The Electromagnetic Shift of Energy Levels

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The effect of the interaction with the radiation field in changing the energy levels of an electron in an external field is calculated using the conventional form of perturbation theory. The infinite self-energy of the electron which occurs in the same approximation is removed by subtracting from the Hamiltonian a "mass operator" M. The criteria used in deriving M are that it should correctly give the self-energy for a free electron in the absence of an external field and that the amended Hamiltonian should give a properly covariant form for the level shift in an external field. It is pointed out that M is uniquely determined by these requirements.

The results give 1051 mc/sec. for the $2s_{\frac{1}{2}}-2p_{\frac{1}{2}}$ separation in hydrogen and also show the surplus magnetic moment of the electron as $\alpha/2\pi$ Bohr magnetons, as first found by Schwinger. The amended Hamiltonian can be used for determining the radiative corrections for other processes.

I. INTRODUCTION

RECENT measurements, for example the work of Lamb and Retherford[1] on the fine structure of hydrogen, indicate that calculations made on the basis of the Dirac theory of the electron should be modified.

This, however, does not necessarily indicate a failure of the theory. Many calculations, for example of atomic energy levels, are made without taking into account the interaction of charged particles with the radiation field. We might reasonably expect that when the radiative coupling is properly considered the energy values would in general be different from those calculated in the usual way. The difficulty, however, is that the energy values thus calculated are found to involve divergent integrals. Thus each energy is infinite and moreover the difference of energy between two levels will in general be also infinite. It is for this reason of course that the interaction is omitted in the usual calculation. It is, however, significant that the energy levels calculated with omission of this interaction are only very slightly different from those observed experimentally. Thus though the radiative coupling gives rise to infinite level shifts, yet in a very real sense it behaves as a small perturbation.

A separate aspect of the problem is due to the fact that the ordinary one-electron Dirac theory is incapable of explaining positron phenomena and is moreover plagued by the negative energy solutions. The Dirac hole theory disposes of these difficulties and it is on the basis of hole theory that the present calculations will be made.

It is well known that the introduction of hole theory reduces the degree of the divergences which are encountered. In place of linear divergences which are encountered in the one-electron theory we meet only logarithmic divergences. This has the important consequence that when we carry through the renormalization of electron mass and charge, to be discussed below, we shall get finite observable effects due to radiative coupling. This would not be true in the one-electron theory.

The other main result of introducing hole theory is the phenomenon of vacuum polarization in the presence of an external field, which also leads to logarithmically divergent expressions. It has long been recognized that the main effect of the vacuum polarization is to effectively increase the electron charge from e_0 to e where

$$e=e_0(1+\delta), \quad \delta\sim(e^2/\hbar c)K.$$

Here K is a logarithmically divergent integral. We argue that when the entire theory is eventually modified for very high momentum exchanges a result will be to make K convergent and of order unity or smaller. Without this belief indeed the application of a perturbation theory would be senseless.

But now with the argument that the measured charge e includes the additional "induced" charge we see that the term in the Hamiltonian of the system which corresponds to this renormalization should be omitted. The residue is then interpreted as the physically observable part of the interaction which produces the vacuum polarization.[2,3]

In precisely the same way it has been argued[4–9] that the infinite energy which arises when we consider the radiative coupling (without the vacuum polarization terms) is mainly due to a term which

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represents an increase in the effective electron mass. The radiative coupling will be considered as a small perturbation, its smallness being characterized by the fact that in a future theory the electromagnetic mass should be of the order $(e^2/\hbar c)m_0$ where m_0 is the mechanical mass. Arguing as above we should therefore subtract this mass renormalization term from the Hamiltonian. The residue will then give rise to observable effects. The separation of the mass renormalization term is however more difficult than the separation of the charge term. We shall consider the self-energy of a free electron (which should of course be exclusively a mass renormalization term) and we shall be guided by this in finding the renormalization operator (the "mass" operator) in the case of an external field. The fact that the self-energy of a free electron diverges, introduces ambiguities in the determination of the mass operator. The choice is determined in this paper by requiring that the results be Lorentz invariant. This is discussed in Section III of this paper. Some evidence can be given that this determination is unambiguous.

In the present work we shall consider an electron in an external time-independent electromagnetic field. We shall evaluate to the first non-vanishing order the perturbation energy which results when we add the radiative coupling to the Hamiltonian of the system. We shall then separate out the part which may be regarded as energy due to electromagnetic mass and also the terms corresponding to a charge renormalization. The residue which will be finite will be regarded as the true level shift.

II. THE PERTURBATION ENERGY

1. We consider an electron in a stationary state ψ_0 of an external time-independent electromagnetic field. The state of the vacuum will be that where all negative energy states are filled; the physical situation in which we are interested will have all the negative energy states and one positive energy state ψ_0 filled.

We regard the radiative coupling as a perturbation. As the significant energy we take the difference between the energies of the two systems: vacuum plus one electron in ψ_0, vacuum.

$$W=W_{\text{vac}+1}-W_{\text{vac}}. \tag{1}$$

We shall call W the perturbation energy. At first we shall consider separately, in the usual way, the transverse (electrodynamic) and longitudinal (electrostatic) parts of the perturbation energy.

2. *The electrodynamic energy* (W^D). We expand $\mathbf{A}$, the transverse vector potential of the radiation field, in plane waves. We regard the wave function function of the entire system as a function of the photon and electron occupation numbers. For physical states where no photons are present we then find that $H_{\text{int}}=-e\sum_{\text{electrons}}\boldsymbol{\alpha}\cdot\mathbf{A}$ transforms as follows:

$$H_{\text{int}}\rightarrow -eh(2\pi c)^{\frac{1}{2}}\sum_{\substack{r,s,\mathbf{k}\\ \lambda=1,2}}\frac{1}{(k)^{\frac{1}{2}}}\langle\psi_r^*\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}_\lambda \times\exp(\pm i\mathbf{k}\cdot\mathbf{r}/h)\psi_s\rangle a_r^* a_s$$

where a^*, a are the usual electron creation and destruction operators and $\langle\ \rangle$ denotes a matrix element. We take the $+$ or $-$ sign in the exponential for transitions involving respectively, absorption or emission of a photon. $\boldsymbol{\varepsilon}_\lambda$ is a unit polarization vector and $\boldsymbol{\varepsilon}_1\cdot\mathbf{k}=\boldsymbol{\varepsilon}_2\cdot\mathbf{k}=\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}_2=0$. We shall often write $\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}_\lambda=\alpha_\lambda$.

A second-order perturbation calculation with this interaction gives

$$W=\frac{\alpha c^2}{4\pi^2}\int\frac{d\mathbf{k}}{k}\sum_{\substack{rs\\ \lambda=1,2}}\left\{\frac{A^\lambda{}_{srrs}N_s(1-N_r)}{E_s-E_r-ck}-\frac{A^\lambda{}_{rrss}N_sN_r}{ck}\right\}, \tag{2}$$

where N_s are the electron occupation numbers and

$$A^\lambda{}_{klmn}=\langle\psi_k^*\alpha_\lambda\exp(-i\mathbf{k}\cdot\mathbf{r}/\hbar)\psi_l\rangle \times\langle\psi_m^*\alpha_\lambda\exp(+i\mathbf{k}\cdot\mathbf{r}/\hbar)\psi_n\rangle. \tag{3}$$

We then get

$$W^D=W^{DX}+W^{DN}, \tag{4}$$

$$W^{DX}=\frac{\alpha c^2}{4\pi^2}\int\frac{d\mathbf{k}}{k}\sum_{\substack{J\\ \lambda=1,2}}\frac{A^\lambda{}_{0J0}}{E_0-E_J-ck\delta_J}, \tag{5}$$

$$W^{DN}=\frac{-\alpha c}{2\pi^2}\int\frac{d\mathbf{k}}{k^2}\sum_{\substack{J^-\\ \lambda=1,2}}A^\lambda{}_{00JJ}, \tag{6}$$

where $\sum_{J^-}$ means a sum over negative energy states and $\delta_J=E_J/|E_J|=\pm1$. We call W^{DX} the electrodynamic exchange energy and W^{DN} the non-exchange energy. The reason for this separation and nomenclature will become obvious later.

3. *The Electrostatic Energy* (W^S). The electrostatic energy which represents the Coulomb interaction energy and self-energy of the electrons can be written as the average value of the operator

$$\tfrac{1}{2}e^2\int\{\chi^*(\mathbf{r})\chi(\mathbf{r})\}\{\chi^*(\mathbf{r}')\chi(\mathbf{r}')\}\frac{d\mathbf{r}d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|},$$

where $\chi=\sum a_s\psi_s$ and as before the a_s and ψ_s are the electron operators and eigenfunctions. The product $\{\ \}$ is the spin scalar product.

We now find as before that W^S can be divided into exchange and non-exchange parts.

$$W^S = W^{SX} + W^{SN}, \tag{7}$$

$$W^{SX} = \frac{\alpha c}{4\pi^2} \int \frac{d\mathbf{k}}{k^2} \sum A^4{}_{0JJ0}\delta_J, \tag{8}$$

$$W^{SN} = \frac{\alpha c}{2\pi^2} \int \frac{d\mathbf{k}}{k^2} \sum_{J-} A^4{}_{00JJ}, \tag{9}$$

where $A^4{}_{klmn}$ is as defined before with the understanding that $\alpha_4 = 1$.

4. *Symmetric Form of the Perturbation Energy.* The expressions above for W^D and W^S are in different form corresponding to the fact that we have divided the total field into two parts and treated each part in a different fashion. We may now write W^S in a form closely analogous to W^D. This is of little importance in calculating W^S and W^D since the two methods will of course give the same result. But the difference will be found to be of importance when we later consider the subtraction procedure to eliminate the energy due to electromagnetic mass.

If H be the one-electron Hamiltonian, ψ_n and E_n its eigenfunctions and energy values, we have

$$\langle \psi_n^* [H \exp(\pm i\mathbf{k}\cdot\mathbf{r}/\hbar) - \exp(\pm i\mathbf{k}\cdot\mathbf{r}/\hbar) H] \psi_m \rangle = (E_n - E_m) \langle \psi_n^* \exp(\pm i\mathbf{k}\cdot\mathbf{r}/\hbar) \psi_m \rangle.$$

But

$$H \exp(\pm i\mathbf{k}\cdot\mathbf{r}/\hbar) - \exp(\pm i\mathbf{k}\cdot\mathbf{r}/\hbar) H = \pm c\boldsymbol{\alpha}\cdot\mathbf{k} \exp(\pm \mathbf{k}\cdot\mathbf{r}/\hbar).$$

So

$$(E_n - E_m) \langle \psi_n^* \exp(\pm i\mathbf{k}\cdot\mathbf{r}/\hbar) \psi_m \rangle = \pm ck \langle \psi_n^* \alpha_k \exp(\pm i\mathbf{k}\cdot\mathbf{r}/\hbar) \psi_m \rangle, \tag{10}$$

where $\alpha_k = (1/k)\boldsymbol{\alpha}\cdot\mathbf{k}$.

Using this we find that we may write

$$W^{SX} = \frac{\alpha c^2}{4\pi^2} \int \frac{d\mathbf{k}}{k} \sum_J \frac{1}{(E_0 - E_J - ck\delta_J)} \times \{A^3{}_{0JJ0} - A^4{}_{0JJ0}\}, \tag{11}$$

where in $A^3{}_{klmn}$, α_λ is replaced by α_k.

Adopting now the convention that

$$\sum_\lambda{}' F(\lambda) = F(1) + F(2) + F(3) - F(4), \tag{12}$$

we have

$$W^X = W^{SX} + W^{DX} = \frac{\alpha c^2}{4\pi^2} \int \frac{d\mathbf{k}}{k} \sum_{\lambda J}{}' \frac{A^\lambda{}_{0JJ0}}{(E_0 - E_J - ck\delta_J)}, \tag{13}$$

In evaluating the sum over λ we can most simply take $\alpha_\lambda = \alpha_1, \alpha_2, \alpha_3, 1$.

To combine the non-exchange terms we note that $A^3{}_{00JJ} = 0$ (by an application of (10)). We find too that

$$\int (d\mathbf{k}/k^2) A^\lambda{}_{klmn} = 2\pi^2 \hbar B^\lambda{}_{klmn}, \tag{14}$$

where

$$B^\lambda{}_{klmn} = \int \{\psi_k^*(\mathbf{r}) \alpha_\lambda \psi_l(\mathbf{r})\} \times \{\psi_m^*(\mathbf{r}') \alpha_\lambda \psi_n(\mathbf{r}')\} (d\mathbf{r} d\mathbf{r}'/|\mathbf{r} - \mathbf{r}'|). \tag{15}$$

Then

$$W^N = W^{DN} + W^{SN} = -e^2 \sum_{\lambda J-}{}' B^\lambda{}_{00JJ} = \int [\rho_0(\mathbf{r}) \rho_{\text{neg}}(\mathbf{r}') - (1/c^2) \mathbf{J}_0(\mathbf{r}) \cdot \mathbf{J}_{\text{neg}}(\mathbf{r}')] \times (d\mathbf{r} d\mathbf{r}'/|\mathbf{r} - \mathbf{r}'|). \tag{16}$$

The last step follows from the definition of $B^\lambda{}_{klmn}$ and the identification of $ec\boldsymbol{\alpha}$ as the current operator. ρ_0 and $\mathbf{J}_0$ are here the charge and current densities due to the electron in ψ_0; ρ_{neg} and $\mathbf{J}_{\text{neg}}$ are the densities due to the electrons in the negative energy states.

To summarize we have now, for the total perturbation energy, $W^X + W^N$ given by Eqs. (13), (16). We now examine each of these separately.

5. *The Non-Exchange Energy.* The non-exchange energy has been considered by many authors, first by Uehling [10] for the case of an external electrostatic field. When the eigenfunctions occcurring in W^N (Eq. (16)) are expanded in powers of the external potential the zero order term gives simply the interaction with the unperturbed vacuum electrons which is of no physical interest. The first order term contains a divergent charge renormalization energy and a finite term ΔW^N which we regard as an observable effect. The higher powers are of no interest to us. We get then

$$\Delta W^N = -\alpha/15\pi(\hbar/mc)^2 [\nabla^2 V - e\boldsymbol{\alpha}\cdot\nabla^2\mathbf{A}]_{\text{Av}}, \tag{17}$$

where $\phi = V/e$ and $\mathbf{A}$ are the potentials for the external field.

In the case of no external field, only the zero order term remains and thus we may say that the non-exchange energy vanishes for a free electron.

[10] E. A. Uehling, Phys. Rev. **48**, 55 (1935).

6. *The Exchange Energy.* We now put W^X (Eq. (13)) in a form suitable for calculation. We treat ψ_0 exactly but expand the intermediate states in powers of $U=e\phi-e\boldsymbol{\alpha}\cdot\mathbf{A}$ where $\phi, \mathbf{A}$ refer to the applied field. This procedure is clearly not accurate for low-lying intermediate states. We therefore restrict ourselves to values of k (the momentum exchange) which are greater than $\delta\mu$ where we consider $\delta\sim 1/137$ and $\mu=mc$. For $k<\delta\mu$ we shall be able to use the non-relativistic calculation of Bethe.[4] The two parts of the result will join correctly.

We expand ψ^J as

$$\psi^J=\phi^J(\mathbf{q})+\sum\frac{U_{q'q}{}^{J'J}}{E_J(q)-E_{J'}(\mathbf{q}')}\phi^{J'}(\mathbf{q}')+\cdots, \quad (18)$$

where $\phi^J(\mathbf{q})=u^J(\mathbf{q})\exp(i\mathbf{q}\cdot\mathbf{r}/\hbar)$ is a Dirac free electron solution and $U_{qp}{}^{rs}=\langle\phi^{*r}(\mathbf{q})U\phi^s(\mathbf{p})\rangle$.

We insert this expansion for the intermediate states into W^X and collect together the terms of different orders in U. We have then

$$W^X=W_0{}^X+W_1{}^X+\cdots \quad (19)$$

where, for example,

$$W_0{}^X=\frac{-\alpha c^2}{4\pi^2}\int\frac{d\mathbf{k}}{k}\sum_{\lambda J,\mathbf{q}}{}'\,\delta_J\frac{\langle\psi_0{}^*\alpha_\lambda\exp(i\mathbf{k}\cdot\mathbf{r}/\hbar)\phi^J(\mathbf{q})\rangle\langle\phi^{*J}(\mathbf{q})\alpha_\lambda\exp(-i\mathbf{k}\cdot\mathbf{r}/\hbar)\psi_0\rangle}{(E(\mathbf{q})-E_0\delta_J+ck)},$$

where E_0 is the energy of ψ_0 and

$$E(q)=+[c^2q^2+\mu^2c^2]^{\frac{1}{2}}.$$

We introduce now the projection operators $G^\pm$ which we shall use all throughout the calculation. These operators when operating on free electron states of momentum $\mathbf{p}-\mathbf{k}$ select those of $\pm$ energy.

$$G^\pm=\tfrac{1}{2}\left[1\pm\frac{c\boldsymbol{\alpha}\cdot(\mathbf{p}-\mathbf{k})+\beta\mu c}{E(\mathbf{p}-\mathbf{k})}\right]=\tfrac{1}{2}[1\pm H/E]. \quad (20)$$

We write

$$(E(\mathbf{p}-\mathbf{k})\mp E_0+ck)=B_\pm. \quad (21)$$

We can now write the terms of W^X as expectation values in ψ_0 of certain operators. We get

$$W_0{}^X=-\frac{\alpha c^2}{4\pi^2}\int\frac{d\mathbf{k}}{k}\left[\sum_\lambda{}'\,\alpha_\lambda\left\{\frac{G^+}{B^+}-\frac{G^-}{B^-}\right\}\alpha_\lambda\right]_{Av}, \quad (22)$$

$$\begin{aligned}W_1{}^X=\frac{\alpha c^2}{4\pi^2}\int\frac{d\mathbf{k}}{k}\Bigg[\sum_\lambda{}'\,\alpha_\lambda\Bigg\{&\frac{G^+U}{B_+}\frac{G^+}{B_+}+\frac{G^-}{B_-}\frac{UG^-}{B_-}\\&-\frac{G^+}{B_+}\frac{UG^-}{B_-}-\frac{G^-}{B_-}\frac{UG^+}{B_+}\Bigg\}\alpha_\lambda\Bigg]_{Av}\\&-\frac{\alpha c^2}{4\pi^2}\int\frac{d\mathbf{k}}{k}\Bigg[\sum_\lambda{}'\,\alpha_\lambda L\Bigg\{\frac{G^+}{B_+}\frac{UG^-}{B_-}\\&+\frac{G^-}{B_-}\frac{UG^+}{B_+}\Bigg\}\alpha_\lambda\Bigg]_{Av}. \quad (23)\end{aligned}$$

In the second part of $W_1{}^X$ we have

$$L=\frac{2ck}{E(\mathbf{p}-\mathbf{k})+E(\mathbf{p}'-\mathbf{k})}, \quad (24)$$

where it is to be understood that p stands to the left of U and p' to the right. Both $W_0{}^X$ and $W_1{}^X$ diverge. The higher terms however are convergent. It will become plain later that the higher terms will not contribute to the level shift and it is for this reason that we have written no terms of higher order than $W_1{}^X$ above.

III. THE ELECTROMAGNETIC MASS OPERATOR

We must now separate from W^X that part W^M which is due to electromagnetic mass; or, more generally (for use e.g. in scattering problems), we should find an operator M which represents the e.m. mass effect. This operator, which should be proportional to the Dirac operator β, will then be subtracted from the one-electron Hamiltonian in accordance with our idea of mass renormalization. For level shifts we subtract from W^X the average value W^M of this operator in ψ_0.

We proceed as follows; let us find an operator R such that for a free electron in the state ϕ the self-energy is $\langle\phi^*R\phi\rangle$. We would then like to argue that R is the desired mass operator M. There are however ambiguities. The self-energy of a free electron involves only the diagonal elements of an operator (in terms of free electron solutions) and thus from the self-energy the non-diagonal elements cannot be determined. Thus M is determined by the above procedure only to within an operator with no diagonal elements.

The ambiguity may be removed by definitely choosing M so that it is a multiple of $\beta\mu$. We meet here the fact that M is divergent and we encounter all the difficulties involved in handling infinite quantities in an unambiguous fashion. (For example the usual calculation for the free electron self-energy does give for the divergent part a term in $\beta\mu_{Av}$ but there are finite terms not of this form.)

One method of treating the problem is the relativistic "cut-off" procedure introduced by Feynman.[8] This method avoids the ambiguity by modifying the electromagnetic interaction so that the divergent integrals become finite. We shall not discuss the procedure at this point though in part V we shall give a simple application of it.

Alternately we shall use as a criterion the requirement that the result, for the level shift in an external field be in a relativistically invariant form. Specifically we shall calculate the level shift in both an electrostatic and a magnetic field and combine the results to give the level shift in a general e.m. field. We will find that this may be written as the average value of the following operator

$$c_1e\phi-c_2e\boldsymbol{\alpha}\cdot\mathbf{A}+c_3\tfrac{1}{2}(\hbar/mc)^2SV+c_4(e\hbar/2mc)\beta\boldsymbol{\sigma}\cdot\mathbf{H}$$
$$+(\hbar/mc)^2\nabla^2(c_5\phi-c_6e\boldsymbol{\alpha}\cdot\mathbf{A}), \quad (25)$$

where SV is the operator which enters in the interaction of a magnetic moment with the electrostatic field. It is defined later by Eq. (36) and its relativistic equivalent is $(mc/\hbar)i\beta\boldsymbol{\alpha}\cdot\nabla V$. The c's are numerical constants of order α. For an invariant result we must have $c_1=c_2$, $c_3=c_4$ (also $c_5=c_6$ but we shall not however calculate c_6). We shall require that M be so chosen that these conditions are satisfied. It will be noted that when $c_1=c_2$ the first two terms simply form a charge renormalization term of no significance. It will however turn out that $c_1=c_2=0$. The identity of c_3 and c_4 insures that the correction to the magnetic moment will be identical if measured by the interaction with an electrostatic field or with a magnetic field.

We see from Eq. (22) that we can write the self-energy of an electron with momentum $\mathbf{p}$ as

$$W(\text{self})=\frac{-\alpha c^2}{4\pi^2}\int\frac{d\mathbf{k}}{k}\Big[\sum_\lambda{}'\Big\{\frac{\alpha_\lambda G^+\alpha_\lambda}{E(\mathbf{p}-\mathbf{k})-E(\mathbf{p})+ck}$$
$$-\frac{\alpha_\lambda G^-\alpha_\lambda}{E(\mathbf{p}-\mathbf{k})+E(\mathbf{p})+ck}\Big\}\Big]_{\text{Av}}. \quad (26)$$

We therefore define

$$M=-\frac{\alpha c^2}{4\pi^2}\int\frac{d\mathbf{k}}{k}\sum_\lambda{}'\Big[\alpha_\lambda G^+\alpha_\lambda\frac{1}{E(\mathbf{p}-\mathbf{k})-H(p)+ck}$$
$$-\alpha_\lambda G^-\alpha_\lambda\frac{1}{E(\mathbf{p}-\mathbf{k})+H(p)+ck}\Big], \quad (27)$$

where $H(p)=c\boldsymbol{\alpha}\cdot\mathbf{p}+\beta\mu c$ and is to be interpreted in the obvious fashion.

We note that consideration of W(self) does not distinguish between $E(p)$ and $H(p)$ in the denominators above. It has been pointed out to the authors by Professor F. J. Belinfante that the requirement of symmetry between electrons and positrons demands the use of $H(p)$ and not $E(p)$. We shall use M as given above for calculating the level shift. Strictly speaking, the mass operator should be written as $\frac{1}{2}(M+M^+)$ where M^+ is the adjoint to M and differs from (27) by having the denominators placed to the left side of $\alpha_\lambda G^\pm\alpha_\lambda$. This is of no account for the Lamb shift calculation, but is important for scattering problems. There is still an ambiguity to be resolved. If we had written the self-energy not in the form above but, for example, in the more usual form given by Eqs. (5), (8) with the electrostatic and electrodynamic energies separated, then clearly we would have written a different mass operator (say M') which would have different non-diagonal elements than M, though its diagonal elements would coincide with W(self). With some manipulation the difference between M and M' can be evaluated. Neglecting terms of order higher than $(p/\mu)^4$ we find that $M-M'=T$,

$$T=\alpha/(3\pi m)[\beta p^2-\mu\boldsymbol{\alpha}\cdot\mathbf{p}]. \quad (28)$$

It has indeed been pointed out by Kroll and Lamb[9] that any operator, not higher in order than $(p/\mu)^4$, whose diagonal elements are zero is equal to a multiple of T. Thus any operator which has the same diagonal elements as the correct mass operator must differ from it only by a multiple of T. This fact can be used to justify the mass operator (27).

For an applied electromagnetic field given by ϕ, $\mathbf{A}$ (where $V=e\phi$) we find by use of the Pauli approximation to the Dirac equation that

$$\langle\psi_0^*T\psi_0\rangle=(\alpha/6\pi)(\hbar/mc)^2\langle\psi_0^*SV\psi_0\rangle$$
$$+(\alpha/3\pi)\langle\psi_0^*e\boldsymbol{\alpha}\cdot\mathbf{A}\psi_0\rangle. \quad (29)$$

Thus the addition of a multiple of T to any mass operator changes the values of c_2 and c_3 in the resultant level shift (25) but leaves c_1 and c_4 unaltered. We may choose this multiple of T so that $c_3=c_4$. If at the same time we find $c_1=c_2$ we may conclude that the resultant mass operator M is the correct one and that this choice of M is unambiguous since a different choice must necessarily lead to a non-invariant result.

We remark finally that M, defined by (27) does give the desired invariant result. This will be shown in IV and V; it will be found that the mag-

netic moment term will be invariant (i.e., $c_3=c_4$) without using any special precautions in handling divergent integrals which enter in the calculation. More care must be taken with the charge term and we shall indeed use the Feynman cut-off[8] to evaluate it. Its value will then be zero.

IV. THE LEVEL SHIFT IN AN ELECTROSTATIC FIELD

It is convenient to consider separately the cases of an external electrostatic and a magnetic field. We now briefly outline the detailed calculation of the level shift for the former case. We have for the level shift (exclusive of ΔW^N given by Eq. (17)).

$$\Delta W^X = W^X - W^M = W^X - [M]_{Av}. \tag{30}$$

In anticipation we may say that the order of the level shift is

$$(\alpha/\mu^2)[p^2V]_{Av} \sim (\alpha/\mu c)[V^2]_{Av}.$$

Thus in the expansion of W^X we need terms up to $W_2{}^X$. On the other hand the only terms in $W_2{}^X$ not smaller than the order of the level shift are terms in $[V^2]_{Av}$; we would expect such terms to be canceled exactly since V^2 is not a gauge invariant quantity. It has been verified that this is so, and for this reason the expansion of W^X is now carried only to first order. We have

$$\Delta W^X = W_0{}^X + W_1{}^X - W^M. \tag{31}$$

We combine $W_0{}^X$ and W^M and then to eliminate the $H(p)$ in the denominator we introduce the operators $H^{\pm}$ which decompose ψ_0 into its positive and negative energy parts.

$$H^{\pm} = \frac{1}{2}\left[1 \pm \frac{c\boldsymbol{\alpha}\cdot\mathbf{p} + \beta\mu c}{E(p)}\right] = \frac{1}{2}\left[1 \pm \frac{H(p)}{E(p)}\right]. \tag{32}$$

We get then $W_0{}^X - W^M = W_1{}' + W_2{}'$,

$$W_1{}' = \frac{-\alpha c^2}{4\pi^2}\int \frac{d\mathbf{k}}{k}\left[\sum_\lambda{}'\left\{\alpha_\lambda \frac{G^+}{B_+{}^2}\alpha_\lambda H^+ + \alpha_\lambda \frac{G^-}{B_-{}^2}\alpha_\lambda H^+ + \frac{H^-}{B_+B_-}\right\} V\right]_{Av}. \tag{33}$$

$W_2{}'$ is of second order in V. (It is $W_2{}'$ which cancels the V^2 terms in $W_2{}^X$.) Combining $W_1{}^X$ and $W_1{}'$ we now find that we can write ΔW^X as the sum of four separately convergent terms.

$$\begin{aligned}\Delta W^X = {}& \frac{\alpha c^2}{4\pi^2}\int \frac{d\mathbf{k}}{k}\left[\sum_\lambda{}'\left\{\alpha_\lambda \frac{G^+}{B_+{}^2}\alpha_\lambda + \alpha_\lambda \frac{G^-}{B_-{}^2}\alpha_\lambda - \frac{1}{B_+B_-}\right\} H^- V\right]_{Av} \\ & + \frac{\alpha c^2}{4\pi^2}\int \frac{d\mathbf{k}}{k}\left[\sum_\lambda{}'\alpha_\lambda\left\{\frac{G^+}{B_+{}^2}(EV-VE)\frac{1}{B_+} + \frac{G^-}{B_-{}^2}(EV-VE)\frac{1}{B_-}\right\}\alpha_\lambda\right]_{Av} \\ & - \frac{\alpha c^2}{4\pi^2}\int \frac{d\mathbf{k}}{k}\left[\sum_\lambda{}'\alpha_\lambda L\left\{\frac{G^+}{B_+}\frac{VG^-}{B_-} + \frac{G^-}{B_-}\frac{VG^+}{B_+}\right\}\alpha_\lambda\right]_{Av} \\ & - \frac{\alpha c^2}{4\pi^2}\int \frac{d\mathbf{k}}{k}\left[\sum_\lambda{}'\alpha_\lambda\left\{\frac{G^+}{B_+}VG^- + \frac{G^-}{B_-}VG^+\right\} \times\left\{\frac{1}{B_+}+\frac{1}{B_-}\right\}\alpha_\lambda\right]_{Av}. \end{aligned} \tag{34}$$

In (34) and elsewhere we have

$$E = E(\mathbf{p}-\mathbf{k}) = c[\mu^2 + (\mathbf{p}-\mathbf{k}^2)]^{\frac{1}{2}}$$

and $G^{\pm}$, B, are given by (20), (21). Each of these terms is now calculated. The procedure is to expand, B, E in powers of $\mathbf{k}\cdot\mathbf{p}$ keeping as many terms as will contribute to the effect (no higher than p^2 in any case). Such expansions are valid for all k. We then integrate over directions of $\mathbf{k}$ and finally over the magnitude of $\mathbf{k}$. In the last step we meet only convergent elementary integrals of the form $\int k^n/(Q^m A_+{}^r A_-{}^s)$ where

$$Q = c[\mu^2 + k^2]^{\frac{1}{2}}, \quad A_\pm = Q \mp E_0 + ck. \tag{35}$$

Since we concern ourselves only with non-relativistic ψ_0 it is convenient to use the Pauli approximation to the Dirac equation to reduce various operators to their non-relativistic equivalents. We then find that all operators occurring reduce to a combination of the two

$$\nabla^2 V, \quad SV = (1/\hbar)\nabla V\cdot\boldsymbol{\sigma}\times\mathbf{p} - \tfrac{1}{2}\nabla^2 V. \tag{36}$$

In SV we understand that $\nabla V\cdot\boldsymbol{\sigma}\times\mathbf{p}=0$ for s states. The relativistic form of SV is $(mc/\hbar)i\beta\alpha\cdot\nabla V$.

We find the following result for the four terms of ΔW^X. Writing each term as

$$\alpha/3\pi(\hbar/mc)^2[\rho_1\langle\nabla^2 V\rangle_{Av} + \rho_2\langle SV\rangle_{Av}], \tag{37}$$

we find, respectively, for the four terms of ΔW^X (34)

$$\begin{array}{ll}\rho_1=0, & \rho_2=\frac{3}{2}\ln\delta+\frac{3}{2}\ln 2-\frac{3}{4},\\ \rho_1=\frac{1}{2}\ln\delta+\frac{1}{4}\ln 2+\frac{1}{8}, & \rho_2=0,\\ \rho_1=-\frac{7}{4}\ln 2+35/24, & \rho_2=-\frac{1}{2}\ln 2+\frac{1}{2},\\ \rho_1=-\frac{3}{2}\ln\delta+\frac{1}{2}\ln 2-9/8, & \rho_2=-\frac{3}{2}\ln\delta-\ln 2-\frac{1}{2},\end{array} \tag{38}$$

where, as mentioned above, $\delta\mu$ is the lower limit of the k integration and we consider $\delta\sim 1/137$.

Now adding the four terms above, together with the Bethe non-relativistic result[4] which joins on correctly and ΔW^N given by Eq. (17), we get for the total level shift

$$\Delta W=\frac{\alpha}{3\pi}\left(\frac{\hbar}{mc}\right)^2\Big[\langle\nabla^2 V\rangle_{\text{Av}} \times\left\{\int_{k_0}^{\mu}\frac{dk}{k}-\ln 2+\frac{11}{24}-\frac{1}{5}\right\}-\frac{3}{4}\langle SV\rangle_{\text{Av}}\Big], \tag{39}$$

where ck_0 is the Bethe lower limit.

We record also the results of separate calculations of the shifts due to electrodynamic exchange terms ($\lambda=1, 2$) and the electrostatic terms ($\lambda=3, 4$). Writing as above, we have:

$$\text{For } \Delta W^{DX},\ \rho_1=\int_{k_0}^{\mu}(dk/k)-\ln 2+\tfrac{5}{8},\ \rho_2=\tfrac{1}{4}.$$

$$\text{For } \Delta W^{SX},\ \rho_1=-\tfrac{1}{6},\ \rho_2=-1.$$

The spin term in the level shift has the form of the interaction of a magnetic moment with the electrostatic field. It gives a surplus magnetic moment of the electron $\alpha/2\pi$ Bohr magnetons, a result first given by Schwinger.[5]

For hydrogenic atoms of principal quantum number n we have

$$\begin{aligned}(\hbar/mc)^2\langle\nabla^2 V\rangle_{\text{Av}}&=8\alpha^2(T^4/n^3)Ry,\quad (l=0),\\ &=0,\quad (l\neq 0),\\ (\hbar/mc)^2\langle SV\rangle_{\text{Av}}&=-4\alpha^2\frac{T^4}{n^3}Ry\frac{1}{(l+1)(2l+1)},\\ &\qquad (j=l+\tfrac{1}{2}),\\ &=+4\alpha^2\frac{T^4}{n^3}Ry\frac{1}{l(2l+1)},\quad (j=l-\tfrac{1}{2}).\end{aligned} \tag{40}$$

Then for the $n=2$ states of hydrogen we use the value for $\int_{k_0}^{\mu}dk/k$ given by Bethe[11] ($=7.6876$) and we find the following level shifts, which we write in terms of the corresponding frequency:

$$\begin{aligned}\nu(2s_{1/2})&=+1034\text{ mc/sec.}\\ \nu(2p_{1/2})&=-17\text{ mc/sec.}\\ \nu(2p_{3/2})&=+8\text{ mc/sec.}\end{aligned}$$

Thus the $2s_{1/2}$ state, which by the Dirac theory is degenerate with the $2p_{1/2}$ state, will on the present theory be higher by an energy corresponding to 1051 mc/sec. This is in agreement with the measurement of Lamb and Retherford.[1]

V. THE LEVEL SHIFT IN A MAGNETIC FIELD

We repeat now the calculations for the case of an external magnetic field. We shall be content with verifying the magnetic moment result derived in IV and shall not calculate the term in $\boldsymbol{\alpha}\cdot\nabla^2\mathbf{A}$. We have for the applied field $U=-e\boldsymbol{\alpha}\cdot\mathbf{A}$. The order of magnitude of the level shift will be

$$\Delta W^X\sim\alpha[\boldsymbol{\mu}\cdot\mathbf{H}]_{\text{Av}}\sim\alpha[e\boldsymbol{\alpha}\cdot\mathbf{A}]_{\text{Av}}$$

where $\boldsymbol{\mu}$ is the magnetic moment operator. We may thus neglect terms in $\mathbf{A}^2$ and $p^2\mathbf{A}$.

Expanding the intermediate states in powers of U, it will be sufficient to consider terms of order zero and one. Since α_λ does not commute with U, the manipulations used in IV to write ΔW^X as the sum of individually convergent terms will not be useful.

As before $\Delta W^X=W_1{}^X+W_1'$, where $W_1'=W_0{}^X-W^M$. $W_1{}^X$ is found by taking $U=-e\boldsymbol{\alpha}\cdot\mathbf{A}$ in Eq. (23) and W_1' comes from Eq. (33) by changing V to $-e\boldsymbol{\alpha}\cdot\mathbf{A}$.

We now calculate each term using the same procedure as in IV. We consider non-relativistic ψ_0 and, in various convergent terms, we use the Pauli approximation to the Dirac theory, to reduce various operators to their non-relativistic equivalents. We find one essential difference in the calculation. Whereas for the electrostatic field no divergent integrals entered the actual calculation, this is not the case for the magnetic field. This will have an important consequence to be seen later.

W_1' is easily evaluated. Much more work is required for $W_1{}^X$. We write each term in the form

$$e\alpha/2\pi[\rho_1\boldsymbol{\alpha}\cdot\mathbf{A}+\rho_2(\hbar/mc)\boldsymbol{\sigma}\cdot\mathbf{H}+\rho_3(1/mc)(\mathbf{A}\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{A})]_{\text{Av}}. \tag{41}$$

We find

for W_1':

$$\rho_{11}=2\int kdk\left[\frac{1}{A_+{}^2}+\frac{1}{A_-{}^2}\right],$$

$$\rho_2=\rho_3=2\ln\delta+2\ln 2, \tag{42}$$

[11] H. A. Bethe, Solvay Report, 1948.

for W_1^X (1st term)

$$\rho_{12}=-\frac{1}{3}\int\frac{k^3}{Q^2}\left[\frac{1}{A_+^2}+\frac{1}{A_-^2}+\frac{2}{A_+A_-}\right],$$

$$\rho_2=-\ln\delta-\tfrac{5}{6}\ln 2-\tfrac{3}{4},$$
$$\rho_3=-\ln\delta-\tfrac{5}{6}\ln 2-(5/12), \tag{43}$$

for W_1^X (2nd term)

$$\rho_{13}=-2\int\frac{k^2dk}{QA_+A_-}-\frac{2}{3}\int\frac{k^4}{Q^3A_+A_-},$$

$$\rho_2=-\tfrac{1}{6}\ln 2,$$
$$\rho_3=-\tfrac{1}{6}\ln 2+\tfrac{1}{6}. \tag{44}$$

The integrals appearing in each ρ_1 are logarithmically divergent. Q and $A\pm$ are as given in (35). The range of integration is from $\delta\mu(\delta\sim 1/137)$ to ∞. If now we evaluate the integrals without taking special precautions because of their divergent nature we find the following values:

$$\rho_{11}=\int(dk/Q)-2\ln\delta-2\ln 2-\tfrac{1}{2},$$

$$\rho_{12}=-\tfrac{1}{3}\int(dk/Q)+\tfrac{1}{3}\ln 2,$$

$$\rho_{13}=-\tfrac{2}{3}\int(dk/Q)-\tfrac{1}{3}\ln 2+\tfrac{2}{3}.$$

We note, of course, that the divergent integrals cancel. We now add the terms and use the Pauli approximation:

$$2\mu\langle\boldsymbol{\alpha}\cdot\mathbf{A}\rangle_{Av}=[\mathbf{A}\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{A}+\hbar\boldsymbol{\sigma}\cdot\mathbf{H}]_{Av}.$$

We now find that we may write, for the level shift,

$$\Delta W^X=\frac{\alpha}{2\pi}\left[\frac{-e\hbar}{2mc}\boldsymbol{\sigma}\cdot\mathbf{H}\right]_{Av}-\frac{\alpha}{6\pi}[e\boldsymbol{\alpha}\cdot\mathbf{A}]_{Av}. \tag{45}$$

We thus get once again $\alpha/2\pi$ Bohr magnetons for the surplus magnetic moment of the electron. The magnetic moment term in the level shift is therefore invariant or in terms of (25) we have $c_3=c_4$. (Note that the Pauli approximation which we have used does not distinguish between $\beta\boldsymbol{\sigma}\cdot\mathbf{H}$ and $\boldsymbol{\sigma}\cdot\mathbf{H}$.) However comparison of (45) with the electrostatic level shift (39) shows that the condition $c_1=c_2$ is not satisfied. This can be easily remedied by a better treatment of the divergent integrals appearing above in the ρ_1's. We shall use the relativistic cut-off procedure described by Feynman.[8] For our purpose this may be described as follows: In every integral appearing we replace k by $\omega=(k^2+\lambda^2)^{\frac{1}{2}}$ wherever k represents an energy: We now evaluate the integrals for the two cases: $\lambda=0$ and λ large (say not smaller than 137μ). In doing this we take K as the upper limit of the integration where $K\gg\lambda$. We then subtract the result for large λ from that for $\lambda=0$ and then allow $K\to\infty$. Finally we may allow $\lambda\to\infty$.

A little consideration will show that if an individual integral is convergent this procedure will not change its value. Thus the results of IV will be unchanged. We must consider the divergent integrals in ΔW^X (magnetic case). For $\lambda=0$ the values will be as found before with an upper limit K on the integrals $\int dk/Q$. For large λ we have $A\pm\to(k+\omega)$, $Q\to k$, $d\mathbf{k}/k\to d\mathbf{k}/\omega$ and thus $kdk\to(k^2dk/\omega)$.

Then each separate integral in W_1' and the first term of W_1^X becomes

$$\int_\delta^K[k^2dk/\omega(k+\omega)^2]=I_1 \text{ say.}$$

The total contribution of these terms for large λ is thus

$$\alpha/2\pi\langle e\boldsymbol{\alpha}\cdot\mathbf{A}\rangle_{Av}\{(8/3)I_1\}.$$

The structure of the integrals in the second term of W_1^X is different. For these integrals come from terms involving $L=[2ck/E(\mathbf{p}-\mathbf{k})+E(\mathbf{p}'-\mathbf{k})]\sim\omega/k$. Thus each integral here becomes, for large λ,

$$\int_\delta^K[kdk/(k+\omega)^2]=I_2 \text{ say}$$

and the total contribution for large λ is

$$\alpha/2\pi\langle e\boldsymbol{\alpha}\cdot\mathbf{A}\rangle_{Av}\{-(8/3)I_2\}.$$

Thus we must subtract from ΔW^X given above (Eq. (39)) the following term

$$4\alpha/3\pi\langle e\boldsymbol{\alpha}\cdot\mathbf{A}\rangle_{Av}\{I_1-I_2\}. \tag{46}$$

I_1 and I_2 are easily evaluated. Neglecting terms which do not contribute in the limit we find

$$I_1=\tfrac{1}{4}\ln(2K/\lambda)-\tfrac{3}{16},\quad I_2=\tfrac{1}{4}\ln(2K/\lambda)-\tfrac{1}{16},$$
$$I_1-I_2=-\tfrac{1}{8}. \tag{47}$$

The term to be subtracted from ΔW^X is then $-(\alpha/6\pi)\langle e\alpha\cdot A\rangle_{Av}$. We thus get

$$\Delta W^X=\alpha/2\pi[-(e\hbar/2mc)\boldsymbol{\sigma}\cdot\mathbf{H}]_{Av} \tag{48}$$

giving in (25) $c_2=0=c_1$ and clearing up the difficulty of non-invariance. It might be noted that the divergent terms which enter the calculation involve only the operators $U=V$ or $-e\boldsymbol{\alpha}\cdot\mathbf{A}$ and thus any

error due to improper handling of divergent integrals will result only in a violation of the condition $c_1=c_2$ in (25). An error of this sort can be corrected by simply omitting any V and $e\boldsymbol{\alpha}\cdot\mathbf{A}$ terms in the result.

VI. COMPARISON WITH OTHER WORK

The results derived here are in agreement with those of Kroll and Lamb.[9] Standard perturbation procedures are used in both papers. Kroll and Lamb do not introduce a mass operator however, but calculate instead the value of the perturbation energy (1) and omit afterwards all terms independent of the external electrostatic field. Their treatment of the ambiguities differs from ours only in the fact that they do not use their own methods to calculate the energy shift in a magnetic field but use the criterion that the surplus magnetic moment in an electrostatic field should be equal to that previously measured by Kusch and Foley[12] and calculated by Schwinger.[5]

An expression for the level shift has also been derived by Feynman.[8] His procedure differs from that used here in that the ambiguities in handling the divergent quantities which normally appear are removed by introducing a relativistic cut-off. For processes in which the actual value of the Feynman cut-off is unimportant (e.g., the level shift), his results should agree with ours. Apart however from the non-exchange level shift ΔW^N (the Uehling terms given by (17)) which Feynman deliberately omits, there is a discrepancy, of value $\frac{1}{6}(\alpha/3\pi)\times(\hbar/mc)^2[\nabla^2 V]_{\text{Av}}$, between the two values (Eq. (39) above and Feynman's Eq. (19)). This is due to an incorrect joining of Feynman's result with the non-relativistic result of Bethe. In order to join at a momentum k_0, Feynman introduces a small light quantum mass λ and then integrates down to $k=0$. The correspondence between λ and k_0 is given as $\ln\lambda=\ln 2k_0-1$. Examination of ΔW^X (34) shows that the correct transformation is $\ln\lambda=\ln 2k_0-\frac{5}{6}$ and when this is applied to Feynman's Eq. (18) the result agrees with the exchange level shift given above.

The formalism introduced by J. Schwinger,[13] in order to express quantum electrodynamics in a more obviously relativistic form, leads to results identical with ours for the electromagnetic level shift.

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[12] P. Kusch and H. M. Foley, Phys. Rev. **72**, 1256 (1947).

[13] J. Schwinger, Phys. Rev. **74**, 1439 (1948).