also well suited for presentation to students who possess a limited knowledge of the theory of determinants and matrices.

II. PROOF OF THE THEOREM

Theorem. *If A and B are hermitian matrices of order n, A being positive definite, the rank of the matrix $B-\lambda_i A$ is exactly $n-k$, where k is the multiplicity of the root λ_i of the secular equation $|B-\lambda A|=0$.*

Let the rank of $B-\lambda_i A$ be $n-r$. Then the equation

$$(B-\lambda_i A)\xi=0 \tag{1}$$

has r independent solutions, say $\xi_1, \cdots, \xi_r$. These solutions can be so chosen[1,2] that they also satisfy the orthonormality relations

$$\xi_i' A \xi_j=\delta_{ij}. \tag{2}$$

By selecting arbitrarily $n-r$ additional vectors,[6] say $\xi_{r+1}, \cdots, \xi_n$, so that the entire set of n vectors is orthonormal in the sense of (2), one obtains a non-singular matrix $X=[\xi_1, \cdots, \xi_n]$ such that $X'AX=I$. In view of this relation and the fact that the first r columns of X satisfy (1), the matrix $X'BX$ has the form

$$\begin{pmatrix} \lambda_i & & \cdot & & \\ & \cdot & \cdot & & \\ & \cdot & \cdot & 0 & \\ & & \lambda_i \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & & \\ & & \cdot & & \\ & 0 & \cdot & B_1 & \\ & & \cdot & & \end{pmatrix},$$

where B_1 is hermitian and of order $n-r$. It follows that $X'(B-\lambda A)X$

$$=\begin{pmatrix} \lambda_i-\lambda & & \cdot & & \\ & \cdot & \cdot & & \\ & \cdot & \cdot & 0 & \\ & & \lambda_i-\lambda \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & & \\ & & \cdot & & \\ & 0 & \cdot & B_1-\lambda I & \\ & & \cdot & & \end{pmatrix}. \tag{3}$$

Since the roots of the equation $|X'(B-\lambda A)X|=0$ are the same as the roots of the secular equation and, in view of (3), λ_i is a root of the equation $|X'(B-\lambda A)X|=0$ of multiplicity r at least, it follows that r cannot exceed the multiplicity k of the root λ_i for the secular equation. But if r is less than k then λ_i is necessarily a root of the equation $|B_1-\lambda I|=0$. This is impossible since the rank of $X'(B-\lambda_i A)X$ is equal to the rank of $B-\lambda_i A$, which is $n-r$ by assumption, and by (3) the rank of $X'(B-\lambda_i A)X$ is also equal to the rank of $B_1-\lambda_i I$, which is less than $n-r$ if $|B_1-\lambda_i I|=0$. It follows that $r=k$ and the rank of $B-\lambda_i A$ is $n-k$ as asserted.

[6] This device has been used by other authors to prove similar theorems. See, for example, P. R. Halmos, "Finite dimensional vector spaces," *Annals of Mathematics Studies* (Princeton University Press, Princeton, 1942), No. 7, pp. 125–126.

PHYSICAL REVIEW VOLUME 75, NUMBER 7 APRIL 1, 1949

Remarks on the Evolution of the Expanding Universe*,†

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(Received December 27, 1948)

The relativistic energy equation for an expanding universe of non-interconverting matter and radiation is integrated. The above result, together with a knowledge of the physical conditions that prevailed during the element forming process in the early stages of the expansion, is used to determine the time dependences of proper distance as well as of the densities of matter and radiation. These relationships are employed to determine the mean galactic diameter and mass when formed as 2.1×10^3 light years and 3.8×10^7 sun masses, respectively. Galactic separations are computed to be of the order of 10^6 light years at the present time.

I. INTRODUCTION

WITH the experimental and theoretical information now available it is possible to give a tentative description of the structure and evolution of the universe. Investigations of cosmological models of various types have been carried out which explain many of the features of the observed

* The work described in this paper was supported by the Bureau of Ordnance, U. S. Navy, under Contract NOrd-7386.

† A preliminary account of this work was given at the New York meeting of the American Physical Society, January, 1949.

universe.[1] It does not appear to have been possible to complete these speculations principally for lack of sufficient physical data. Recent studies of the origin and relative abundances of the elements have yielded new information concerning the physical state of the universe at the very early time during which the elements were apparently formed.[2–9] According to this theory the ylem (the primordial substance from which the elements were formed) consisted of neutrons at a high density and temperature. Protons were formed by neutron decay, and the successive capture of neutrons led to the formation of the elements. In order to predict the observed relative abundances of the elements, it is necessary to stipulate the magnitude and the time dependence of the temperature, and density of matter during the period of element formation.

On the basis of a simplified version of the neutron capture theory, namely, one which involves the building up of deuterons only, Gamow[8] has examined the state of the universe at early times and traced the evolution of the universe through the formation of galaxies. For reasons which will be discussed later, Gamow's formulation gives rise to certain difficulties.

We have reformulated this problem from a somewhat different point of view, following some of Gamow's basic ideas.[4, 8] This reformulation, which is the main purpose of this paper, involves the use of the general non-static relativistic cosmological model together with knowledge of the physical conditions of matter and radiation which prevail now and also those which are required to predict the observed relative abundances of the nuclear species formed during the very early stages of the universe. As a consequence, it is possible to obtain the functional dependence of both the density of matter and radiation on time. On the basis of the foregoing, the formation of galaxies and other cosmological consequences are considered.

II. FORMULATION OF THE PROBLEM

The model of the expanding universe that we shall discuss is one in which there is a homogeneous and isotropic mixture of radiation and matter, assumed to be non-interconverting. This mixture is treated as a perfect fluid. If the pressure due to matter is neglected, one may write the relativistic energy equation for the non-static model in the following form:[1]

$$d[\exp(\tfrac{1}{2}g(t))]/dt = \pm[(8\pi/3)\rho\exp(g(t)) - R_0^{-2}]^{\frac{1}{2}}, \quad (1)$$

which is in relativistic units. The cosmological constant Λ is taken equal to zero. In Eq. (1), ρ is the density of mass and the radius of curvature, R, is given by $R = R_0\exp(g(t))$, where $\exp(g(t))$ is the time-dependent factor in the spatial portion of the line element. Now,

$$\exp(\tfrac{1}{2}g(t)) = l/l_0 = R/R_0, \quad (2)$$

where l is any proper distance, and l_0, the unit of length, together with R_0, must be determined from the boundary conditions for Eq. (1). It should be pointed out that solutions of Eq. (1) involve l/l_0 and not l alone. The density of mass ρ, which determines the geometry of the space, is the sum of the density of matter, ρ_m, and the density of radiation, ρ_r. If matter is to be conserved we must have

$$\rho_m l^3 = A = \text{constant}. \quad (3a)$$

Furthermore, if the universal expansion is adiabatic, the temperature, T, must vary[1] as l^{-1}. If one assumes that the universe contains blackbody radiation, then

$$\rho_r l^4 = B = \text{constant}. \quad (3b)$$

It is to be noted that energy is not conserved in models of this type. Equations (3a) and (3b) obviously may be written as

$$\rho_r\rho_m^{-4/3} = \text{constant}. \quad (4)$$

It is clear that this relationship must hold throughout the universal expansion and that the density of mass at any time is

$$\rho = \rho_m + \rho_r = Al^{-3} + Bl^{-4}, \quad (5)$$

providing, as stated earlier, there is no interconversion of matter and radiation. If we substitute Eqs. (2) and (5) into Eq. (1), and convert to c.g.s. units, we obtain

$$dl/dt = +[(8\pi G/3)(Al^{-3} + Bl^{-4})l^2 - c^2l_0^2/R_0^2]^{\frac{1}{2}}, \quad (6)$$

where the positive sign is taken to indicate expansion and c and G are the velocity of light and the gravitational constant, respectively. Equation (6) can be integrated and the result given in the form

$$t = K_1 + K_2^{-1}[\gamma\rho_{r''} + \gamma\rho_{m''}L + K_2L^2]^{\frac{1}{2}} - (\gamma\rho_{m''}/2K_2^{\frac{3}{2}})\ln\{[\gamma\rho_{r''} + \gamma\rho_{m''}L + K_2L^2]^{\frac{1}{2}} + K_2^{\frac{1}{2}}L + (\gamma\rho_{m''}/2K_2^{\frac{1}{2}})\}, \quad (7)$$

where

$$K_1 = (\gamma\rho_{m''}/2K_2^{\frac{3}{2}})\ln[(\gamma\rho_{r''})^{\frac{1}{2}} + (\gamma\rho_{m''}/2K_2^{\frac{1}{2}})] - (\gamma\rho_{r''}/K_2^2)^{\frac{1}{2}}. \quad (8)$$

[1] R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Clarendon Press, Oxford, 1934).
[2] G. Gamow, Phys. Rev. **70**, 572 (1946).
[3] R. A. Alpher, H. A. Bethe, and G. Gamow, Phys. Rev. **73**, 803 (1948).
[4] G. Gamow, Phys. Rev. **74**, 505 (1948).
[5] R. A. Alpher, R. C. Herman, and G. Gamow, Phys. Rev. **74**, 1198 (1948).
[6] R. A. Alpher, Phys. Rev. **74**, 1577 (1948).
[7] R. A. Alpher and R. C. Herman, Phys. Rev. **74**, 1737 (1948).
[8] G. Gamow, Nature, **162**, 680 (1948).
[9] R. A. Alpher and R. C. Herman, Nature **162**, 774 (1948).

In Eqs. (7) and (8), $L=l/l_0$, $\gamma=(8\pi G/3)$, $K_2=(c^2/|R_0^2|)$, and $\rho_{m''}$ and $\rho_{r''}$ are the densities of matter and radiation when $L=1$. In order to integrate Eq. (6) and evaluate the integration constant, it is necessary to specify the parameter R_0 and consequently l_0, which gives the units in which R_0 is measured. Examination of Eq. (6) indicates that R_0 can be determined only if it is possible to specify $[(dl/dt)/l]_{l=l_0}$, ρ_m, and ρ_r at any given time. Since $[(dl/dt)/l]_{l=l_0}$ is the expansion rate of space as determined by Hubble[10] and known, therefore, only at the present time, since ρ_m is also known now, and if we assume that $\rho_m \gg \rho_r$ now, one may evaluate R_0 and K_2. Introducing the value of the present expansion rate of the universe $[(dl/dt)/l]_{l=l_0}=1.8\times10^{-17}$ sec.$^{-1}$, taking $\rho_{m''}=10^{-30}$ g/cm^3 and $l=l_0=10^{10}$ cm, i.e., l_0 is the side of a cube containing one gram of matter now, one obtains $R_0=1.7\times10^{27}(-1)^{\frac{1}{2}}$ cm and $K_2=3.2\times10^{-34}$ sec.$^{-2}$. The constants appearing in Eqs. (7) and (8) involve the present densities of matter and radiation. Clearly, in utilizing Eqs. (7) or (8) one may introduce the density values at any other time providing one specifies a value of L at that time which leads to the present value of the density of matter. For convenience we have chosen l_0 to be the side of a cube containing one gram of matter at the present time, so that $L=1$ now. Furthermore, we have again for convenience assumed that $L=0$ at $t=0$. While Eq. (6) has a singularity at $t=0$ which is physically unreasonable, we have employed the solutions in such a manner that the singularity is of no consequence.

For purposes of computation it is convenient to employ an approximate form for Eq. (7) which is valid for early t, i.e., when

$$L[(\rho_{m''}/\rho_{r''})+(K_2/\gamma\rho_{r''})L]<1. \tag{9}$$

The expansion of Eq. (7) which satisfies the above inequality is

$$t=(4\gamma\rho_{r''})^{-\frac{1}{2}}L^2+(\rho_{m''}/6\gamma^{\frac{1}{2}}\rho_{r''}^{\frac{3}{2}})L^3+(8\gamma^{\frac{3}{2}}\rho_{r''}^{\frac{3}{2}})^{-1}\times[(3\gamma\rho_{m''}^2/4\rho_{r''})-K_2]L^4+\cdots. \tag{10}$$

The validity of Eqs. (7) or (10) is questionable for very early times, i.e., in the vicinity of the singularity at $t=0$, when the energy of light quanta was comparable to the rest mass of elementary particles. In fact, Einstein[11] has pointed out that there is a difficulty at very early times because of the separate treatment of the metric field (gravitation) and electromagnetic fields and matter in the theory of relativity. For large densities of field and of matter, the field equations and even the field variables which enter into them will have no real significance. However, since we do not concern ourselves with the "beginning" this difficulty is obviated. In addition to the fact that the relativistic energy equation is not valid for very early times, there are the problems of angular momentum of matter in the universe, as well as certain physical factors involved in the formation of the elements, which we cannot handle satisfactorily at present.

In order to utilize the above equations, it is necessary to specify $\rho_{m''}$, $\rho_{r''}$, and K_2. While it may appear that one need specify the matter and radiation densities at the present time only, because of Eq. (4), specifying $\rho_{m''}$ and $\rho_{r''}$ is equivalent to specifying $\rho_{m'}$ and $\rho_{r'}$, these being the densities at a time during the period of element formation. This time is to be specified later. (The primed quantities should not be confused with the running variables.) It must be remembered that the value of R_0 employed is that calculated from the present value of dL/dt.

III. PHYSICAL CONDITIONS DURING THE EXPANSION

Some information is available regarding the values of the matter and radiation densities at the present time and, recently, studies of the relative abundances of the elements have indicated values for these densities prevailing very early in the universe during the period of element formation. Because of Eq. (4) a knowledge of $\rho_{m'}$ and $\rho_{r'}$ during the element forming period together with $\rho_{m''}$ fixes a value for $\rho_{r''}$, the present radiation density, which is perhaps the least well-known quantity.

In a recent paper Gamow,[8] by considerations which are different than those we have employed, found a set of physical conditions which prevailed during the early stages of the universe. He studied the formation of deuterons only, by the capture of neutrons by protons, taking into account the universal expansion. Equations for the formation of deuterons were integrated from $t=0$, subject to the condition that there were neutrons at the start (unit concentration by weight) and that the final concentration by weight of protons and deuterons was 0.5. This solution determined a parameter α which in turn defined the magnitude of the matter density,[12] $\rho_m=\rho_0 t^{-\frac{3}{2}}$.

[10] E. P. Hubble, *The Observational Approach to Cosmology* (Clarendon Press, Oxford, 1937).

[11] A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, 1945).

[12] The expression for the parameter α, as given by Gamow in reference 8, has been found to be incorrect (see reference 9). We find that $\alpha(=\rho_m v\sigma t/m$, where $\rho_m=\rho_0 t^{-3/2}$ is the density of matter, v is the mean velocity of particles of mass m, and σ is the capture cross section of protons for neutrons) is correctly given by

$$\alpha=\frac{2^{9/4}\pi^{5/4}G^{1/4}a^{1/4}e^2\hbar}{3^{1/4}m^{9/2}c^{11/2}k}(|\mu_P|+|\mu_N|)^2(\epsilon^{1/2}+\epsilon_0^{1/2})\epsilon^{3/2}\rho_0.$$

In this expression all the quantities have been defined by Gamow in reference 8 except μ_P and μ_N, the magnetic moments in nuclear magnetons of proton and neutron, respectively, ϵ_0,

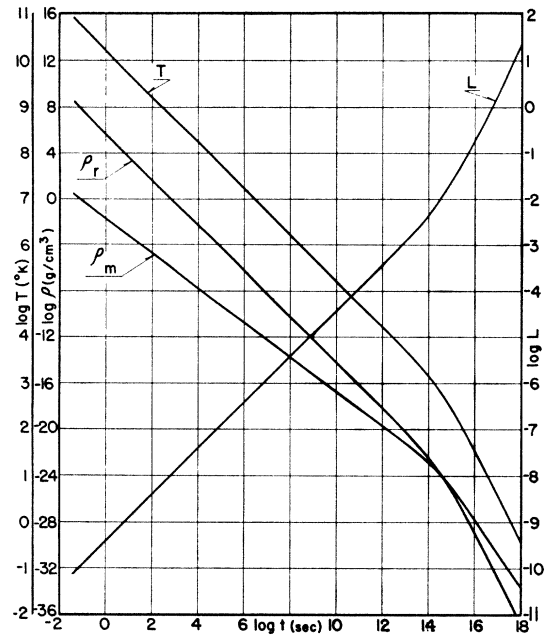

FIG. 1. The time dependence of the proper distance L, the densities of matter and radiation, ρ_m, and ρ_r, as well as the temperature, T, are shown for the case where $\rho_{m''} \leqq 10^{-30}$ g/cm³, $\rho_{r''} \leqq 10^{-32}$ g/cm³, $\rho_{m'} \leqq 10^{-6}$ g/cm³, and $\rho_{r'} \leqq 1$ g/cm³. [See Eq. (12).]

We believe that a determination of the matter density on the basis of only the first few light elements is likely to be in error. Our experience with integrations required to determine the relative abundances of all elements[6,7] indicates that these computed abundances are critically dependent upon the choice of matter density. Furthermore, all formulations of the neutron capture process which have been made thus far neglect the thermal dissociation of nuclei, which is one of the important competing processes during the element forming period if elements are formed from a very early time.

In order to clarify the difficulties associated with the singularity at $t=0$, we digress here for an examination of the equations employed to describe the formation of the elements. These equations, recently given by the authors,[7] include neutron decay and universal expansion but do not take into account the effects of nuclear evaporation or any processes other than radiative capture of neutrons. In terms of concentrations by weight, $x_j = m_j n_j/\rho_m$, rather than particle concentrations, n_j, Eqs. (6)–(8) of reference 7 may be written as

$$dx_0/dt = -\lambda x_0 - \sum_{j=1}^{J} (p_j \rho_m/m_j) x_j x_0, \qquad (11a)$$

$$dx_1/dt = \lambda x_0 - (p_1 \rho_m/m_1) x_1 x_0, \qquad (11b)$$

and

$$dx_j/dt = j(p_{j-1}\rho_m/m_{j-1})x_{j-1}x_0 - j(p_j\rho_m/m_j)x_j x_0, \quad j=2, 3, \cdots, J, \qquad (11c)$$

where x_0, x_1, and x_j are the concentrations by weight of neutrons, protons, and nuclei of atomic weight $2 \leqq j \leqq J$, respectively, m_j the nuclear mass, ρ_m the density of matter, λ the neutron decay constant, and p_j the effective neutron capture volume swept out per second by nuclei of species j. Gamow[8] has solved Eqs. (11a) and (11b) numerically, taking $J=1$, and thereby describing the building up of deuterons only. In general, Eqs. (11) have a singularity at the origin because when $t \to 0$, $\rho_m \to \infty$ as $t^{-\frac{1}{2}}$. In the approximation used by Gamow this singularity is reduced because a relation for the capture cross section of protons for neutrons is employed which makes $p_1\rho_m (= \sigma_1 v \rho_m)$ vary as t^{-1}.

It may be seen readily that Eq. (11c) can be written in the form

$$dx_j/dz = (p_{j-1}/\lambda m_{j-1})x_{j-1} - (p_j/\lambda m_j)x_j, \quad j=2, 3, \cdots, J, \qquad (11d)$$

where

$$z = \int_{\tau_0}^{\tau} j\rho_m(\tau) x_0(\tau) d\tau, \qquad (11e)$$

and

$$\tau = \lambda t.$$

In general, the integrand in Eq. (11e) is singular at $\tau=0$, so that one must take $\tau_0 > 0$. This implies the choice of an initial time at which the element forming process started. Physically, one may not speak of an initial time because there were competing processes which became unimportant as the neutron capture process became important. Competing processes such as photo-disintegration and nuclear evaporation fall off approximately exponentially with time so that neutron capture would become significant rather rapidly, say in a time of the order of 10^2 seconds. The inclusion of this type of competing process in principle could be handled and would yield a better estimate of the relative abundances of the elements. However, without a better knowledge of cosmology at very early t it does not appear to be possible to avoid the above-mentioned difficulty. Finally, if Eqs. (11a), (11b), and (11c) are solved simultaneously for $J=4$, the remaining equations for $j>4$ are given by Eq. (11d) which is a simple first-order linear differential equation with constant coefficients. Nevertheless, Eqs. (11a) and (11b), which are the controlling equations for the process, are not reduced to a simple form and must still be solved in their present form. Because of the above difficulties we find it necessary to introduce the concept of a starting

the binding energy of the virtual triplet state of the deuteron, and the radiation density constant $a = 7.65 \times 10^{-15}$ erg cm^{-3} deg.$^{-4}$. Our expression differs from that originally given by Gamow because of algebraic errors contained in his results and because he neglected the magnetic moment factor.

time for the element forming process. Equations (11) have not yet been solved but are given to illustrate the singularity. So far as we know, any formulation of a theory of element building which includes the type of cosmology discussed will reflect these same difficulties.

In what follows we continue the discussion of the physical conditions employed in the solutions of the relativistic energy equation. The mean density of matter in the universe at the present time has been determined by Hubble[10] to be

$$\rho_{m''} \leqq 10^{-30} \text{ g/cm}^3. \tag{12a}$$

An estimate of the density of matter, $\rho_{m'}$, prevailing at the start of the period of element formation is obtained by integration of the equations for the neutron capture theory of the formation of the elements. Integrations in which neutron decay is explicitly included, but in which the expansion of the universe is not included, yield a matter density of 5×10^{-9} g/cm^3. Preliminary investigations of the equations, including the universal expansion, indicate that this density should be increased by a factor roughly of the order of 100 in order that one may correctly determine the relative abundance of the elements with the universal expansion taken into account. In fact, we have numerically integrated for the light elements the complete equations (see Eqs. (11)) with an "initial" density about 100 times the density used in obtaining solutions without the universal expansion.[7] We find that the above factor of $\sim$100 is roughly what might be required. Accordingly, we have taken

$$\rho_{m'} \leqq 10^{-6} \text{ g/cm}^3. \tag{12b}$$

As discussed elsewhere,[6,7] the temperature during the element-forming process must have been of the order of 10^8–10^{10}°K. This temperature is limited, on the one hand, by photo-disintegration and thermal dissociation of nuclei and, on the other hand, by the lack of evidence in the relative abundance data for resonance capture of neutrons. For purposes of simplicity we have chosen

$$\rho_{r'} \leqq 1 \text{ g/cm}^3, \tag{12c}$$

which corresponds to $T \leqq 0.6\times10^{9}$°K at the time when the neutron capture process became important.

In accordance with Eq. (4), the specification of $\rho_{m''}$, $\rho_{m'}$, and $\rho_{r'}$ fixes the present density of radiation, $\rho_{r''}$. In fact, we find that the value of $\rho_{r''}$ consistent with Eq. (4) is

$$\rho_{r''} \leqq 10^{-32} \text{ g/cm}^3, \tag{12d}$$

which corresponds to a temperature now of the order of 5°K. This mean temperature for the universe is to be interpreted as the background temperature which would result from the universal expansion alone. However, the thermal energy resulting from the nuclear energy production in stars would increase this value.

Since we have $\rho_{r'} \gg \rho_{m'}$ at early time the energy relation given in Eq. (6) may be integrated in a simpler form, with the result

$$T = [(32\pi Ga)/(3c^2)]^{-\frac{1}{4}} t^{-\frac{1}{2}} \text{°K} = 1.52\times10^{10} t^{-\frac{1}{2}} \text{°K}. \tag{13a}$$

The density of radiation, ρ_r, may be found from $\rho_r = (a/c^2)T^4$, or

$$\rho_r = 4.48\times10^{5} t^{-2} \text{ g/cm}^3. \tag{13b}$$

These expressions for T and ρ_r at early time are the consequence of the assumption of an adiabatic universe filled with blackbody radiation. It can also be shown that with the densities chosen in Eq. (12) we have for early time

$$\rho_m = 1.70\times10^{-2} t^{-\frac{3}{2}} \text{ g/cm}^3. \tag{13c}$$

Using l and l_0 as already defined, we may determine the constants A and B in Eq. (3). With the densities discussed above we find $A=1$ g and $B=10^8$ g cm. These values of A and B fix the dependence of ρ_m and ρ_r on time through $L(=l/l_0)$. Using these values of A and B, we have computed L, ρ_m, ρ_r, and T. These quantities are plotted on a logarithmic scale in Fig. 1. It should be noted in Fig. 1 that all the quantities plotted bear simple relationships with the time to within several orders of magnitude

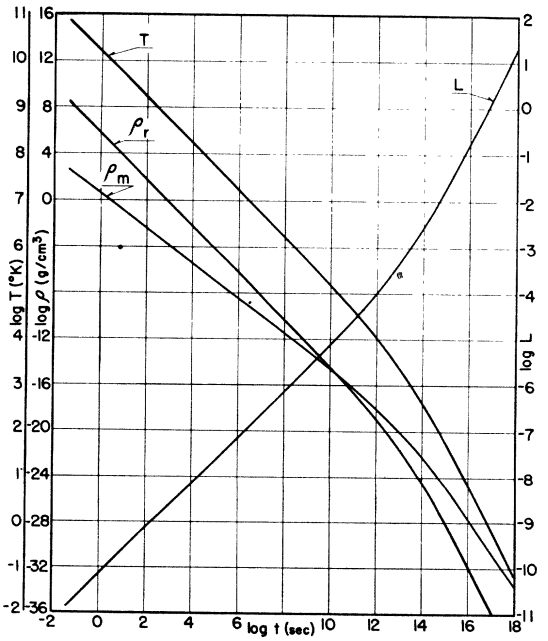

FIG. 2. The time dependence of the proper distance L, the densities of matter and radiation, ρ_m, and ρ_r, as well as the temperature, T, are shown for the case where $\rho_{m''} \leqq 10^{-30}$ g/cm^3, $\rho_{r''} \leqq 10^{-35}$ g/cm^3, $\rho_{m'} \leqq 1.8\times10^{-4}$ g/cm^3, and $\rho_{r'} \leqq 1$ g/cm^3. [See Eq. (15).]

of the time when the universal expansion changes from one controlled by gravitation to one of free escape. This transition occurs in the region of about 10^{13}–10^{14} sec. Following this transition the quantities L, ρ_m, ρ_r, and T again are simple functions of the time. The relations for large t are as follows:

$$\begin{aligned} L &= K_2^{\frac{1}{2}}t, \\ \rho_m &= (\rho_{m''}/K_2^{\frac{3}{2}})t^{-3}, \\ \rho_r &= (\rho_{r''}/K_2^{2})t^{-4}, \end{aligned} \tag{14}$$

and,

$$T = (c^2\rho_{r''}/aK_2^{2})t^{-1}.$$

It is to be noted that in the region of transition to free escape the densities of matter and radiation become equal so that, in fact, prior to the transition the expansion is controlled chiefly by radiation and subsequent to the transition by matter. The universe is now in the freely expanding state, and, since the radius of curvature is imaginary, is of the open, hyperbolic type.

In order to study how sensitive this model is to the choice of densities, we have considered the following additional set of density values which satisfy Eq. (4):

$$\begin{aligned} \rho_{m'} &\cong 1.78\times10^{-4}\ \text{g/cm}^3, \\ \rho_{r'} &\cong 1\ \text{g/cm}^3, \\ \rho_{m''} &\cong 10^{-30}\ \text{g/cm}^3, \end{aligned} \tag{15}$$

and

$$\rho_{r''} \cong 10^{-35}\ \text{g/cm}^3.$$

The value obtained for $\rho_{r''}$ in this case corresponds to a present mean temperature of about 1°K. The constants A and B are found to be 1 g and 10^5 g cm, respectively. In Fig. 2 we have plotted the time dependence of the quantities of interest. One finds that the transition occurs at an earlier time than in the previous case, namely, at $\sim10^{10}$ sec., which implies that this universe would have been in a state of free expansion for a considerably longer time. Apparently the behavior of the model is extremely sensitive to the choice of density conditions. However, the simple type of relations for L, ρ_m, ρ_r, and T that were given previously still apply, but with different constants and different regions of validity.

The time at which $\rho_m=\rho_{m'}$ and $\rho_r=\rho_{r'}$ for both sets of densities given in Eqs. (12) and (15) are found from Eq. (13b) to be 6.7×10^2 seconds, with a corresponding temperature of 0.59×10^9°K. We have chosen $\rho_{r'}\cong1$ g/cm³ in both cases because the corresponding temperature is seen by independent considerations to be that required for the element forming process. As will be seen later, the densities given in Eq. (15) with $\rho_{m'}\cong1.78\times10^{-4}$ g/cm³ do not yield a satisfactory description of the size and mass of galaxies. On the other hand, as already stated a density $\rho_{m'}\cong100(5\times10^{-9}$ g/cm³) is apparently enough to overcome the effect of the universal expansion and give the correct relative abundances of the elements. Thus, on the basis of these considerations one is led to the conclusion that when $t\cong6.70\times10^2$ sec., and $\rho_{r'}\cong1$ g/cm³ we have

$$5.0\times10^{-7}\ \text{g/cm}^3 \lesssim \rho_{m'} \lesssim 1.8\times10^{-4}\ \text{g/cm}^3.$$

While it is not particularly germane to the study reported in this paper, it is interesting to note that one may find the dependence of the universal expansion rate on the time in this type of model. This rate is the percentage change in proper distance per unit time determined by Hubble[10] from the red-shift in spectra of nebulae, and is given in $V=Hd$, where V is the velocity of recession of a nebula at a distance d. In our notation, we have, in general,

$$H=(dL/dt)/L=L^{-1}(\gamma\rho L^2+K_2)^{\frac{1}{2}}. \tag{16}$$

For early time this reduces to

$$H=(2t)^{-1}, \tag{16a}$$

and, for late time, to

$$H=t^{-1}. \tag{16b}$$

For early and late t, the value of H does not depend upon the choice of densities. However, in the transition region where the functional form of H changes, the manner of change does depend on the existing density conditions. The universal expansion rate is the reciprocal of the age of the universe if measured during the period of free expansion.

IV. THE FORMATION OF GALAXIES

In his discussion of the evolution of the universe, Gamow[8] suggested that galactic formation occurred at the time when the densities of matter and radiation were equal. He assumes that the Jeans' criterion of gravitational instability may be applied at this time and as a consequence derives expressions for the galactic diameter and mass.[13] We have carried out calculations[9] based on Gamow's formulation using the corrected expressions for D and M given in footnote 13. We find that $\rho_m=\rho_r$ when $t_c\cong0.86\times10^{18}$ sec., which is greater than the age of the universe. This arises out of the fact that, in addition to the difficulties with density determinations mentioned earlier, there is involved an extra-

[13] Using the corrected form of α described in footnote 12, we find for the galactic diameter, D, and mass, M, the following corrected expressions according to Gamow's formulation:

$$D^2=\frac{10\pi^2e^2\hbar}{3Gm^{11/2}c^5\alpha}(|\mu_P|+|\mu_N|)^2(\epsilon^{1/2}+\epsilon_0^{1/2})\epsilon^{3/2}t_c,$$

and

$$M\cong\rho_mD^3=\frac{5^{3/2}\pi^{7/4}e\hbar^{1/2}k}{2^{3/4}3^{5/4}G^{7/4}a^{1/4}m^{15/4}c^2\alpha^{1/2}}(|\mu_P|+|\mu_N|)(\epsilon^{1/2}+\epsilon_0^{1/2})\epsilon^{3/4},$$

where t_c is the time at which the densities of matter and radiation were equal.

polation of relations valid only for early t past their region of validity. It is evident from our choice of densities that the densities of matter and radiation must be equal at a time t_c which is earlier than now. We have retained Gamow's basic idea of galactic condensation at t_c and have applied the Jeans' criterion,[14]

$$D^2=(5\pi kT_c)/(3Gm\rho_{m,c}), \tag{17}$$

where T_c and $\rho_{m,c}$ are taken at $t=t_c$. We may write

$$D=KB^{9/8}, \tag{18}$$

where

$$K=[(5\pi kc^{\frac{1}{2}})/(3a^{\frac{1}{2}}Gm)]^{\frac{1}{8}}, \tag{18a}$$

$$B=\rho_r\rho_m^{-4/3}, \tag{18b}$$

and

$$M\cong\rho_m D^3=K^3B^{3/8}. \tag{19}$$

For the set of density conditions as given in Eq. (12), we obtain for D and M the values 2.1×10^3 light years and 3.8×10^7 sun masses, respectively. When the densities of matter and radiation were equal, $t_c\cong3.5\times10^{14}$ sec.$\cong10^7$ years, $\rho_{m,c}\cong10^{-24}$ g/cm^3 and $T_c\cong5.9\times10^{2}$°K. For the set of densities given in Eq. (15) we obtain $D\cong1$ light year, $M\cong2.8\times10^5$ sun masses, $t_c\cong1.8\times10^{10}$ sec.$\cong6\times10^2$ years, $\rho_{m,c}\cong10^{-15}$ g/cm^3, and $T_c\cong10^5$°K. In the former case we find values for the galactic mass, diameter and density which are roughly of the order of magnitude observed for the average nebula. In the latter case the values differ by many orders of magnitude. Thus, the values one obtains for the galactic mass and diameter appear to be extremely critical to the choice of densities. One might interpret the large discrepancy in the latter case as arising from the fact that the density conditions chosen appear to be incompatible with the neutron-capture theory of the formation of the elements.

The Jeans' criterion of gravitational instability was derived by the consideration of an acoustic wave propagating in a static medium. If the Jeans' criterion is satisfied, regions of condensation whose size is of the order of D, D being the acoustic wavelength, would have separated and would have been gravitationally stable. The separation between condensations would then also have been of the order of D. The separation distance would increase with time, however, because of the universal expansion, whereas the condensations, being gravitationally stable units, would not expand. Subsequently, stars would evolve in these condensations and nebular configurations would be established.[14] From the time variation of proper distance the separation between galaxies is computed to be about 10^6 light years at the present time, in general agreement with observed separations.

The applicability of Jeans' criterion of gravitational instability to this situation must be seriously questioned since it does not contain the possible effects of universal expansion, radiation, relativity, and low matter density. However, it seems reasonable to attach some significance to the time at which radiation and matter densities are equal, because beyond this time the expansion is free and it would become increasingly difficult to form condensations.[15] It should be mentioned that Lifshitz[16] has considered the problem of gravitational instability associated with infinitesimal perturbations of an arbitrary nature in a general relativistic expanding universe and has found that the system is stable and the perturbations do not grow. Until such time as a physically satisfactory criterion for the formation of galaxies is found, it does not appear to be profitable to delve further into such questions as the variation in galactic mass and size with time of formation.

V. ACKNOWLEDGMENTS

Our thanks are due Dr. G. Gamow for his interest in this work and for many stimulating discussions. We wish to express our appreciation to Dr. R. B. Kershner for his interest and invaluable aid concerning some mathematical questions. We wish also to thank Drs. J. W. Follin, Jr., and F. T. McClure for valuable discussions, and Miss S. Thomas for her aid during the preparation of the manuscript.

[14] J. H. Jeans, *Astronomy and Cosmogony* (Cambridge University Press, London, 1929). Except for a numerical factor, Jeans' criterion may be obtained by equating the average thermal energy of a particle with the gravitational potential energy of this particle on the surface of a sphere of diameter D.

[15] See G. Gamow and E. Teller, Phys. Rev. **55**, 654 (1939). In this paper it is shown that if galaxies were formed during a period of free expansion then

$$G\rho_m[(4\pi/3)(D/2)^3]/(D/2)\geqq(H^2/2)(D/2)^2,$$

where H is Hubble's expansion rate and D is the diameter of the condensation. This condition sets a lower limit to ρ_m, namely $\rho_m\geqq(3H^2/8\pi G)=0.6\times10^{-27}$ g/cm^3 and is satisfied by the density value we obtain for galaxies at the time of formation.

[16] E. Lifshitz, J. Phys. U.S.S.R. **10**, 116 (1946).