On the Quantization of a Unitary Field Theory

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In a unitary field theory particles appear not as singularities but as small volumes in which energy and charge of the field are concentrated. In a theory of this nature, which is necessarily non-linear, all properties of the particles, such as their equations of motion, follow from the field equations. It is pointed out in this paper that the non-linearity present in the usual classical field theories is sufficient to permit the existence of particle-like solutions under certain conditions and that it is therefore possible to derive a class of classical unitary theories by postulating that solutions of the usual classical theories are physically admissible only if they are free from singularities. We have studied the quantization of such a theory without specializing the Lagrangian. Momentum (G_α) and angular momentum ($M_{\alpha\beta}$) appear naturally and a relativistic definition of the position (X_α) of a particle may be given in terms of $M_{\alpha\beta}$ and G_α. The commutators of these particle observables (X_α, G_α, $M_{\alpha\beta}$) with each other are calculated and found similar to those postulated in the Snyder formalism which quantizes space-time; these commutators reduce to the usual ones for non-relativistic velocities. (The quantized Born-Infeld theory did not agree with the usual quantum theory of particles even in the non-relativistic limit, because it was not strictly unitary.) The connection between velocity and momentum, and the equation of motion of particles which follow from the field equations are the usual ones. Operators for mass and charge may be defined; these mutually commute and permit the usual classification of the elementary particles according to mass and charge. Magnetic moment and charge commute with the total electromagnetic field. Particle observables and observables describing the external field may not commute in general.

INTRODUCTION

CURRENT theories of matter are based on the concept of elementary particles, which are described either as point singularities or as extended sources of field. The point source models lead to infinities which must be removed by subtraction formalisms; they are unsatisfactory in their present forms either because of the arbitrariness associated with the subtraction recipes or because of their reliance on a future theory which is expected to permit the calculation of certain "infinite" integrals.[1] On the other hand, the extended source models, which correspond to cut-off and strong coupling theories, are not relativistically covariant. We are here interested in a theory which avoids, at least at the classical level, the difficulties characteristic of both the point and extended source particles. In the theory to be discussed the field is everywhere finite, single-valued, and continuous. The particles appear only as small regions where energy-momentum and charge current of the field are concentrated. The particles, being no longer associated with singularities, do not have an independent existence. Such a theory is for this reason termed unitary and ascribes to the elementary particles a finite extension whenever the concept of particle is adequate. On the other hand, in this kind of theory one must abandon the particle concept in certain situations such as a high energy collision, in which the interaction is so intimate that charge, spin, and rest energy are exchanged, and the number of product particles need not be equal to the number of colliding ones; for it is certainly meaningless to speak of the existence of extended and distinct particles during such a collision. It is, however, still possible to describe collisions of this sort in terms of the energy-momentum and charge current densities in the space where the interaction occurs. Of course, it is likely that such a detailed description of the interaction is not physically meaningful; in that case it can be hoped that investigations along these lines will reveal which elements are essential and which may be omitted. In a similar spirit one may interpret the structure which a field theory ascribes to the elementary particles: only those structural details which are observable, such as mass and charge, should be taken seriously.

In a unitary theory the equations of motion of the particle follow from the field equations; in this respect it differs from a theory in which the particles are represented by singularities in the field, for in that case the field equations break down at the singularities, and therefore cannot in general determine the motion of the particles. A unitary theory is necessarily non-linear, for one particle could not influence another if the equations were linear. A second essential role of the non-linearity is to stabilize the concentrations of energy which are interpreted as particles. Since the current field theories are non-linear before field quantization, one is led to investigate the possibilities which arise, when one takes over the Lagrangians of the usually accepted fields—Maxwell, Dirac, Yukawa and Einstein—and the usually discussed couplings between them, and adds the postulate that a solution is physically admissible only if it has no singularities. The coupling between these fields leads to a non-linearity in the field equations which

[1] Mass and charge renormalization appears to be a unique procedure, but (1) the replacement of infinite integrals by finite quantities needs to be justified by the "future theory," and (2) mass and charge are associated with the singularities in a semi-empirical way.

permits in the one case which has been investigated the existence of static and spherically symmetric solutions, which can be interpreted as particles, since they concentrate charge and energy in a small region. The Lagrangian which has been investigated is

$$L=-\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu}+(D_\mu\psi)(D_\mu{}^*\psi^*)+\sigma^2\psi\psi^*, \tag{1}$$

where

$$D_\mu\psi=(\partial_\mu-i\epsilon\varphi_\mu)\psi, \qquad \mu=1\cdot\cdot 4,$$
$$D_\mu{}^*\psi^*=(\partial_\mu+i\epsilon\varphi_\mu)\psi^*, \quad x_4=ict. \tag{1a}$$

This case was considered by Rosen,[2] who was led to it as the simplest modification of the Maxwellian Lagrangian which contains the potential explicitly and is still gauge invariant. Except for a sign difference, however, this Lagrangian represents simply the usual Maxwell field coupled in the ordinary way to the usual scalar (or pseudoscalar) mesic field. The signs are so chosen that the energy of the free electromagnetic field is positive, as it should be, while that of the free mesic field is negative. This choice of signs is physically meaningful if the net energy of a particle constructed out of this mixture is positive; we return to this question later.

Before considering the Lagrangian (1) in more detail, we wish to point out that the Maxwell-Dirac case, which is the most thoroughly investigated field from the standpoint of quantum perturbation theory, has not yet been considered from this point of view. Other combinations of known fields are possible, and it is possible that if a field theory of matter exists at all, one can approach it by considering simple combinations of fields already believed to exist. The cases to be discussed in the next paragraph are only illustrative and have no physical application as they stand; we wish to point out mainly that there is a class of unitary theories which are very close to the current theories and which may be arrived at by rejecting singular solutions and rigorously satisfying the self-consistency conditions of current theory.

THE MAXWELL-YUKAWA AND THE SCALAR-SCALAR FIELD

The Lagrangian (1) leads to the field equations

$$(D_\lambda D_\lambda-\sigma^2)\psi=0, \tag{2}$$

and

$$F_{\mu\lambda,\lambda}=S_\mu, \tag{3}$$

where

$$S_\mu=i\epsilon(\psi^*D_\mu\psi-\psi D_\mu{}^*\psi^*) \tag{3a}$$

is the current. We look for solutions of the type

$$\psi=\theta(r)e^{i\omega t}, \quad \varphi_\mu=(0,\, i\varphi).$$

[2] Nathan Rosen, Phys. Rev. **55**, 94 (1939). The particle-like solutions corresponding to this Lagrangian were independently found by H. Jehle whose calculations have, however, not been published.

Then the field equations become

$$r^{-2}\frac{d}{dr}r^2\frac{d\theta}{dr}+\left(\frac{\omega}{c}+\epsilon\varphi\right)^2\theta-\sigma^2\theta=0, \tag{4a}$$

$$r^{-2}\frac{d}{dr}r^2\frac{d\varphi}{dr}+2\epsilon\left(\frac{\omega}{c}+\epsilon\varphi\right)\theta^2=0. \tag{4b}$$

Change to dimensionless variables

$$z=\sigma^{-1}(\omega/c+\epsilon\varphi), \quad y=\sqrt{2}\sigma^{-1}\epsilon\theta, \quad x=\sigma r, \tag{5a}$$

and take

$$\eta=xy, \quad \zeta=xz. \tag{5b}$$

Then the non-linear equations

$$\eta''+(\zeta^2/x^2-1)\eta=0, \tag{6a}$$

$$\zeta''+\eta^2x^{-2}\zeta=0, \tag{6b}$$

must be satisfied simultaneously subject to the boundary conditions

$$z,\, y \text{ finite at } x=0,$$

$$\frac{dz}{dx}=\frac{dy}{dx}=0 \text{ at } x=0,$$

or

$$\zeta(0)=\eta(0)=0, \quad \zeta'(0)=z(0), \quad \eta'(0)=y(0). \tag{7a}$$

These conditions exclude a discontinuity in slope of θ and φ and a corresponding discontinuity in the electric field at $x=0$. In addition we require

$$\lim_{r\to\infty}\begin{cases}\varphi=0\\ \theta=0\end{cases} \quad\text{or}\quad \lim_{x\to\infty}\begin{cases}\zeta/x=\omega/c\sigma\\ \eta/x=0.\end{cases} \tag{7b}$$

If one of the slopes $\zeta'(0)$ or $\eta'(0)$ is assigned arbitrarily, the other is fixed by the condition of finiteness at $x=\infty$.

Thus there is a one parameter family of solutions to (6) and (7). The value of this parameter may be assigned as follows. Asymptotically one has by (6b)

$$\zeta=\epsilon Q+Bx,$$

where Q and B are integration constants. To satisfy (7) we take $B=\omega/c\sigma$. We may regard Q as the parameter to be fixed. One sees that $\phi=Q/r$ at large distances, and also that

$$\int S_0 d\mathbf{r}=4\pi Q. \tag{8}$$

Hence the constant of integration, Q, corresponds to the charge of the particle, and may be fixed by assigning the charge. It may now be shown that (6) and (7) possess nodeless solutions[2] depending on the four physical constants (c, σ, ϵ, Q); the frequency ω has a corresponding eigenvalue.

The energy momentum tensor follows uniquely from the equation

$$\delta L = -\tfrac{1}{2}\int \theta_{\mu\nu}\delta g^{\mu\nu}(-g)^{\frac{1}{2}}d\tau, \qquad (9a)$$

where δL is the variation in the Lagrangian when only the $g^{\mu\nu}$ are varied. In the simple case considered here one finds

$$-\theta_{\mu\nu} = 2\frac{\partial L}{\partial g^{\mu\nu}} - Lg_{\mu\nu}. \qquad (9b)$$

According to this definition of $\theta_{\mu\nu}$ the integral, $-\int\theta_{44}d\mathbf{r}$ is equivalent to the gravitational rest mass. The canonical tensor,

$$-\sum \frac{\partial L}{\partial(D_\mu\psi)}(D_\nu\psi) + Lg_{\mu\nu} = T_{\mu\nu}$$

in general must be corrected to agree with (9a). One finds here

$$\begin{aligned}-\theta_{\mu\nu} = \tfrac{1}{4}F_{\lambda\sigma}F_{\lambda\sigma}g_{\mu\nu} - F_{\alpha\mu}F_{\alpha\nu} \\ +(D_\mu{}^*\psi^*)(D_\nu\psi) + (D_\nu{}^*\psi^*)(D_\mu\psi) \\ -[(D_\mu{}^*\psi^*)(D_\mu\psi) + \sigma^2\psi^*\psi]g_{\mu\nu}. \qquad (9c)\end{aligned}$$

This leads to the rest mass

$$Mc^2 = -\int\theta_{44}d\mathbf{r}. \qquad (10)$$

M is a function of the four physical quantities (c, ϵ, σ, Q). For example, if ϵQ is small, $Mc^2 \sim -\epsilon^{-1}Q\sigma$. The sign of M is negative for the Lagranian (1); it cannot be made positive by adding a divergenceless tensor to $\theta_{\mu\nu}$ without violating (9a) which relates θ_{44} to the density of gravitational mass. The sign of M is a fundamental defect of the theory just described, but there are ways of avoiding this difficulty. For example, consider the Lagrangian describing a scalar-scalar field with pair coupling

$$L = -(\partial_\lambda\psi)(\partial_\lambda\psi^*) - \sigma^2\psi\psi^* - (\partial_\lambda\varphi)(\partial_\lambda\varphi) + g\psi\psi^*\varphi^2, \quad (1)'$$

where $g>0$. This leads to the equations of motion

$$\begin{aligned}\nabla^2\theta + (\omega^2/c^2 - \sigma^2)\theta = -g\theta\varphi^2, \\ \nabla^2\varphi = -g\theta^2\varphi,\end{aligned} \qquad (4)'$$

where $\psi = \theta e^{i\omega t}$ and $\partial\varphi/\partial t = 0$. These may be brought to the form (6) by the substitutions

$$\eta = g^{\frac{1}{2}}r\theta;\quad \zeta = g^{\frac{1}{2}}r\varphi;\quad x = ar;\quad a^2 = \sigma^2 - \omega^2/c^2.$$

The boundary conditions at infinity are $\zeta/x = \eta/x = 0$. These correspond to (7b) if we put $\omega = 0$. Since the boundary value problem described by Eqs. (6) and (7) has a solution when $\omega = 0$, it may be concluded that (4)′ also have particle-like solutions.[3] It is easily shown that the energy of such a particle is positive (the energy density is positive), and so it is possible to display at least one unitary theory which leads to particles of positive mass.[4]

A unitary theory also has the following property: if the charge, Q, is fixed, a discrete spectrum of rest masses follows uniquely, and one can hope to identify the masses of the elementary particles with this spectrum.[2] On the other hand, in the unquantized theory which we have described there is no condition to fix the charge; it will be shown later that the field quantization imposes a relation between the minimum value of Q and the coupling constant, ϵ, namely: $\epsilon = Q_{\min.}/\hbar c$. It is not clear whether the number $Q_{\min.}{}^2/\hbar c$ is also determined.

If one limits himself to classical theory, it is natural to consider other Lagrangians, in particular, the Maxwell-Dirac case, and to study solutions for which $\varphi \neq 0$.[5] However, one would expect major changes to be introduced by the field quantization, and we therefore turn to this question. The theory corresponding to the Lagrangians (1) and (1)′ has been given because it is the only unitary theory which, as far as we know, has been carried to a point where the existence of particle-like solutions could be ascertained. It should be emphasized, however, that the work to be given in the remainder of this paper has not been specialized to a particular Lagrangian, and so is not vulnerable to criticism directed specifically against the forms (1) and (1)′.

QUANTIZATION

Since the field operators are fundamental and all particle observables are derived in a unitary scheme, the quantum theory of particles is already contained in the quantized field theory; or in more formal terms, the commutation relations between the particle observables are already determined by the postulated commutators of the field operators. On the other hand, all the commutators of the new theory must agree with those of the current theory in the correspondence limit. Hence in this limit the new theory must postulate the field commutators of current theory and derive the particle commutators which are now known to be valid. It was one of the defects of the quantized Born-Infeld theory that this condition was not met; in particular the position and momentum of a particle were commutable. The cause of this failure can be traced to the fact that the corresponding classical theory was

[3] N. Rosen and A. Menius, Phys. Rev. 62, 436 (1942). Solutions are found for which $\omega = 0$.

[4] I wish to thank D. Bohm for the remark that one can expect to form a particle of positive mass by locking the Dirac field to a scalar Yukawa field, and that one could perhaps get a useful theory of the nucleon in this way.

[5] In unpublished calculations H. Jehle has found that the Maxwell-Yukawa Lagrangian, with the *usual* choice of signs, leads to particles of positive mass for the case of $\varphi \neq 0$.

not strictly unitary insofar as particles were still associated with field singularities. Since the positions of the singularities did not follow from the field equations, it was necessary to regard their coordinates, which were also the particle coordinates, as independent of the field observables. But since the theory was unitary insofar as momentum resided exclusively in the field, it followed that the momentum, depending on the field observables only, commuted with position, which was independent of the field variables. An attempt to overcome the difficulty was made by Pryce,[6] who ascribed to the particles an intrinsic momentum which did not reside in the field; but after this amendment the theory was no longer unitary in even a restricted sense. It will be shown here that the quantization of a classical theory which is strictly unitary does not lead to such difficulties. This is a general fact which does not depend on either the Lagrangian or the commutation relations obeyed by the fields.

COMMUTATORS FOR FIELD VARIABLES

The field may be quantized according to either Einstein-Bose or Fermi-Dirac rules. The former are

$$\begin{aligned}[\pi^{(\sigma)}(x), \psi^{(\sigma')}, (x')]_- &= -i\hbar\delta_{\sigma\sigma'}\delta(x-x'),\\ [\pi^{*(\sigma)}(x), \psi^{*(\sigma')}(x')]_- &= -i\hbar\delta_{\sigma\sigma'}\delta(x-x').\end{aligned} \quad (11a)$$

All other commutators vanish. The latter are

$$\begin{aligned}[\psi^{(\sigma)}(x), \psi^{*(\sigma')}(x')]_+ &= \delta_{\sigma\sigma'}\delta(x-x'),\\ [\psi^{(\sigma)}(x), \psi^{(\sigma')}(x')]_+ = [\psi^{*(\sigma)}(x), &\psi^{*(\sigma')}(x')]_+ = 0.\end{aligned} \quad (11b)$$

In the following Einstein-Bose rules are to be understood unless the contrary is explicitly stated.

MOMENTUM AND ANGULAR MOMENTUM

The density of energy-momentum in space is defined as $(-iT_{4\nu}/c)$, where

$$T_{\mu\nu} = -\sum_\alpha\left[\frac{\partial L}{\partial(D_\mu\psi^{(\alpha)})}(D_\nu\psi^{(\alpha)}) + \frac{\partial L}{\partial(D_\mu{}^*\psi^{(\alpha)*})}(D_\nu{}^*\psi^{(\alpha)*})\right] + Lg_{\mu\nu}. \quad (12)$$

The sum over α means the sum over all components of all fields assumed to be present, but D_μ is to be replaced by ∂_μ in those terms corresponding to the Maxwell field. A symmetrizing term must be added to $T_{\mu\nu}$ if (12) does not agree with (9a). The density of angular momentum is similarly defined as $(-iM_{4\mu\nu}/c)$, where[7]

$$M_{\lambda\mu\nu} = T_{\lambda\mu}x_\nu - T_{\lambda\nu}x_\mu + \sum_{\alpha,\beta}\frac{\partial L}{\partial(D_\lambda\psi^{(\alpha)})}S_{\mu\nu}{}^{(\alpha\beta)}\psi^{(\beta)} + {}^*, \quad (13)$$

where

$$\delta\psi^{(\alpha)} = \sum_{\lambda<\mu}\sum_\beta S_{\lambda\mu}{}^{(\alpha\beta)}\delta\omega_{\lambda\mu}\psi^{(\beta)},$$

and

$$\delta x_\lambda = \sum_\mu \delta\omega_{\lambda\mu}x_\mu$$

is an infinitesimal Lorentz transformation. The third term in (13) can be interpreted as the density of spin angular momentum. One has $T_{\lambda\nu,\lambda}=0$, $\theta_{\lambda\nu,\lambda}=0$, and $M_{\lambda\mu\nu,\lambda}=0$. Then if $\theta_{\mu\nu}$ and $M_{\lambda\mu\nu}$ vanish outside a 3-dimensional volume, the integrals

$$icG_\nu = \int T_{4\nu}d\mathbf{x} = \int \theta_{4\nu}d\mathbf{x},$$

$$icM_{\mu\nu} = \int M_{4\mu\nu}d\mathbf{x} = \int (\theta_{4\mu}x_\nu - \theta_{4\nu}x_\mu)d\mathbf{x}$$

over that volume have the transformation properties of a 4-vector and an antisymmetric tensor, respectively. Hence if there are solutions free from singularities for which $\theta_{\mu\nu}$ and $M_{\lambda\mu\nu}$ are concentrated in a small volume, then this small region behaves like a particle whose momentum is G_ν and whose angular momentum is $M_{\mu\nu}$.

In canonical variables the momentum now is

$$G_\mu = \int \theta_\mu d\mathbf{x} \quad (14)$$

and the angular momentum

$$M_{\mu\nu} = \int (g_\mu x_\nu - g_\nu x_\mu)d\mathbf{x} + \int S_{\mu\nu}{}^{(\alpha\beta)}\pi^{(\alpha)}\psi^{(\beta)}d\mathbf{x}, \quad (15)$$

where

$$g_k = -\sum\nolimits_\sigma(\pi^{(\sigma)}D_k\psi^{(\sigma)} + D_k{}^*\psi^{*(\sigma)}\pi^{*(\sigma)}), \quad (16a)$$

$$g_4 = -ic^{-1}L - \sum\nolimits_\sigma(\pi^{(\sigma)}D_4\psi^{(\sigma)} + D_4{}^*\psi^{*(\sigma)}\pi^{*(\sigma)}), \quad (16b)$$

where

$$ic\pi = \partial L/\partial(D_4\psi) = \partial L/\partial(\partial_4\psi).$$

The "true" density of momentum, i.e., the density agreeing with gravitational theory, is, on the other hand, $\theta_\alpha \equiv -i\theta_{4\alpha}/c$, where $\theta_{\alpha\beta}$ is the symmetric energy momentum tensor. In terms of θ_α the angular momentum is

$$M_{\mu\nu} = \int (\theta_\mu x_\nu - \theta_\nu \mathbf{x}_\mu). \quad (15a)$$

The momentum and angular momentum so defined are obviously gauge invariant. The density

[6] M. H. L. Pryce, Proc. Roy. Soc. **159**, 355 (1937).
[7] Wolfgang Pauli, Rev. Mod. Phys. **13**, 204 (1941). The notation for $T_{\mu\nu}$ and $M_{\lambda\mu\nu}$ is slightly different here.

of momentum can be written

$$g_\mu = p_\mu + \rho\varphi_\mu, \qquad (16c)$$

where p_μ is the form of g_μ when no electromagnetic field is simultaneously present, and ρ is the charge density operator:

$$\rho = -i\epsilon \sum_{(\sigma)} (\pi^{(\sigma)}\psi^{(\sigma)} - \psi^{*(\sigma)}\pi^{*(\sigma)}). \qquad (16d)$$

From these definitions and the commutation rules (11) one finds

$$[G_k, f(x)] = i\hbar\frac{\partial f}{\partial x_k} + \int \varphi_k'[\rho', f]d\mathbf{x}',$$

where ρ' means $\rho(x')$. If f is gauge invariant, then

$$[\rho', f] = 0$$

and the integral vanishes. In the applications to be made here f is gauge invariant. The commutator with respect to G_4 is similarly the x_4 derivative but this last relation, giving the equations of motion, is not a consequence of the commutation relations (11). One may write

$$[G_\alpha, f] = i\hbar(\partial f/\partial x_\alpha) \quad \alpha = 1\cdot\cdot 4. \qquad (17)$$

Similarly one shows that the space components of G_α commute among themselves; and if $T_{\mu\nu}$ vanishes outside the volume of the particle, i.e., if there is no external field, then $[G_i, G_4] = 0$ also. Hence in this case

$$[G_\alpha, G_\beta] = 0 \quad \alpha, \beta = 1\cdot\cdot 4. \qquad (18)$$

The angular momentum ($M_{\mu\nu}$) consists according to (15), of two parts

$$G_{\mu\nu} = \int (g_\mu x_\nu - g_\nu x_\mu)d\mathbf{x}, \qquad (19a)$$

and

$$s_{\mu\nu} = \int S_{\mu\nu}{}^{(\alpha\beta)}\pi^{(\alpha)}\psi^{(\beta)}d\mathbf{x}. \qquad (19b)$$

$G_{\mu\nu}$ and $s_{\mu\nu}$, the orbital and spin contributions, are not separately tensors although their sum is.

One now shows by the rules (11) that

$$[G_{ij}, G_{jk}] = i\hbar G_{ik}, \quad [G_{ij}, G_{kl}] = 0,$$
$$[s_{ij}, s_{jk}] = i\hbar s_{ik}, \quad [s_{ij}, s_{kl}] = 0,$$
$$[G_{ij}, s_{kl}] = 0, \text{ including the case } k = j.$$

For example

$$[s_{ij}, s_{jk}] = \sum_{\alpha\beta\gamma\delta} \int\int S_{ij}{}^{(\alpha\beta)} S_{jk}{}^{(\gamma\delta)} [\pi^{(\alpha)}\psi^{(\beta)}, \pi^{(\gamma)\prime}\psi^{(\delta)\prime}]d\mathbf{x}d\mathbf{x}'$$
$$= i\hbar \sum_{\alpha\lambda\delta} \int (S_{ij}{}^{(\alpha\lambda)} S_{jk}{}^{(\lambda\delta)} - S_{jk}{}^{(\alpha\lambda)} S_{ij}{}^{(\lambda\delta)})\pi^{(\alpha)}\psi^{(\delta)}d\mathbf{x}$$
$$= i\hbar s_{ik}.$$

It follows that the total angular momentum obeys the rules

$$[M_{ij}, M_{jk}] = i\hbar M_{ik}, \quad i, j, k = 1, 2, 3.$$

Now by making a Lorentz transformation on this equation one can show that the same rules are also valid for the $i4$ components.

All of these commutators can be summarized in the tensor equation:

$$[M_{\alpha\beta}, M_{\gamma\delta}] = (g_{\alpha\delta}M_{\beta\gamma} + g_{\beta\gamma}M_{\alpha\delta} - g_{\alpha\gamma}M_{\beta\delta} - g_{\beta\delta}M_{\alpha\gamma})i\hbar. \qquad (20)$$

Likewise one finds

$$[M_{\alpha\beta}, G_\gamma] = (G_\alpha g_{\beta\gamma} - G_\beta g_{\alpha\gamma})i\hbar. \qquad (21)$$

Equations (20) and (21) depend simply on the facts that G_α is a displacement operator and that $M_{\alpha\beta}$ is an infinitesimal representation of the Lorentz group. The Eqs. (18), (20), (21) show that the momentum and angular momentum associated with a small volume of space according to the definitions (14) and (15) indeed obey the commutation laws *ordinarily assumed for the corresponding properties of particles.*

NON-RELATIVISTIC DEFINITIONS OF POSITION

According to the last paragraph, the momentum and angular momentum of the derived particles appear very naturally in a field theory. The concept of position is less fundamental. We give two illustrations of non-relativistic definitions before considering the relativistic case. If the particle results from the locking of a Maxwell field to a Yukawa field, or to some other field, in the way discussed earlier, a simple definition of position X_i, suggests itself, namely:

$$X_i \int \rho d\mathbf{x} = \int x_i \rho d\mathbf{x}. \qquad (22)$$

For ρ one can take the charge density (Eq. 16d). It then follows from the commutation rules (11a) that

$$[G_k, X_i] \int \rho d\mathbf{x} = -i\hbar\delta_{ki} \int \rho d\mathbf{x}. \qquad (23)$$

One can put $\int \rho d\mathbf{x} = 1$ when (as for the Lagrangian (1)) this integral is a constant of the motion. One can also get (23) by taking $\rho = \psi^*\psi$.

To prove (23) we calculate

$$J_{ik} = \int\int x_i[\rho, g_k']d\mathbf{x}d\mathbf{x}'.$$

Note that $[\rho, g_k'] = [\rho, p_k'] = [\rho, (\pi'\partial_k\psi' + (\partial_k\psi^*)'\pi^{*\prime})]$,

$$\int\int x_i[\rho, \pi'(\partial_k\psi)']d\mathbf{x}d\mathbf{x}' = \epsilon\hbar \int\int x_i\{\pi(\partial_k\psi)'\delta(x-x') - \pi'\psi\partial_k'\delta(x-x')\}d\mathbf{x}d\mathbf{x}'$$
$$= \epsilon\hbar \int x_i\partial_k(\pi\psi)d\mathbf{x},$$

$$\int\int x_i[\rho, \pi^{*\prime}(\partial_k\psi^*)']d\mathbf{x}d\mathbf{x}' = -\epsilon\hbar \int x_i\partial_k(\pi^*\psi^*)d\mathbf{x}.$$

Hence

$$J_{ik}=i\hbar\delta_{ik}\int\rho d\mathbf{x}.$$

Likewise if one considers a particle built out of the Dirac-Maxwell field, a natural definition of position is

$$X_i\int(\psi^*1\psi)d\mathbf{x}=\int x_i(\psi^*1\psi)d\mathbf{x}. \tag{24}$$

The weighting function in (24) is again the charge density. Let us take a Dirac-Lagrangian[8] unsymmetric in ψ and ψ^*, such that $\pi^*=0$ and $\pi=i\hbar\psi^*$. Then (24) is a special case of (22) and (16d), and (23) is, of course true if one quantized the Dirac field according to the canonical rules. Equation (23) is moreover still true if one quantizes according to the anticanonical rules (11b).

To show this one finds

$$\left[\int x_i'\rho'd\mathbf{x}', G_k\right]=\left[\int x_i'\rho'd\mathbf{x}', p_k\right],$$

since $[\rho,\rho']=0$. The hermitized momentum density is

$$p_k=(i\hbar/2)[\psi^*\partial_k\psi-(\partial_k\psi)^*\psi]. \tag{25}$$

One therefore calculates

$$J_{ik}=-(i\hbar/2)\sum_{\alpha,\beta}\int\int x_i'[\psi_\alpha^{*\prime}\psi_\alpha', \psi_\beta^*\partial_k\psi_\beta]d\mathbf{x}d\mathbf{x}',$$

and the hermitian conjugate term. The following identity may be used

$$[ab,cd]_-=a[c,b]_+d-c[a,d]_+b+ca[b,d]_+-[c,a]_+bd.$$

Then by (11b)

$$J_{ik}=-(i\hbar/2)\sum_\alpha\int\int x_i\{\psi_\alpha^{*\prime}\delta(x-x')\partial_k\psi_\alpha -\psi_\alpha^*\psi_\alpha'\partial_k\delta(x-x')\}d\mathbf{x}d\mathbf{x}'$$

$$=(i\hbar/2)\delta_{ik}\sum_\alpha\int\psi_\alpha^*\psi_\alpha d\mathbf{x}.$$

Since $J_{ik}^*=J_{ik}$, the total commutator is $2J_{ik}$.

Thus, starting from the non-relativistic definition of position (22), one arrives at the usual commutators whether one quantizes according to F. D. or E. B. brackets.

RELATIVISTIC DEFINITION OF POSITION (CLASSICAL)

One can give a relativistic definition of position (X_μ) in terms of G_μ, $M_{\mu\nu}$, and a new tensor, $m_{\mu\nu}$, in the following way

$$M_{\mu\nu}=G_\mu X_\nu-G_\nu X_\mu+m_{\mu\nu}. \tag{26}$$

Here $m_{\mu\nu}$ is to be interpreted as the intrinsic spin of the particle. Let us first consider (26) as a classical equation; then if one assigns X_4, there are six relations to determine (X_1, X_2, X_3) and the six components of $m_{\mu\nu}$. It is hence necessary to have three other relations; these may be chosen as follows. We require that in the proper system $(G_i=0)$,

$$m_{i4}=0. \tag{27a}$$

The motivation for the choice (27a) is the following. One can put the definition (26) in a different form by starting from

$$M_{i4}=G_iX_4-G_4X_i+m_{i4}.$$

In the proper system this becomes

$$M_{i4}=-G_4X_i.$$

By (15a)

$$M_{i4}=G_ix_4-\int\theta_4x_id\mathbf{x}$$

$$=-\int\theta_4x_id\mathbf{x}.$$

Hence

$$X_i=(G_4)^{-1}\int\theta_4x_id\mathbf{x}. \tag{28}$$

That is, in the proper system (26) reduces to a center of mass definition[9] since θ_4 is just the energy density. This simple interpretation of (28) indicates that the choice of the auxiliary conditions (27a) is a natural one. These three conditions on $m_{\mu\nu}$ can be written in tensor form

$$G_\mu m_{\mu\nu}=0. \tag{27b}$$

Equation (26) may be solved for X_μ in any coordinate system by multiplying by G_ν and summing on ν.

$$X_\nu=G^{-1}(G_\mu M_{\mu\nu}+G_\nu T), \tag{29}$$

where

$$G\equiv G_\mu G_\mu,\quad T\equiv G_\mu X_\mu=G_4{}^pX_4{}^p=G^{\frac{1}{2}}X_4{}^p,$$

where the index p refers to the proper system. It is still necessary to connect $X_4{}^p$ with the field time; we do this by identifying $-iX^p{}_4$ with the field time in the proper system. Hence

$$T=iG^{\frac{1}{2}}\tau, \tag{29a}$$

where τ is proper field time.

The Eqs. (29) and (29a) define the X_μ explicitly. The $m_{\mu\nu}$ now follow from (26) and (29).

$$m_{\mu\nu}=M_{\mu\nu}+G^{-1}(G_\nu G_\alpha M_{\alpha\mu}-G_\mu G_\alpha M_{\alpha\nu}). \tag{30}$$

If there is no spin angular momentum in the field, $s_{\mu\nu}=0$, but it does not follow that $m_{\mu\nu}=0$. Equations (26) are consistent with $m_{\mu\nu}=0$ only if

$$\epsilon_{\alpha\beta\gamma\delta}G_\beta M_{\gamma\delta}=0. \tag{31}$$

[8] G. Wentzel, *Quantentheorie der Wellenfelder* (Frany Deuticke, Wein, 1943), Chap. V.

[9] Møller has also used a similar definition in a paper to appear soon (Communications of Institute for Advanced Study, Dublin).

The conditions (31) are in general not satisfied for extended particles, even if $s_{\mu\nu}=0$.

RELATIVISTIC DEFINITION OF POSITION AFTER FIELD QUANTIZATION

Let us now regard (29) and (30) as operator equations defining the operators X_μ and $m_{\mu\nu}$ in terms of $M_{\mu\nu}$, G_μ, and T. It is assumed that these equations have been hermitized. $M_{\mu\nu}$ and G_μ are defined as operators in (14) and (15). It is now necessary to define T as an operator and, just as in the classica case, to connect it with the field time. We therefore postulate the following connection between T and the field time

$$T=iG^{\frac{1}{2}}\tau, \tag{32}$$

where τ is the field time in the coordinate system for which the expectation values of the G_i vanish.

$$\langle G_i\rangle=0. \tag{32a}$$

The operator Eq. (32) agrees with the corresponding classical equation in the limit $\hbar\rightarrow 0$.

From (32) it follows that T obeys the following commutation rules

$$[T, G_\alpha]=0. \tag{33}$$

Further, since T is an invariant (for Lorentz rotations) it commutes with the angular momentum

$$[T, M_{\alpha\beta}]=0. \tag{34}$$

One can now calculate from (21), (29), and (33)

$$[G_\alpha, X_\beta]=-i\hbar(g_{\alpha\beta}-G_\alpha G_\beta G^{-1}). \tag{35}$$

Using also (34) one finds

$$\begin{aligned}[X_\alpha, X_\beta]=&[G^{-1}G_\mu M_{\mu\alpha}, G^{-1}G_\sigma M_{\sigma\beta}]\\&+[G^{-1}G_\mu M_{\mu\alpha}, G^{-1}G_\beta T]\\&+[G^{-1}G_\alpha T, G^{-1}G_\mu M_{\mu\beta}]\\=&[G^{-1}G_\mu M_{\mu\alpha}, G^{-1}G_\sigma M_{\sigma\beta}]\\=&G^{-2}[G_\mu M_{\mu\alpha}, G_\sigma M_{\sigma\beta}].\end{aligned}$$

And by (20)

$$[X_\alpha, X_\beta]=i\hbar G^{-1}M_{\alpha\beta}, \tag{36}$$

and using (26) one gets

$$[m_{\alpha\beta}, G_\gamma]=0, \tag{37a}$$

$$[m_{\alpha\beta}, m_{\gamma\delta}]=i\hbar(m_{\beta\gamma}g_{\alpha\delta}'+m_{\alpha\delta}g_{\beta\gamma}'-m_{\alpha\gamma}g_{\beta\delta}'-m_{\beta\delta}g_{\alpha\gamma}'), \tag{37b}$$

where $g_{\alpha\beta}'=g_{\alpha\beta}-G_\alpha G_\beta G^{-1}$. According to Eqs. (37) the $m_{\alpha\beta}$ do not quite obey the commutation laws of a spin angular momentum. The Eqs. (18), (35), and (36) are very similar to commutation rules appearing in a recently proposed formalism which quantizes space.[10] According to our view, however, these new uncertainty relations, stemming from the commutation rules for the underlying fields, describe the particles rather than space itself. The fundamental length appearing in (35) and (36), corresponding to limits of observability for position, is simply the Compton wave length of the particle. For the electron this length is rather large but is not in disagreement with Dirac electron theory where position is uncertain in the region of the *zitterbewegung*.

We note that these results also apply to the case in which there are several particles in the volume for which the momentum and the angular momentum are considered. In this situation Eq. (26) is a center of mass definition for the set of particles, and (35) and (36) then apply to the coordinates of the center of mass.

Although we wish to interpret G_α and $M_{\alpha\beta}$ as the momentum and angular momentum of a particle-like field, it should be noted that the commutators (35) and (36) depend only on general properties of the momentum and angular momentum, and the definition of position in terms of them.

EQUATIONS OF MOTION

The angular momentum of a free particle is conserved

$$\frac{dM_{ij}}{dt}=\frac{\partial M_{ij}}{\partial t}+\frac{1}{\hbar}[G_4, M_{ij}]$$

because $\partial M_{ij}/\partial t$ and $[G_4, M_{ij}]$ vanish separately. The M_{i4} components are also conserved since

$$\begin{aligned}\frac{dM_{i4}}{dx_4}&=+G_i-\int\frac{dg_4}{dx_4}x_i=G_i+\int\frac{\partial g_j}{\partial x_j}x_i\\&=G_i-G_j\delta_{ij}=0.\end{aligned}$$

We now show that the relativistic connection between velocity and momentum of a free particle is preserved by the definition (29). By (29)

$$dX_i/dt=(dT/dt)G_iG^{-1}, \tag{38}$$

since dG_μ/dt and $dM_{\mu\nu}/dt$ separately vanish and by (32) this becomes

$$\langle dX_i/dt\rangle=\langle G_iG_4^{-1}\rangle \tag{39}$$

as was to be shown.

One of the important advantages of a unitary theory over a dualistic one is that the equations of motion of the particle are in the unitary case deducible from the field equations. This has been shown in the classical limit.[2,6] One can show that

[10] Hartland S. Snyder, Phys. Rev. 71, 38 (1947).

the equations of motion also follow from the field equations in the quantum theory. For this purpose assume that there is a weak external field. Then one must add an interaction term, H^{int}, which depends on the external field, to the Hamiltonian. The change of momentum is caused by this term alone.

$$\frac{dG_i}{dt}=\frac{i}{\hbar}[H^{\text{int}}, G_i]=\frac{i}{\hbar}\int[\eta', G_i]d\mathbf{x}',$$

where $\eta'=\eta(x')$ is the density of the interaction Hamiltonian

$$\frac{i}{\hbar}[\eta, G_i]=\sum_{(\sigma)}\frac{\partial\eta}{\partial\psi^{(\sigma)}}\frac{\partial\psi^{(\sigma)}}{\partial x_i}+\frac{\partial\eta}{\partial\pi^{(\sigma)}}\frac{\partial\pi^{(\sigma)}}{\partial x_i}$$

$$=\frac{d\eta}{dx}-\left(\frac{d\eta}{dx}\right)_{\text{ext}},$$

where $(d\eta/dx)_{\text{ext}}$ is the partial gradient associated with the dependence of η on the external field. Then

$$\frac{dG_i}{dt}=\int\left[\frac{d\eta}{dx_i}-\left(\frac{d\eta}{dx_i}\right)_{\text{ext}}\right]d\mathbf{x}=-\int\left(\frac{d\eta}{dx_i}\right)_{\text{ext}}d\mathbf{x} \quad (40)$$

since the first term can be removed by an integration. This is the equation of motion: the rate of change of momentum is the integral of the force over the volume of the particle.

MASS AND CHARGE

We define the rest mass by the operator

$$-M^2=G_\alpha G_\alpha=G_4{}^2+\mathbf{G}^2. \quad (41)$$

Then

$$\hbar\dot{M}=[G_4, M]=0. \quad (42a)$$

Similarly,

$$[G_i, M]=0, \quad (42b)$$

and

$$[M_{\alpha\beta}, M^2]=0. \quad (43)$$

Hence the rest mass is a constant of the motion and can be specified simultaneously with the momentum and angular momentum. From Eq. (35) it follows that M^2 also commutes with position

$$[M^2, X_\alpha]=0. \quad (44)$$

It is possible to ascribe a charge to the particle. Although $\int S_\alpha d\mathbf{x}$ is not a 4-vector, $\int S_4 d\mathbf{x}$ is a scalar since $\partial S_\alpha/\partial x_\alpha=0$. This is defined to be the charge

$$Q=\int s_4 d\mathbf{x}.$$

In general

$$S_\alpha=-i\epsilon\sum_{(\sigma)}\left[\frac{\partial L}{\partial(\partial\psi^{(\sigma)}/\partial x_\alpha)}\psi^{(\sigma)}-\frac{\partial L}{\partial(\partial\psi^{(\sigma)*}/\partial x_\alpha)}\psi^{*(\sigma)}\right] \quad (45)$$

so that the operator for the charge is

$$Q=-i\epsilon\sum_{(\sigma)}\int(\pi^{(\sigma)}\psi^{(\sigma)}-\pi^{*(\sigma)}\psi^{*(\sigma)})d\mathbf{x}. \quad (46)$$

The charge is conserved; it also commutes with the momentum and angular momentum, since they are gauge-invariant and as one may verify by the use of (46).

$$[Q, G_\alpha]=0, \quad (47)$$

$$[Q, M_{\alpha\beta}]=0, \quad (48)$$

and by (41)

$$[Q, M^2]=0. \quad (49)$$

Since Q and M are commuting observables, the usual classification of the elementary particles according to charge and mass remains valid.

The charge current vector of the particle may be defined in the customary way

$$J_\mu=Q(dX_\mu/dS) \quad (50)$$

although $\int s_\mu d\mathbf{x}$ cannot be used for this purpose.

CONNECTION BETWEEN CHARGE AND COUPLING CONSTANT

The solutions of Eqs. (4) contain two constants, Q and ϵ. Q is defined as the integral of the charge density (Eq. (8)), whereas ϵ is a constant appearing in the Lagrangian in the combination $\partial_\nu-i\epsilon\varphi_\nu$. In the classical theory Q and ϵ are unrelated, but we shall now show that the field quantization imposes a connection between them. The charge after field quantization is given by Eq. (46).

By the usual transformation

$$\psi=(\hbar/2)^{\frac{1}{2}}(\alpha+\beta^*), \quad \pi=i(\hbar/2)^{\frac{1}{2}}(\alpha^*-\beta). \quad (51)$$

Equation (46) may be rewritten as

$$Q=(\epsilon\hbar/2)\int[\alpha\alpha^*+\alpha^*\alpha-\beta\beta^*-\beta^*\beta]d\mathbf{x}, \quad (52)$$

where

$$[\alpha(x), \alpha^*(x')]=\delta(x-x'),$$
$$[\beta(x), \beta^*(x')]=\delta(x-x'), \quad (52a)$$
$$[\alpha(x), \beta(x')]=0.$$

The transformation (51) is usually made in momentum space but it is also useful in configuration space since the charge operator contains no gradients. Now replace the continuum by a discrete set of points, each commanding the volume, v. Then $\delta(x'-x)\rightarrow v^{-1}\delta_{ij}$ where the indices i, j replace

x and x'; and by the usual argument

$$Q=(\epsilon\hbar/2)\sum\{(2N_a+1)-(2N_b+1)\}=\hbar\epsilon\sum(N_a-N_b), \quad (53)$$

where the sum extends over all lattice points. If e is defined to be the minimum positive value of Q, then

$$\epsilon=e/\hbar. \quad (54)$$

Hence the connection between the coupling constant ϵ and the elementary charge cannot be imposed on the theory in an arbitrary way, since it follows from the field quantization. In particular the connection $\epsilon=1/e$ suggested by Rosen[2] in an interpretation of the Lagrangian (1), is not consistent with the field quantization.

ELECTRIC FIELD, MAGNETIC MOMENT, AND CHARGE

The electromagnetic field commutes with the charge and magnetic moment of a particle. This may be shown as follows. The commutators of the field with itself are

$$[F_{\alpha\beta}, F_{\gamma\epsilon}']=-i\hbar c^2\{\delta_{\beta\epsilon}\partial_{\alpha\gamma'}+\delta_{\alpha\gamma}\partial_{\beta\epsilon'}-\delta_{\beta\gamma}\partial_{\alpha\epsilon'}-\delta_{\alpha\epsilon}\partial_{\beta\gamma'}\}D(x-x', t-t'),$$

where

$$\partial_{\alpha\gamma'}=\partial^2/\partial x_\alpha\partial x_\gamma'.$$

If s_γ is the current density one has, since $F_{\gamma\epsilon,\epsilon}=s_\gamma$

$$[F_{\alpha\beta}, S_\gamma']=-i\hbar c^2\{\partial_{\alpha\gamma'\beta'}-\delta_{\beta\gamma}\partial_\alpha\Box'-\partial_{\alpha'\beta\gamma'}+\delta_{\alpha\gamma}\partial_\beta\Box'\}D(x-x', t-t')$$
$$=i\hbar c(\partial_{\alpha\gamma'\beta'}-\partial_{\alpha'\beta\gamma'})D(x-x', t-t'),$$

since $\Box D=0$. Since also $\partial=-\partial'$, it follows that

$$[F_{\alpha\beta}, S_\gamma']=0.$$

Hence

$$[F_{\alpha\beta}, Q]=\left[F_{\alpha\beta}, \int S_4'd\mathbf{x}\right]=0. \quad (55)$$

Since the magnetic moment is

$$\mu_{ij}=\int(x_is_j-x_js_i)d\mathbf{x},$$

one also has

$$[F_{\alpha\beta}, \mu_{ij}]=0, \quad (56)$$

REPLACEMENT OF CLASSICAL DIFFERENTIAL EQUATIONS BY OPERATOR EQUATIONS

Since the G_α are displacement operators according to (17), the classical wave equation, for example, can be replaced by the quantal equation

$$[G_\alpha, [G_\alpha, \psi]]=0.$$

Maxwell's equations have been similarly rewritten in terms of commutators by Snyder[10] and Yukawa.[11] The rule $i\hbar\partial_\alpha F\rightarrow[G_\alpha, F]$ was proposed by the former in order to define differentiation in a lattice space. In a unitary scheme, however, this substitution is not an additional rule but an elementary consequence of the theory.

EXTERNAL FIELDS

In the usual dualistic theory the following relations are satisfied

$$[X_\alpha{}^p, A]=0, \quad (57a)$$

$$[G_\alpha{}^p, A]=0, \quad (57b)$$

where $X_\alpha{}^p$ and $G_\alpha{}^p$ are the position and momentum of a particle and A is any field quantity: Markow[12] has proposed a modified theory which replaces (57a) by a non-vanishing commutator, and Yukawa has suggested a scheme in which the second relation becomes $[G_\alpha{}^p, A]=i\hbar\partial_\alpha A$. Both of these proposals can be regarded as consequences of a unitary theory since the particle observables there become function of the field observables, and hence the relations (57) may not hold. The precise form of these commutators depends on the way the external field is defined. We hope to discuss this question in a later paper.

I wish to thank Professor C. Møller for having pointed out an error in (37b) as it appeared in the manuscript. The remaining differences between our commutators stem from the fact that he takes t to be a c-number, whereas in this paper the proper time is taken to be a c-number.

[11] H. Yukawa, Prog. Theor. Phys. **2**, 209 (1947).
[12] M. Markow, J. Phys. U.S.S.R. **2**, 453 (1940).