' are given by Eq. (16) and are equal to our roots k' and k'' when we substitute the value for k' and k'' into κ in Eq. (16). The contour of integration is deformed at the poles k' and k'' on the real axis as described in Part I. The contour is further deformed to avoid any contribution to the integral from singularities off the real axis. This can be done as we have assumed $\kappa(z)$ to be analytic at $z=\infty$. According to the residue theorem, integrating about the pole $k^0=k'$ gives a term

$$[(1+\kappa v_0^2)(k'-k_0'')(1-(\partial k_0'/\partial k^0))]^{-1}_{k^0=k'}\equiv 1/2k\Lambda\gamma, \tag{22}$$

where the root k' is inserted for k^0 into κ after the derivative is evaluated, as indicated. Here Λ and γ are defined as

$$\begin{aligned}&\Lambda\equiv[(1+\kappa v_0^2)-\kappa b^2]^{\frac{1}{2}},\\&\gamma\equiv(1-(\partial k_0'/\partial k^0)_{k^0=k'},\\&k\equiv|\mathbf{k}|,\quad\text{and}\quad b\equiv(\mathbf{v}\cdot\mathbf{k})k^{-1}.\end{aligned} \tag{23}$$

The expression for Λ is obtained from $(k'-k_0'')$ using Eq. (16). The pole at $k^0=k''$ gives

$$[(1+\kappa v_0^2)(k''-k_0')(1-(\partial k_0''/\partial k^0))]^{-1}_{k^0=k''}.$$

On replacing $\mathbf{k}$ by $-\mathbf{k}$ this expression goes over into the negative of Eq. (22), since $k''(-\mathbf{k})=-k'(\mathbf{k})$. Collecting the results, we can write the D function as

$$D(y)=\tfrac{1}{2}(2\pi)^3 i\int(d^3k/k\Lambda\gamma)[\exp[i(\mathbf{k}\cdot\mathbf{y}-k'y^0)]-\exp[-i(\mathbf{k}\cdot\mathbf{y}-k'y^0)]]. \tag{24}$$

This differs from that obtained in Part I only in the occurrence of the factor γ and in that κ is now a function of $\mathbf{k}$. Since the D function has the properties of a Green's function, it will determine the manner of propagation of a light signal in the medium. In particular, we wish to so choose the contour of integration in Eq. (24) that the wave-front velocity of a light signal is just c, its vacuum velocity—a problem similar to that of Sommerfeld and Brillouin.[3]

In appendix I the following results are shown concerning the analytic structure of the quantities in the integrand of Eq. (24):

(i) k', Λ, κ, $1/\gamma$, and μ are analytic functions of k in a neighborhood of $k=\infty$ for all values of the angular variables.

(ii) k' is an odd function of k, whereas Λ, κ, $1/\gamma$, and μ are even in k.

We first illustrate the problem by considering the propagation of a signal $f(x^0)(=0$ for $x^0<0)$ initiated at the point $\mathbf{x}=0$ at the time $x^0=0$ in the coordinate system for which the medium is at rest $(\mathbf{v}=0)$. Consider the function

$$\begin{aligned}f(\mathbf{x},x^0)&=((2\pi)^{-3}/2)i\int(d^3k/kn\gamma)\\&\quad\times[f(k')\exp[i(\mathbf{k}\cdot\mathbf{x}-k'x^0)]\\&\quad-f(-k')\exp[-i(\mathbf{k}\cdot\mathbf{x}-k'x^0)]],\end{aligned} \tag{25}$$

where $f(k')$ is the Fourier transform of $f(x^0)$ and n is the refractive index of the medium. As n and γ

[3] A. Sommerfeld, Ann. d. Physik **44**, 177 (1914); L. Brillouin, Ann. d. Physik **44**, 203 (1914). A discussion is given by J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), p. 335.

are even and k' is odd in k, we can write Eq. (25) as

$$(f\mathbf{x},x^0)=((2\pi)^{-3}/2)i\int_{-\infty}^{+\infty}(kdk/n\gamma)f(k')\times\int d\Omega\,\exp[i(\mathbf{k}\cdot\mathbf{x}-k'x^0)]$$
$$=(1/4\pi)(1/r)(1/2\pi)\int_{-\infty}^{+\infty}dk'f(k')\times[\exp[ik'(nr-x^0)]-\exp[-ik'(nr+x^0)]], \quad (26)$$

where $r=|\mathbf{x}|$, $d\Omega$ is an element of solid angle, and $k=k'n$. We have also used the relation $dk/n\gamma=dk'$, which is demonstrated later (see Eq. (90)). Choosing the contour of integration to lie above the singularities of the integrand for outgoing waves and below the singularities for incoming waves, we see that $f(\mathbf{x},x^0)$ vanishes outside its light cones, or for

$$|\mathbf{x}|>|x^0|.$$

Going to the limit of an impulsive signal for which $f(k')=1$, we obtain just the D function for the case $\mathbf{v}=0$. When $\mathbf{v}\neq0$ we can expect the same result, as the light cones are invariant under a Lorentz transformation. However, we wish to investigate this general case in more detail.

Consider the tensor

$$\sum\nolimits_{\lambda\mu}(y)=(2\pi)^{-4}\int_C d^4k\Gamma_{\lambda\mu}\times[\exp[ik_\sigma y^\sigma]/[k_\rho k^\rho-\kappa(v_\rho k^\rho)^2]], \quad (27)$$

where $\Gamma_{\lambda\mu}$ is a tensor which may depend explicitly on κ and μ, but does not otherwise depend on k_ρ, and is non-singular as $\kappa\to0$, $\mu\to1$. C represents the particular contour of integration that we shall choose. Because of the covariant form of Eq. (27), a Lorentz transformation will give an equivalent expression in the transformed variables k_ρ, v_ρ, and y_ρ. Since k_ρ enters only in an invariant manner into the integrand, the contour C will transform into an equivalent contour in the new variables.

As $\Gamma_{\lambda\mu}$ is assumed to involve only κ and μ as variable quantities, we can perform the k^0-integration in such a manner as to avoid any contribution to the integral from the singularities of $\Gamma_{\lambda\mu}$. This gives

$$\sum\nolimits_{\lambda\mu}(y)=((2\pi)^{-3}/2)i\int(d^3k/k\Lambda\gamma)\Gamma_{\lambda\mu}(v^0k'-\mathbf{v}\cdot\mathbf{k})\times[\exp[i(\mathbf{k}\cdot\mathbf{y}-k'y^0)]-\exp[-i(\mathbf{k}\cdot\mathbf{y}-k'y^0)]], \quad (28)$$

as in Eq. (24). Again, since κ, Λ, γ are even and k' is odd in k, we can write this as

$$\sum\nolimits_{\lambda\mu}(y)=\tfrac{1}{2}(2\pi)^{-3}i\int_{-\infty}^{+\infty}kdk\int(d\Omega/\Lambda\gamma)\Gamma_{\lambda\mu}\times\exp[i(\mathbf{k}\cdot\mathbf{y}-k'y^0)]. \quad (29)$$

From our assumption that in the limit as $k\to\infty$, $\kappa\to0$, and $\mu\to1$, the integrand in Eq. (29) then goes over into that of the usual vacuum case. Thus on performing the angular integrations, we obtain two terms which behave asymptotically as simple incoming and outgoing waves, respectively. Since the integrand is analytic in k in a neighborhood of $k=\infty$ before the angular integrations are performed, it will remain so afterward. Then choosing the k-contour above the singularities of the term that behaves asymptotically as an outgoing wave and below the singularities for the asymptotically incoming wave term, we see that $\sum_{\lambda\mu}$ vanishes outside its light cones, or for

$$|\mathbf{y}|>|y^0|.$$

Replacing $\Gamma_{\lambda\mu}$ by unity in the above analysis gives just the D function.

A further important relation can be established:

$$(\partial/\partial y^0)D(y)|_{y^0=0}=\delta(y). \quad (30)$$

For

$$\frac{\partial}{\partial y^0}D(y)|_{y^0=0}=\frac{(2\pi)^{-3}}{2}\int d^3k\left[\left(\frac{\kappa(\mathbf{k})bv^0}{\Lambda\gamma}+\frac{1}{\gamma}\right)\times\left(\frac{1}{1+\kappa v_0^2}\right)\right](\exp[i\mathbf{k}\cdot\mathbf{y}]+\exp[-i\mathbf{k}\cdot\mathbf{y}]). \quad (31)$$

The quantity in brackets can be written in the form $1+f(\mathbf{k})$ where $f(\mathbf{k})\to0$ as $k\to\infty$. The singularities of $f(\mathbf{k})$ are among those discussed above, so the integral of the $f(\mathbf{k})$ term vanishes, when we choose the contour above (or below) the singularities of the integrand for terms that behave asymptotically as $\exp(ik|\mathbf{y}|)$(or $\exp(-ik|\mathbf{y}|)$). The "1" term gives just the δ-function.

When the electromagnetic field interacts with an electron field whose current-charge density is j^λ, Eq. (14) must be generalized. Writing the field equation in terms of $H_{\mu\lambda}$ defined by Eq. (8), we have

$$\partial_\mu H^{\mu\lambda}=-j^\lambda \quad (32)$$

making use of the subsidiary condition. That the subsidiary condition can be imposed in the case of interaction follows from the continuity equation $\partial_\lambda j^\lambda=0$, as in Part II.

B. THE QUANTUM THEORY OF THE DISPERSIVE FIELD

Our next problem is to develop a quantum theory of the electromagnetic field discussed in the first section. In particular we shall suppose that there are N electrons at the space points $\bar{\mathbf{z}}_n (n=1,2,\cdots N)$ and shall develop the quantum dynamical theory of the interacting electron and electromagnetic fields. The coordinate representation of the electron field will be used for simplicity, as we are primarily interested in the quantization of the dispersive electromagnetic field.

The dispersive field is of the non-localized type discussed by several authors.[4] That is, the field variables $H^{\mu\lambda}$ depend on the values of the $F^{\mu\lambda}$ not at a point but over all space-time. This introduces certain difficulties from the standpoint of the usual canonical development of field theories. Whereas we can still derive the field equations from the variation of a Lagrangian density,[2] we find that the space-time symmetry of the integral operators having the form given by Eq. (9) does not readily lend itself to an unsymmetric space-time canonical treatment. We must accordingly develop the theory along other lines. In discarding the energy-momentum densities of the canonical method we lose a prescription for constructing a Hamiltonian formulation but not quantities of physical significance, since these latter quantities occur only as the integrated form of the densities of energy and momentum. Noting that the integral operators occurring in our theory are replaced by multiplicative factors in k-space, we can suspect that the theory can be more readily developed in a k-space representation. Therefore, we choose a development similar to that of Pauli[5] in discussing the λ-limiting process, which will later be shown to be a special case of the present theory.

We can write immediately the Hamiltonian for the N electrons in the electromagnetic field, for it will be of the usual form:

$$H_e = H_p + H',$$

where

$$H_p = \sum_{n=1}^{N} (\boldsymbol{\alpha}\cdot\mathbf{p}_n + \beta m),$$

$$H' = -\int j_\lambda \phi^\lambda d^3x = \sum_{n=1}^{N} e_n(\phi^0(\mathbf{z}_n) - \boldsymbol{\alpha}\cdot\boldsymbol{\phi}(\mathbf{z}_n)). \quad (33)$$

Here j^λ, the current density of the electron field, is $j^\lambda = \sum_n e_n \alpha^\lambda \delta(\mathbf{x}-\mathbf{z}_n)$, with e_n the charge on the nth electron, $\alpha^k (k=1, 2, 3)$ the Dirac matrices, and $\alpha^0 = 1$ the unit matrix. Equation (33) clearly leads to the correct equations of motion for the electrons, as these are the same as those for the usual vacuum case.

It remains to quantize the field variables ϕ^σ and to find the part of the Hamiltonian, H_0, corresponding to the free electromagnetic field. We postulate commutation rules for this (non-interacting) field:

$$i[\phi_\lambda(x),\quad \phi_\sigma(x')] = \Sigma_{\lambda\sigma}(x-x'), \quad (34)$$

where $\Sigma_{\lambda\sigma} = \Sigma_{\sigma\lambda}$ are real c-number functions to be determined. As we are interested in a k-space representation, we introduce the Fourier decomposition of $\Sigma_{\lambda\sigma}$:

$$\Sigma_{\lambda\sigma}(y) = (2\pi)^{-4}\int d^4k \Gamma_{\lambda\sigma}(k)\frac{\exp[ik_\sigma y^\sigma]}{k_\nu k^\nu - \kappa(v^\nu k_\nu)^2}$$

$$= i\frac{(2\pi)^{-3}}{2}\int \frac{d^3k}{k\Lambda\gamma}\Gamma_{\lambda\sigma}(\mathbf{k})[\exp[i(\mathbf{k}\cdot\mathbf{y} - k'y^0)] - \exp[-i(\mathbf{k}\cdot\mathbf{y} - k'y^0)]], \quad (35)$$

where $\Gamma_{\lambda\sigma} = \Gamma_{\sigma\lambda}$ are real unknown functions of $\mathbf{k}$ yet to be determined. The last form follows if we assume that we can exclude any contribution from the singularities of $\Gamma_{\lambda\sigma}$ to the k^0-integration, as will be justified from the final choice of that function. We note that this last form is consistent with the field equations (14).

Performing a Fourier decomposition of the field variables ϕ_λ, we have

$$\phi_\lambda = ((2\pi)^{-\frac{3}{2}}/\sqrt{2})\int d^3k T(k) \times [A_\lambda(k)\exp[i(\mathbf{k}\cdot\mathbf{x} - k'x^0)] + A_\lambda^+(k)\exp[-i(\mathbf{k}\cdot\mathbf{x} - k'x^0)]], \quad (36)$$

where $T(k)$ is a normalizing factor and the $A_\lambda(k)$ and $A_\lambda^+(k)$ represent dynamical variables in our theory. The form of Eq. (36) is such that ϕ_λ satisfies Eq. (14) with the A_λ and A_λ^+ constants of motion in the free-field case. From Eqs. (34) and (36), we see that if $T(k)$ is

$$T(k) = 1/(k\Lambda\gamma)^{\frac{1}{2}}, \quad (37)$$

then the commutation rules for the A_λ and A_λ^+ are

$$[A_\lambda(k), A_\sigma^+(l)] = \Gamma_{\lambda\sigma}(k)\delta(k-l),$$
$$[A_\lambda(k), A_\sigma(l)] = [A_\lambda^+(k), A_\sigma^+(l)] = 0. \quad (38)$$

These are considered as the fundamental commutation relations of our theory.

There are a number of conditions which must be met by the theory. Since we have imposed a subsidiary conditions $\chi(x)=0$ on the classical theory, we have now

$$\chi(x)\Omega = 0, \quad (39)$$

[4] P. Dirac, Phys. Rev. **73**, 1092 (1948); F. Bopp, Zeits. f. Naturforschung **1**, 53 (1946); W. Heisenberg, Zeits. f. Naturforschung **1**, 608 (1946). The occurrence of the four-vector parameter v^μ in our theory represents a difference between these theories and that of the present paper. For this reason the method of treatment used here cannot be directly applied to the field considered by Bopp and Heisenberg.

[5] W. Pauli, Rev. Mod. Phys. **15**, 194 (1943).

where Ω represents the state vector of the system in a Heisenberg representation and χ is given by Eq. (10). In order that Eq. (39) may not lead to further subsidiary conditions, we must also have

$$[\chi(x),\chi(x')]=0 \quad (40)$$

hold in virtue of the commutation relations (38). Then

$$(d\chi/dx^0)\Omega=0$$

will follow from Eq. (39), as was shown in Part II. With interaction, Maxwell's equation (32) will be

$$[\partial_\mu H^{\lambda\mu}-j^\lambda]\Omega=0. \quad (41)$$

Also, the Hamiltonian and the commutation relations (38) must be consistent with the field equations (17,II), which are for the dispersive field

$$\partial_\lambda H_\mu{}^\lambda-\partial_\lambda\int[\delta_\mu{}^\lambda\delta(x-x') -v_\mu v^\lambda q(x-x')]\chi'(x')d^4x'=j_\mu(x). \quad (42)$$

$q(x-x')$ is given by Eq. (7), and $\chi'(x)$ is derived from $\chi(x)$ by

$$\chi'(x)=(2\pi)^{-2}\int d^4k(1/\mu)\chi(k)\exp[ik_\sigma x^\sigma],$$

where $\chi(k)$ is the Fourier transform of $\chi(x)$. Collecting the terms in Eq. (42) which contain a time derivative, we obtain just $\pi_\mu'(x)$ of Eq. (13). Then we must have

$$i[H_0,\pi_\mu']=-\partial_k H_\mu{}^k+\partial_k\int[\delta_\mu{}^k\delta(x-x') -v_{\mu\nu}kq(x-x')]\chi'(x')d^4x' \quad (43)$$

and

$$i[H',\pi_\mu']=j_\mu. \quad (44)$$

Since the time-dependent operators A_λ, $A_\lambda{}^+$ are constants of motion when there is no interaction, it follows that

$$i[H_0,A_\lambda(k)]=-ik'A_\lambda(k),$$
$$i[H_0,A_\lambda{}^+(k)]=ik'A_\lambda{}^+(k). \quad (45)$$

These equations imply that H^0 must be a quadratic function of the A's determined to within an additive constant to be of the form

$$H_0=\tfrac{1}{2}\int d^3kk'[\daleth^{\lambda\mu}(A_\lambda(k)A_\mu{}^+(k) +A_\mu{}^+(k)A_\lambda(k))], \quad (46)$$

where $\daleth_{\lambda\mu}=\daleth_{\mu\lambda}$ is determined from Eqs. (38), (45), and (46) by

$$\daleth_{\lambda\mu}\Gamma^{\mu\sigma}=\delta_\lambda{}^\sigma. \quad (47)$$

A principal axis transformation on the quadratic form in Eq. (46) will also diagonalize the commutation rules (38). It is thus seen that the eigenvalues of H_0 are of the form $(n+1)k'$ where n is a positive integer (or zero) as usual. The corresponding field momentum vector will be

$$\mathbf{P}=\tfrac{1}{2}\int d^3k\mathbf{k}[\daleth^{\lambda\mu}(A_\lambda(k)A_\mu{}^+(k)+A_\mu{}^+(k)A_\lambda(k))]. \quad (48)$$

Writing the four-vector $(\mathbf{P},H_0)$ as P^μ, we see that the expectation value for this operator for any state is of the form

$$\bar{P}^\mu=\sum_k\sum_{\sigma=1}^{4}(\pm)(n_\sigma(k)+\tfrac{1}{2})k^\mu, \quad (49)$$

where $k^\mu=(\mathbf{k},k')$ and $n_\sigma(k)$ represents the number of photons in the k'th momentum state with "polarization" along the σ'th principal axis of the quadratic form in Eq. (46).

We also note that if $F(x)$ represents any of our field variables,

$$i[F(x),P_\mu]=\partial F(x)/\partial x^\mu \quad (50)$$

because of Eq. (47).

Equation (43) is satisfied in virtue of Eq. (45).

To satisfy conditions (40), (41), and (44) we choose

$$\Gamma_{\lambda\sigma}(k)=[g_{\lambda\sigma}+(\kappa/(1+\kappa))v_\lambda v_\sigma]\mu. \quad (51)$$

These commutation relations are now formally equivalent to those of Parts I and II except that the factor μ is explicitly introduced, since the eigenvalues of the Hamiltonian give the energy directly in the present paper, in contrast to Parts I and II, where the energy was multiplied by the factor μ. It is readily verified that Eq. (40) is satisfied by Eq. (51) if one expresses $\chi(x)$ as a Fourier series in the A's, as is done in the next section.

As we have assumed $\kappa(z)$ and $\mu(z)$ to be analytic in a neighborhood of $z=\infty$, it follows that we may perform the k^0-integration as indicated in Eq. (35).

Since the commutator of the field variables given by Eq. (34) is of the form of Eq. (27), we may conclude that the field variables commute at points outside each other's light cones.

That the commutation rules (44) hold in the non-dispersive case was shown in Part II, except that the factor μ was explicitly introduced. In the dispersive case the commutator in Eq. (44) will have the same form as for the non-dispersive case, except for the factor of $1/\gamma$ in the integrand, and that κ is no longer constant. We are thus led to integrands of the type given in Eq. (31), which give a δ-function as was the case for Eq. (31).

We now have the complete Hamiltonian for our dynamical system. It is of the form

$$H=H_p+H_0+H', \quad (52)$$

where the terms on the right-hand side are defined by Eqs. (33) and (46). By forming the commutators of H with the field and electron variables we obtain the field equations (32) as well as the differential equations describing the electron motion when we apply the subsidiary condition.

C. ELIMINATION OF THE LONGITUDINAL COMPONENTS

Just as in the case of the non-dispersive theory (Part II), we can eliminate the longitudinal components of the field by making use of the subsidiary condition. Since we are developing the theory by a momentum space representation, we must find the subsidiary condition in momentum space. Expanding $\chi(x)$ of Eq. (10) into a Fourier series, we obtain

$$x(k)=k^{\sigma}A_{\sigma}-\kappa v_{\nu}k^{\nu}v^{\sigma}A_{\sigma} \tag{53}$$

and its complex conjugate as the expansion coefficients, if we drop the multiplicative factor $iT(k)$. In the absence of interaction the subsidiary condition is

$$x(k)\Omega=0 \quad \text{and} \quad x^{+}(k)\Omega=0. \tag{54}$$

With interaction, Eq. (54) must be generalized to

$$L\Omega=0, \tag{55}$$

where

$$L=x(k)+f. \tag{56}$$

The quantity f is uniquely determined[5] by the condition that

$$(dL/dx^0)\Omega=i[H,L]\Omega=0 \tag{57}$$

must be a consequence of Eq. (55). Using Eq. (33) the quantity f turns out to be

$$f=\sum_{n=1}^{N}e_n[(2\pi)^{-\frac{3}{2}}/\sqrt{2}]\mu T(k)\exp[-i(\mathbf{k}\cdot\mathbf{z}_n-k'z^0)] \tag{58}$$

as may be readily verified from the commutation relations

$$i[H_0,x(k)]=-ik'x(k),$$
$$i[\phi_\nu(\mathbf{z}_n),x(k)]=-i[(2\pi)^{-\frac{3}{2}}/\sqrt{2}]T(k)\mu k_\nu \times\exp[-i(\mathbf{k}\cdot\mathbf{z}_n-k'z^0)],$$
$$i[\mathbf{p}_n,L(k)]=-i[(2\pi)^{-\frac{3}{2}}/\sqrt{2}]e_n\mathbf{k}\mu T(k) \times\exp[-i(\mathbf{k}\cdot\mathbf{z}_n-k'z^0)], \tag{59}$$

where $\mathbf{p}_n=(1/i)(\partial/\partial\mathbf{z}_n)$.

The elimination of the longitudinal components is straightforward and may be done directly in terms of the A's. It seems to be simpler, however, to introduce emission and absorption operators obtained from diagonalizing the commutation rules (51). The elimination then proceeds in a manner almost identical with that used by Pauli.[5]

Introducing the three unit vectors $\mathbf{e}_1(k)$, $\mathbf{e}_2(k)$, and $\mathbf{e}_3(k)$ of Eq. (61,I), we define a set of operators α_ν by

$$(\mathbf{A}\cdot\mathbf{e}_1)=\alpha_1,$$
$$(\mathbf{A}\cdot\mathbf{e}_2)=[\Lambda/(1+\kappa)^{\frac{1}{2}}]\alpha_2,$$
$$(\mathbf{A}\cdot\mathbf{e}_3)=\frac{\alpha_3}{(|1-\kappa b^2|)^{\frac{1}{2}}}-\left(\pm\frac{\kappa b v_0}{(|1-\kappa b^2|)^{\frac{1}{2}}}\right)\frac{\alpha_0}{\Lambda}+\frac{ab\kappa}{(1+\kappa)^{\frac{1}{2}}}\frac{\alpha_2}{\Lambda},$$
$$A_0=\left(\pm\frac{(|1-\kappa b^2|)^{\frac{1}{2}}}{\Lambda}\right)\alpha_0+\frac{\kappa a v_0}{\Lambda(1+\kappa)^{\frac{1}{2}}}\alpha_2, \tag{60}$$

where $a\equiv\mathbf{v}\cdot\mathbf{e}_2$, $b\equiv\mathbf{v}\cdot\mathbf{e}_3$, and wherever the $(\pm)$ sign ambiguity occurs, the $(+)$ sign is taken when $\kappa b^2<1$ and the $(-)$ sign when $\kappa b^2>1$. The commutation rules satisfied by the α's are

$$[\alpha_\mu(k),\alpha_\nu^{+}(l)]=g_{\nu\mu}\mu\delta(\mathbf{k}-\mathbf{l}) \tag{61}$$

(all other combinations commuting) for $\kappa b^2<1$. When $\kappa b^2>1$ the commutation rules for α_3 and α_0 are interchanged. These become

$$[\alpha_3(k),\alpha_3^{+}(l)]=-\mu\delta(\mathbf{k}-\mathbf{l}),$$
$$[\alpha_0(k),\alpha_0^{+}(l)]=+\mu\delta(\mathbf{k}-\mathbf{l}). \tag{61'}$$

Then according to Eqs. (46) and (47) H_0 becomes

$$H_0=H_0^{tr}+H_0^{\text{long}}, \tag{62}$$

where

$$H_0^{tr}=\frac{1}{2}\int d^3k(k'/\mu)\left[\sum_{r=1}^{2}(\alpha_r\alpha_r^{+}+\alpha_r^{+}\alpha_r)\right],$$
$$H_0^{\text{long}}=(\pm)\frac{1}{2}\int d^3k(k'/\mu)[\alpha_3\alpha_3^{+}+\alpha_3^{+}\alpha_3-\alpha_0\alpha_0^{+}-\alpha_0^{+}\alpha_0]. \tag{63}$$

Again the $(\pm)$ sign is to be chosen according to whether $\kappa b^2 \lesseqgtr 1$. $x(k)$ becomes

$$x(k)=g(\alpha_3+\alpha_0), \tag{64}$$

where $g=\Lambda k'/(|1-\kappa b^2|)^{\frac{1}{2}}$.

H_0^{long} can be written in terms of L as

$$H_0^{\text{long}}=(\pm)\int d^3k(k'/\mu g)\{(\alpha_3^{+}-\alpha_0^{+}-f^*)L+(\alpha_3-\alpha_0-f)L^{+}+2\alpha_0 f^*+2\alpha_0^{+}f+2(ff^*/g)\}. \tag{65}$$

In virtue of the subsidiary condition, the terms involving L and L^+ may be dropped. The interaction term H' can be decomposed into

$$H'=H_1+H'^{\text{long}},$$

where

$$H_1=-\sum_n e_n\boldsymbol{\alpha}\cdot[(2\pi)^{-\frac{3}{2}}/\sqrt{2}]\int d^3kT(k)[(\alpha_1\mathbf{e}_1+(\Lambda/(1+\kappa)^{\frac{1}{2}})\alpha_2\mathbf{e}_2)\exp[-i(\mathbf{k}\cdot\mathbf{z}_n-k'z^0)]+\text{comp. conj.}] \quad (66)$$

and

$$H'^{\text{long}}=-\sum_n e_n[(2\pi)^{-\frac{3}{2}}/\sqrt{2}]\int d^3kT(k)\times\left\{\left[\boldsymbol{\alpha}\cdot\mathbf{e}_3(\mathbf{A}\cdot\mathbf{e}_3)+\left(\pm\frac{(|1-\kappa b^2|)^{\frac{1}{2}}}{\Lambda}\right)\alpha_0+\frac{\kappa a v_0}{\Lambda(1+\kappa)^{\frac{1}{2}}}\alpha_2\right]\exp[i(\mathbf{k}\cdot\mathbf{z}_n-k'z^0)]+\text{comp. conj.}\right\}. \quad (67)$$

Here $\mathbf{A}\cdot\mathbf{e}_3$ is understood as being expressed in terms of the α's by means of Eq. (60). Comparing Eqs. (65) and (67), we see that the α_0 terms in H_0^{long} just cancel the α_0 terms occurring explicitly in Eq. (67).

We now replace $\mathbf{p}_n$ by $\mathbf{p}_n'$, where

$$\mathbf{p}_n'=\mathbf{p}_n-e_n\boldsymbol{\alpha}\cdot((2\pi)^{-\frac{3}{2}}/\sqrt{2})\int d^3kT(k)\mathbf{e}_3\left\{\left[\mathbf{A}\cdot\mathbf{e}_3-\frac{ab\kappa}{\Lambda(1+\kappa)^{\frac{1}{2}}}\alpha_2\right]\exp[i(\mathbf{k}\cdot\mathbf{z}_n-k'z^0)]+\text{comp. conj.}\right\}. \quad (68)$$

But we see that

$$i[\mathbf{p}_n',L]=0,$$
$$i[\mathbf{p}_n',\boldsymbol{\phi}^{tr}]=\partial\boldsymbol{\phi}^{tr}/\partial\mathbf{z}_n,$$
$$i[p_{ni}',z_{mj}]=\delta_{ij}\delta_{nm},$$

and

$$i[\mathbf{p}_n',\mathbf{p}_m']=0. \quad (69)$$

The last relation holds in virtue of our choice of contour of integration which makes the field potentials commute at different space-points. But one can take the observable $\mathbf{p}_n'$ as $(1/i)(\partial/\partial\mathbf{z}_n)$ because of the relations (69) after the subsidiary condition has been applied. Hereafter, we drop the prime on $\mathbf{p}_n'$.

Collecting our results, we see that the Hamiltonian becomes

$$H=H_0^{tr}+H_\rho+H_C+H_1+H_2+H_3, \quad (70)$$

where H_1 is given by Eq. (66) and

$$H_C=(\pm)\int d^3k(k'/\mu)(ff^*/g^2)=\frac{(2\pi)^{-3}}{2}\sum_{n,m}e_ne_m\times\int\frac{d^3k}{k\Lambda\gamma}\frac{\mu(1-\kappa b^2)}{\Lambda^2k'}\exp[i\mathbf{k}\cdot(\mathbf{z}_n-\mathbf{z}_m)] \quad (71)$$

is the Coulomb interaction. H_2 is the last term in Eq. (67) and is

$$H_2=-\sum_n e_n\frac{(2\pi)^{-\frac{3}{2}}}{\sqrt{2}}\int d^3kT(k)\frac{\kappa a v_0}{\Lambda(1+\kappa)^{\frac{1}{2}}}\times[\alpha_2\exp[i(\mathbf{k}\cdot\mathbf{z}_n-k'z^0)]+\text{comp. conj.}]. \quad (72)$$

H_3 comes from the part of $\mathbf{A}\cdot\mathbf{e}_3$ in Eq. (67) that was not included in $\mathbf{p}_n'$ in Eq. (68).

$$H_3=-\sum_n e_n\boldsymbol{\alpha}\cdot[(2\pi)^{-\frac{3}{2}}/\sqrt{2}]\int d^3k\mathbf{e}_3T(k)\times[ab\kappa/\Lambda(1+\kappa)^{\frac{1}{2}}][\alpha_2\exp[i(\mathbf{k}\cdot\mathbf{z}_n-k'z^0)]+\text{comp. conj.}]. \quad (73)$$

Because of the form of the contour of integration, we can evaluate the integral giving the Coulomb energy in Eq. (71). If we drop the terms in Eq. (71) for which $n=m$, we can rewrite it as

$$H_C=\frac{(2\pi)^{-3}}{2}\sum_{n>m}e_ne_m\int\frac{d^3k}{k^2}\frac{\mu}{\Lambda^2\gamma}\left(1+\frac{\kappa b v_0}{\Lambda}\right)\times[\exp[i\mathbf{k}\cdot(\mathbf{z}_n-\mathbf{z}_m)]+\exp[-i\mathbf{k}\cdot(\mathbf{z}_n-\mathbf{z}_m)]]. \quad (74)$$

We assume that the quantities κ and μ are nonsingular at the origin in k-space. We can then displace the k-integral below the origin by a small semicircle, since this won't change the value of the integral. Performing the angular integrations we obtain terms that behave asymptotically with $k\rightarrow\infty$ as simple positive and negative exponentials in $ik|z_n-z_m|$. Since the contour is chosen above the special singularities caused by functional dependence of κ and μ for the positive exponential term and below the singularities for the negative exponential term, we can shrink the contour to a small circle enclosing the origin for the former and a small circle below the origin for the latter term. The integral may thus be evaluated in a neighborhood of the origin in which κ and μ are effectively constant, having the values.

$$\kappa_0=\kappa|_{k=0}, \quad \mu_0=\mu|_{k=0}.$$

We also have $\gamma=1$ when $k=0$. The κ_0bv_0/Λ term will vanish on performing the angular integrations, as it is odd in $\mathbf{k}$. For simplicity, we assume that $\mathbf{v}$ is parallel to the z axis. Then set

$$l_x=k_x, \quad l_y=k_y, \quad l_z=\left(\frac{1+\kappa_0}{1+\kappa_0v_0^2}\right)^{\frac{1}{2}}k_z,$$

$$\xi_x=(\mathbf{z}_n-\mathbf{z}_m)_x, \quad \xi_y=(\mathbf{z}_n-\mathbf{z}_m)_y,$$

$$\xi_z=\left(\frac{1+\kappa_0v_0^2}{1+\kappa_0}\right)^{\frac{1}{2}}(\mathbf{z}_n-\mathbf{z}_m)_z.$$

Thus

$$\int_c l^2 dl \int \frac{d(\cos\theta)d\phi}{l^2} e^{il\xi\cos\theta} = \int_C dl 4\pi\left[\frac{e^{il\xi}-e^{-il\xi}}{2il\xi}\right] = (2\pi)^2(1/\xi), \quad (75)$$

where $\xi = |\xi|$.

Returning to a general $\mathbf{v}$ direction, we have

$$H_C = \frac{1}{4\pi}\sum_{n>m} e_n e_m \frac{\mu_0}{(1+\kappa_0 v_0^2)^{\frac{1}{2}}}\{(\mathbf{z}_n-\mathbf{z}_m)^2(1+\kappa_0) + \kappa_0[\mathbf{v}\cdot(\mathbf{z}_n-\mathbf{z}_m)]^2\}^{-\frac{1}{2}}. \quad (76)$$

For the coordinate system in which $\mathbf{v}=0$, this becomes

$$H_C = (1/4\pi\epsilon_0)\sum_{n>m} e_n e_m/|\mathbf{z}_n-\mathbf{z}_m|, \quad (77)$$

where $\epsilon_0 = \epsilon|_{k=0}$, the static value of the dielectric constant. H_C is thus seen to be the same for a dispersive as for a non-dispersive medium, which is consistent with the notion that it represents a static interaction.

This completes the formal development of the theory. Before applying it to some examples, we must show its relation to the canonical treatment of Parts I and II in the limit of non-dispersive media. We shall find, as a matter of fact, that there exists a canonical transformation which brings the Hamiltonian of Part II into the form of the present Hamiltonian when κ and μ are constants; and, in particular, the Coulomb energy (Eq. (48,II)) will reduce to just Eq. (76), which holds for both dispersive and non-dispersive media.

D. THE LIMITING CASE OF NON-DISPERSIVE MEDIA

ϵ, μ, and κ are now considered to be constants as in the first two parts. We wish to show that we can perform a unitary transformation to bring the Hamiltonian of Part II into the form of Eq. (70).

Since the energy is multiplied by μ in Part II, and the quantity μ does not occur in a simple manner, we will set it equal to unity to simplify the analysis. If desired, we could redefine the A's so that μ did not occur in Eq. (51), and then could show the equivalence of the two forms of H in the general case; however, the added complication is of no interest.

The source of the apparent discrepancy in the two forms of the theory lies in the fact that in the first two parts we diagonalized only the transverse parts of the free-field Hamiltonian, whereas in the present discussion we chose operators α_μ that would diagonalize both the transverse and longitudinal parts of the Hamiltonian simultaneously. The present time-dependent operators α_μ are thus constants of motion when there is no interaction, whereas the quantities a_1 and a_2 defined by Eq. (71,I) have a rather complicated time dependence before the subsidiary condition is imposed.

It is thus necessary to examine the relation between the operators a_1 and a_2 of Parts I and II and the operators α_1 and α_2 appearing in the present treatment. This can probably be done most easily by expressing the operators $P_\nu(k)$ and $Q_\nu(k)$ of Eq. (57,I) in terms of the α's by expressing the potentials ϕ_σ in terms of the α's according to Eqs. (36) and (60), substituting these into Eqs. (22,I) and then comparing the resulting Fourier coefficients in Eq. (57,I) with those of Eqs. (70,I). However, we don't need the general expressions relating the a's to the α's, but just those which hold after the subsidiary condition has been imposed. We then obtain

$$a_1(k) = \alpha_1(k)$$

$$a_2(k) = \alpha_2(k) + \frac{\kappa ab}{\Lambda(1+\kappa)^{\frac{1}{2}}}\frac{(2\pi)^{-\frac{3}{2}}}{\sqrt{2}(\Lambda k)^{\frac{1}{2}}}\frac{1}{k}\sum_n e_n \times \exp[-i(\mathbf{k}\cdot\mathbf{z}_n - k'z^0)]. \quad (78)$$

But the transformation on a_2 in Eq. (78) can be expressed as

$$a_2(k) = S^{-1}\alpha_2(k)S, \quad (79)$$

where

$$S = \exp\left\{-\int d^3k\left[\alpha_2(k)\frac{\kappa ab}{\Lambda(1+\kappa)^{\frac{1}{2}}}\frac{(2\pi)^{-\frac{3}{2}}}{\sqrt{2}(\Lambda k)^{\frac{1}{2}}}\frac{1}{k}\sum_n e_n \times \exp[i(\mathbf{k}\cdot\mathbf{z}_n - k'z^0)] - \text{comp. conj.}\right]\right\}. \quad (80)$$

On performing the transformation (79) from the a's to the α's, the Hamiltonian is transformed as

$$\bar{H}(a) = S^{-1}H(\alpha)S, \quad (81)$$

where $\bar{H}(a)$ represents the Hamiltonian of Part II and $H(\alpha)$ is the same function of the α's. The transformation replaces the a's by the α's according to Eq. (78) in $\bar{H}(a)$ and also the momentum operator $\mathbf{p}_n$ of Part II is replaced by

$$S^{-1}\mathbf{p}_n S = \mathbf{p}_n - e_n\frac{(2\pi)^{-\frac{3}{2}}}{\sqrt{2}}\int\frac{d^3k}{(\Lambda k)^{\frac{1}{2}}}\mathbf{e}_3\frac{\kappa ab}{\Lambda(1+\kappa)^{\frac{1}{2}}} \times[\alpha_2\exp[i(\mathbf{k}\cdot\mathbf{z}_n - k'z^0)] + \text{comp. conj.}]. \quad (82)$$

Comparing this equation with Eq. (73), we see that

$$S^{-1}\left[\sum_n \boldsymbol{\alpha}\cdot\mathbf{p}_n\right]S = \sum_n \boldsymbol{\alpha}\cdot\mathbf{p}_n + H_3. \quad (83)$$

On making the substitutions (78) into the Hamiltonian of Part II, the Coulomb energy and H_2 of that section are transformed into the corresponding expressions (71) and (72). The equivalence of the two Hamiltonian forms is thus demonstrated.

It is interesting to note that by means of a second transformation of the type used here that one can completely eliminate the term H_2 from the Hamiltonian. In this case the Coulomb energy becomes equal to that of the classical theory, which can be directly obtained from Eq. (63,II). This transformation is given in appendix II.

E. ČERENKOV RADIATION

We assume that the medium is at rest and wish to calculate the Čerenkov radiation due to an electron traveling with a velocity u, as in Part II. κ, ϵ, and μ are now functions $(k^0)^2$ only, and we have

$$k^{02}=k^2/(\kappa+1) \tag{84}$$

to determine k^0 as a function of k. As the angles giving the direction of $\mathbf{k}$ are not involved in κ, Λ, γ, etc., the choice of contour of integration is considerably simplified. We must, however, choose it in such a manner that we obtain a δ-function in spite of the $1/\gamma$ term in such equations as (31). The singularities can be avoided by enclosing them in two boxes joined by a line crossing the real axis at a point P sufficiently far from the origin that the condition of energy conservation will make the integrand vanish before we reach it in the final integrations. The probability per unit time for the emission of a photon of momentum $\mathbf{k}$ is, as in Eq. (79,II),

$$dP=2\pi d\rho_F\langle|H_1|^2\rangle_{\text{Av}}. \tag{85}$$

$\langle|H_1|^2\rangle_{\text{Av}}$ is given by Eq. (80,II), except that μ occurs to the first power only (as we are not now multiplying our energy by μ) and that γ (Eq. (23)) occurs in the denominator. We suppose that the initial electron momentum is $\mathbf{p}$ and that after emitting the photon its momentum is $\mathbf{p}-\mathbf{k}$. The initial and final energies are, respectively,

$$E_p=(p^2+m^2)^{\frac{1}{2}},\quad E_F=(k/n)+[(p-k)^2+m^2]^{\frac{1}{2}}, \tag{86}$$

where $n=(\kappa+1)^{\frac{1}{2}}$ is the refractive index. We find that energy and momentum conservation give (Eq. (92,II))

$$\cos\theta=[E_p/pn]+[\kappa/(\kappa+1)][k/2p], \tag{87}$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{p}$.

The density of states is, as before,

$$d\rho_F=k^2(dk/dE_F)d\Omega, \tag{88}$$

where $d\Omega$ is the element of solid angle into which $\mathbf{k}$ is emitted. But

$$dE_F/dk=[d(k/n)/dk]+(k-p\cos\theta)[(\mathbf{p}-\mathbf{k})^2+m^2]^{-\frac{1}{2}}. \tag{89}$$

γ (Eq. (23)) is defined as

$$\gamma=1-k(d(1/n)/dk').$$

Thus

$$dk'/dk=d(k/n)/dk=1/n\gamma \tag{90}$$

in Eq. (89). Using the value of $\cos\theta$ from Eq. (87), we obtain, assuming $k\ll p$,

$$\frac{dk}{dE_F}\cong\frac{\kappa+1}{\kappa}\frac{2E_p}{k}\left[1+\frac{1-\gamma}{n\gamma}\frac{2E_p}{k}\frac{\kappa+1}{\kappa}\right]^{-1}. \tag{91}$$

Similarly, we obtain

$$\frac{d\cos\theta}{dk}\cong\frac{1}{2p}\frac{\kappa}{\kappa+1}\left[1+\frac{1-\gamma}{n\gamma}\frac{2E_p}{k}\frac{\kappa+1}{\kappa}\right], \tag{92}$$

which is valid when $k\ll p$. Combining results, we have

$$dP=\frac{1}{2}\frac{u}{4\pi^2}e^2\mu\left(1-\frac{1}{n^2u^2}\right)\frac{dk}{n\gamma}d\phi, \tag{93}$$

where ϕ is the azimuth angle. The rate at which energy is radiated is

$$\frac{dW}{dt}=\int dP\frac{k}{n}=\frac{e^2u}{4\pi}\int\mu\left(1-\frac{1}{n^2u^2}\right)k'dk' \tag{94}$$

since $dk'=dk/n\gamma$. This agrees exactly with the classical results of Frank and Tamm (references are given in footnotes 3 and 4 of Part II), except that they took $\mu=1$. In this integral n^2 can be expressed in terms of the frequency k' rather than k, which simplifies the integration. The integration in Eq. (94) can be performed entirely along the real axis, since $n^2u^2<1$ before we reach the point P at which the real axis may be cut.

We would like now to investigate the behavior of a charge at rest in a medium moving with velocity $u>c/n$. This is essentially the study of Čerenkov radiation in the coordinate system for which the electron is at rest. We must first modify the classical treatment of Part II to conform with a variable κ. We can start with Eq. (76,II); however, μ will now occur to the first power instead of the second, as the result of our dropping the factor μ in the definition of the energy. κ is a function of

$$z\equiv v^0k^0-\mathbf{v}\cdot\mathbf{k},$$

but in the present electrostatic case $k^0=0$ so

$$\kappa=\kappa[(-\mathbf{v}\cdot\mathbf{k})^2] \tag{95}$$

and similarly for ϵ and μ. When $\kappa v^2<1$ for all values of $\mathbf{k}$, the self-force vanishes as before. Otherwise we have an equation of the type (77,II). However, on integrating over the angles, the dependence of κ on the angles gives us an additional factor

$$\frac{1}{\gamma'}=\left[1+\frac{1}{2}\frac{k}{(\kappa)^{\frac{1}{2}}}\frac{d\kappa}{dz}\bigg|_{z=k/(\kappa)^{\frac{1}{2}}}\right]^{-1}, \tag{96}$$

where z has been written for $-\mathbf{v}\cdot\mathbf{k}$ and is set equal to $k/(\kappa)^{\frac{1}{2}}$ after the derivative $d\kappa/dz$ is evaluated. Then the self-force is given by (Eqs. (77,II) and (78,II))

$$\mathbf{F}=\frac{e^2}{4\pi}\mathbf{v}\int\frac{\mu}{\kappa+1}\frac{(\kappa v^2-1)}{\kappa v^3}\frac{kdk}{\gamma'}. \tag{97}$$

We now turn to the corresponding quantum-mechanical treatment of this problem. The treatment is very similar to that of Part II. We calculate the transition probability for the emission of a photon of momentum $\mathbf{k}$ by an electron at rest in the moving medium. We can assume that $k\ll m$ (since dispersion is assumed to cut off the integrals to satisfy this condition). This probability is (Eq. (99,II))

$$dP'=2\pi d\rho_F\langle|H_2|^2\rangle_{\text{Av}}, \tag{98}$$

where H_2 is now given by Eq. (72). If the electron is initially at rest, it has a momentum $-\mathbf{k}$ in the final state. In working out the condition for energy conservation, we may expand in powers of k/m, keeping only the lowest powers, except that κ must not be expanded as it is assumed to be a rapidly varying function. We thus obtain, as in Part II,

$$\cos\alpha\cong-[v^0/2mv]k-[1/(\kappa v^2)^{\frac{1}{2}}] \tag{99}$$

for the value of the angle between $\mathbf{k}$ and $\mathbf{v}$. Since κ is a function of $z=v^0k'-\mathbf{v}\cdot\mathbf{k}$, this is only an implicit equation for $\cos\alpha$. Equation (99) is obtained from

$$E_F=(k^2/2m)+k'=0,$$

where E_F is the energy of the final state. For γ (Eq. (23)) we have

$$\gamma=1-(\partial k_0'/2k')\cong1+\tfrac{1}{2}(k/(\kappa)^{\frac{1}{2}})(d\kappa/dz) \tag{100}$$

using Eq. (99). From these results, we have

$$dk/dE_F\cong2m/k[1-[2m/k]\times[(\gamma-1)/\gamma][1/(\kappa)^{\frac{1}{2}}v^0]]^{-1} \tag{101}$$

and

$$d\cos\alpha/dk\cong-[v^0/2mv][1-[2m/k]\times[(\gamma-1)/\gamma][1/(\kappa)^{\frac{1}{2}}v^0]]. \tag{102}$$

Then

$$d\rho_F=k^2(dk/dE_F)d\Omega\cong-kdk(v^0/v)d\phi, \tag{103}$$

where ϕ is the azimuth angle. Finally

$$\langle|H_2|^2\rangle_{\text{Av}}=[e^2\mu/2(2\pi)^3][1/v^0]\times[(\kappa v^2-1)/(\kappa)^{\frac{1}{2}}(\kappa+1)][1/\gamma], \tag{104}$$

so

$$dP'=[e^2/2][1/4\pi^2][\mu/(1+\kappa)(\kappa)^{\frac{1}{2}}]\times[1/\gamma][dk/v](\kappa v^2-1)d\phi. \tag{105}$$

The force on the electron is

$$\mathbf{F}=-(\mathbf{v}/v)\int k\cos\alpha dP'$$

$$=(e^2/4\pi)\mathbf{v}\int[(\kappa v^2-1)\mu/(1+\kappa)\kappa v^3][kdk/\gamma]. \tag{106}$$

We see that this agrees exactly with the classical result (97) in our approximation, as $z\equiv v^0k'-\mathbf{v}\cdot\mathbf{k}\simeq(k/(\kappa)^{\frac{1}{2}})$; and thus $\gamma'=\gamma$, as can be seen from Eqs. (96) and (100). That the contour of integration is along the real axis in Eq. (106) follows from arguments similar to those given for the preceding example.

It is apparent from the two examples discussed above that applications of the theory may not be as complicated as would appear from its general formulation. Thus in simple radiation processes such as those considered it is not necessary to solve for k' and κ as functions of $\mathbf{k}$ in general. Rather, the conditions for energy and momentum conservation may lead to quite simple results as is seen above. For instance, if we chose a simple relation for κ such as

$$\kappa=\text{constant}/(1+\lambda k'^2) \tag{107}$$

in the discussion of Čerenkov radiation (when $\mathbf{v}=0$), we would find the k' had eight branch points in the complex k-plane. ϵ, μ would have considerably more, including several poles. By our treatment we need never solve for k' as a function of $\mathbf{k}$, but can substitute Eq. (107) directly into Eq. (94).

F. THE λ-LIMITING PROCESS

If we set $\epsilon\mu=1$, then $\kappa=0$ and the medium velocity no longer appears explicitly in our equations for the field variables or in the Hamiltonian. Its only occurrence is in ϵ and μ, so we may treat it as an arbitrary parameter. In particular, we may set

$$e(\mathbf{v},v^0)=(\boldsymbol{\lambda},\lambda^0)$$

where e is a number.

Then if we take $\mu=\cos(k'\lambda^0-\mathbf{k}\cdot\boldsymbol{\lambda})$, our theory becomes equivalent to the λ-limiting process. For instance, the commutation rules (38) become

$$[A_\nu(k),A_\mu^+(l)]=g_{\nu\mu}\cos(k'\lambda^0-\mathbf{k}\cdot\boldsymbol{\lambda})\delta(\mathbf{k}-\mathbf{l}). \tag{108}$$

G. CONCLUDING REMARKS

We have given a formal extension of the quantum mechanics of the electromagnetic field. This generalization was obtained by subjecting the classical phenomenological field equations of Maxwell to the process of quantization. In doing this it is no longer possible to maintain the principle of relativity. This is only natural since a ponderable medium introduces automatically a preferred coordinate system, namely, the rest system of the medium, and there is no objection to this from a physical point of view.

It might appear to be worth while relaxing some of our restrictive requirements on the functional forms of κ and μ in order to apply the theory to the vacuum case. However, in doing this it seems to be impossible to avoid conflict with the relativity postulate. This conflict would not be of the "crude" type which could be detected by a Michelson-Morley experiment or similar arrangement, since the kinematics is that of the Lorentz group. However, there might occur results for certain scattering cross sections which would depend on the absolute motion of the coordinate system. A consequence of this sort is of such a radical nature that we do not yet feel justified to propose a phenomenological electrodynamics for the vacuum. The same reason prevents us from applying such generalized theories to meson fields.

APPENDIX I

We must justify our being able to choose the contour of integration in such a manner that we can avoid undesirable contributions to the integrals in k-space arising from the special singularities caused by the dependence of ϵ, μ, and κ on $k'v^0-\mathbf{v}\cdot\mathbf{k}$. We have assumed that ϵ and μ approach unity and $\kappa\rightarrow 0$ as their arguments become infinite, and that these quantities are all analytic in a neighborhood of the point at infinity. Writing $1/z\equiv k'v^0-\mathbf{v}\cdot\mathbf{k}$, we have

$$\kappa=\sum_{p=1}^{\infty} a_p z^{2p} \tag{A1}$$

as an absolutely convergent series in a certain neighborhood of $z=0$. Furthermore, we may expand the quantity k_0' of Eq. (16) into an absolutely convergent series in κ, when κ is sufficiently small,

$$k_0'=k\left[1+\sum_{q=1}^{\infty} b_q\kappa^q\right]=k\left[1+\sum_{m=1}^{\infty} c_m z^{2m}\right] \tag{A2}$$

by inserting Eq. (A1) into the first equation in (A2). The final power series in (A2) is again absolutely convergent for sufficiently small z. Finally,

$$z\equiv 1/(k'v^0-\mathbf{v}\cdot\mathbf{k})=[1/k]\left[(1/(v^0-b))+\sum_{n=1}^{\infty} d_n z^{2n}\right] \tag{A3}$$

is obtained by using Eq. (A2). The coefficients d_n in this equation will be expressed in terms of the coefficients c_m of Eq. (A2), which in turn will be expressed in terms of the coefficients a_p of Eq. (A1). The right-hand side of Eq. (A3) is again absolutely convergent in some region about $z=0$. By means of successive approximations, we can solve Eq. (A3) formally for z as a function of k. We obtain

$$z=[1/k(v_0-b)]+\sum_{\lambda=2}^{\infty} f_\lambda(1/k^\lambda). \tag{A4}$$

We must prove that the expression given by Eq. (A4) converges to the solution of Eq. (A3) for large enough k. We have when $|z|<Q$, where Q is some suitable positive number,

$$\sum_{p=1}^{\infty} |d_p|\,|z|^{2p}<D$$

(D is some positive number) by hypothesis. Now let us assume that the $n-1$th approximation, z_{n-1}, is such that

$$|z_{n-1}|<Q'<Q.$$

Then

$$\left|\sum_p{}' d_p(z_{n-1})^{2p}\right|<D,$$

where $\sum'$ means the summation used in finding the nth approximation (in which all powers of z higher than the nth are dropped). Then

$$|z_n|<[1/k][(1/(v^0-b))+D]<Q' \quad \text{when } k>[1/Q'][(1/(v^0-b))+D]. \tag{A5}$$

But this means that all the approximations are bounded and within the circle of convergence of Eq. (A3) (since the first can be so chosen) when k satisfies Eq. (A5). But since z_n differs from z_{n-1} only in powers of k greater than the $(n-1)$th, we can conclude that

$$\lim_{n\rightarrow\infty}|z_n-z_{n-1}|=\lim_{n\rightarrow\infty}\left|\sum_{p=n}^{\infty} d_p(z_{n-1}{}^p-z_{n-2}{}^p)\right|=0.$$

Thus Eq. (A4) converges absolutely for values of k satisfying Eq. (A5).

Then κ, Λ, k', and $1/\gamma$ occurring in the integrals in k-space can all be expanded into convergent power series in z, and thus in k^{-1} for sufficiently large k. It should be noted that it is easily verified that we can make the substitution of Eq. (A4) into absolutely convergent series in z to obtain convergent series when k is sufficiently large. It is also seen that the coefficients in these power series, which involve the angles, have no singularities (as $k'v^0-\mathbf{v}\cdot\mathbf{k}>0$ for real $k>0$). The angular integrations can be performed, leading to analytic functions of k as coefficients of the power series in k^{-1}. Thus the integrands involving the final integration over k will be analytic in a neighborhood of $k=\infty$.

Turning to Eq. (A3), we note that replacing k by $-k$ and z by $-z$ leads to the same equation, so z must be an odd function of k. From Eq. (A2), we see that k' is also an odd function of k. Equation (A1) shows that κ is an even function of k. Also Λ and γ are even functions of k.

APPENDIX II

It was noted above that it is possible to eliminate the H_2 term (Eq. (72)) entirely from the Hamiltonian by means of a unitary transformation (we assume that the medium is non-dispersive for simplicity). This transformation is

$$\alpha_2 = T^{-1} q T, \tag{A6}$$

where q is an operator satisfying the same commutation rules as α_2. Here T is

$$T = e^{D q^+ - D^* q} \tag{A7}$$

with

$$D = \sum_n e_n [(2\pi)^{-\frac{3}{2}}/\sqrt{2}][\kappa a v_0/\Lambda(1+\kappa)^{\frac{1}{2}}] \times [1/(\Lambda k)^2][1/k'] \exp[-i(\mathbf{k}\cdot\mathbf{z}_n - k' z^0)]. \tag{A8}$$

This leads to a Coulomb term

$$H_C = \sum_{n>m} [1-(\kappa/(1+\kappa))v_0^2] e_n e_m \Psi^0(\mathbf{z}_n - \mathbf{z}_m) \tag{A9}$$

in agreement with Eq. (70,II). Here Ψ^0 is defined by Eq. (59,II). We thus see that our Hamiltonian can be put into the form given by the classical theory. (There are some extra c-number vector potential terms, as in Eq. (70,II).)

It is rather curious that the term H_2 can be so eliminated. When $\kappa v^2 > 1$, $\Psi^0(z)$ is infinite everywhere on a cone (Eq. (68,II)), whereas in the previous forms H_C is finite except at points $z_n = z_m$. In the calculation of the self-force on an electron at rest it was shown that H_2 was physically meaningful (as it accounted for the force). However, when we perform the transformation (A6) and if we discard the infinite self-energy electrostatic term corresponding to $n=m$ in Eq. (A12), we lose entirely the effect of H_2 when only one electron is present. This paradox occurs because it is not always a consistent procedure to discard infinite terms just because they are infinite. It is to be recalled that in calculating the self-force classically in Part II that it was necessary to assume an extended source model for the electron, then to let the radius of the source approach zero after the self-force was calculated. Were we to assume an extended source model also in our quantum theory, then $\sum_{n>m}$ in Eq. (A9) could be interpreted as an integration over the charge elements. The self-force would then be given by the expectation value of

$$\sum_n \dot{\mathbf{p}}_n = i[H, \sum_n \mathbf{p}_n],$$

which must be in agreement with the classical result. It appears that a discussion of the problem in terms of point charges (in either classical or quantum theory) must await a more complete theory of the role played by self-energy terms in electrodynamics.