ported here were confirmed, but the latter, obtained from β-ray spectrometer and coincidence measurements, are given in greater detail than in the present paper.

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Boundary Conditions and Range of Force for S State of Two Protons

G. BREIT AND W. G. BOURICIUS
Yale University, New Haven, Connecticut*
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The paper is divided into five sections the first of which is an introduction. In the second the possibilities of describing phase shifts by means of a boundary condition at a distance small compared with $e^2/mc^2=2.8\times10^{-13}$ cm is discussed. It is brought out that the irregular solution of the wave equation in a Coulomb field has a logarithmic infinity which masks the features of the wave function which have to do with phase shifts and therefore with observation. In Table I approximate values of essential quantities are listed. In Section III boundary conditions at moderate distances are studied. It is found that one can replace the "potential energy curve" description by the requirement that the logarithmic derivative of the wave function have an energy independent value at a distance of $\sim0.47\ e^2/mc^2$. Similarly the ^1S proton-neutron interaction can be approximately described by requiring the logarithmic derivative to have an energy independent value at $\sim0.49\ e^2/mc^2$. In the convention of dealing with distance times radial function the values of the logarithmic derivatives are ~0.08, 0.06 for the proton and neutron cases, respectively. It is also possible to require a linear variation of energy for the logarithmic derivative within limits and to retain agreement with experiment. It is pointed out in the introduction that theoretical arguments for considering a failure of the potential energy viewpoint exist and that the agreement of the boundary conditions of Section III with observation may be more than an accident. In Section IV the adjustment of the range of force is treated and evidence for a somewhat smaller value than 2.8×10^{-13} cm, perhaps 2.6×10^{-13} cm is discussed. Use is made of simple relationships between effective depth variation with energy and range. In Section V the function f of BCP is expanded in powers of energy E, the relations for potential energy curves of different shapes are taken up regarding equivalence of range, the deviations from linearity of f with E are discussed from the viewpoint of equivalent error in scattering, and a rapid procedure for finding the equivalent square well range by means of successive approximations is given.

I. INTRODUCTION

THE work reported on in this paper has two objects: (1) To investigate on an empirical basis the possibility of replacing the potential energy point of view for proton-proton scattering by boundary condition requirements; (2) to systematize the treatment of theoretically expected proton-proton scattering by bringing out the way in which different compact potential energy curves can give similar results and to make available convenient ways of adjusting the nuclear potential well parameters to experimental data.

The first of the two objects is related to the general desire of describing the collision process with a minimum of detailed hypothesis concerning the mechanism of the interaction. It has been brought out[1] that a description entirely by means of phase shifts is a possible one and that the isotropy of space implies certain restricting conditions on the possible set of phase shifts. This point of view is closely related to Heisenberg's S matrix.[2] While it is possible to describe the collision process in such a generalized manner, it is difficult to make such a theory quantitatively specific and to establish relations between different phenomena such as those of proton-proton scattering and of the meson field. For this reason it appears advisable not to neglect an approach of intermediate generality with the hope that the investigation might eventually be of help in the formation of a theory of nuclear forces. The question of representing the proton-neutron interaction by means of a suitable boundary condition at zero distance between nuclear particles has been thought about by Wigner about 15 years ago[3] but subsequent calculations on binding energies and other nuclear phenomena have pointed to the potential energy curve of finite width as the more promising temporary expedient. The possibility that the interaction between protons might be confined to distances of negligible amount on the

* Assisted by ONR, Project NR 024-055.
[1] G. Breit, nuclear physics volume of the *University of Pennsylvania Bicentennial Conference*, N-15-15 (University of Pennsylvania Press, Philadelphia, 1941).
[2] W. Heisenberg, Zeits. f. Physik. **120**, 513, 673 (1943).
[3] E. P. Wigner, private communication.

scale of an electronic radius has been borne in mind in the early work on proton-proton scattering[4] and for this reason a formula [Eq. (7.9) of BCP] has been given by BCP[4] by means of which the phase shift expected for an interaction at any energy can be obtained from a preassigned phase shift at one energy. Figures 8, 9, 10, and 11 of BCP show the comparison of the variation of the phase shift according to this formula with variations to be expected for interactions of finite range and the then available experimental material. It was clear from this as well as the later work of BTE that on the hypothesis of zero range of force the phase shift increased much too fast with energy. The relation under discussion was obtained by going to the limit of an interaction potential of zero range which was made to fit the preassigned value of the phase shift at a preassigned energy during the limiting process. The interaction energy in the limit of zero range is infinite.

Although experiment speaks in favor of interaction potentials of finite rather than zero range, one cannot claim that the potential energy curves are more than convenient and temporary expedients. In particular the meson theory of nuclear forces[5] implies that when the protons are close together they exist part of the time as neutrons and positive mesons or as protons and neutral mesons. The meson theory,[5] while similar to quantum electrodynamics differs from the latter in two important respects: (a) the finite rest mass of the meson as contrasted with the zero mass of the photon, (b) the relatively large value of the parameter which corresponds to the fine structure constant $e^2/\hbar c$. Meson theory does not presuppose the existence of interactions between mesons other than the ordinary electromagnetic interaction. It is not known, however, that such interactions do not exist. Supposing for the moment that they are present, one would have a theory of nuclear forces somewhat analogous to that of molecular forces. The ratio of the meson mass to the proton mass being much larger than the corresponding electron-nucleus ratio for the molecular case one expects the picture of a static potential energy curve to be relatively much poorer. For the same kinetic energy the residual proton's or neutron's velocity is only about $(1836/330)^{\frac{1}{2}}=2.4$ times smaller than that of the meson. The conditions of the Born-Oppenheimer approximation are satisfied on this crude picture very poorly. It appears doubtful, therefore, that the usual type of wave mechanical treatment of nuclear particles inside the potential energy hole can be taken literally. This appears to be especially questionable in relation to the assignment of the same effective mass to the protons in the region where the interaction takes place as when they are free. The two-particle wave equation in this region is only a substitute for a many particle equation of a character which cannot be definitely stated until the meson theory is more fully developed. It would not be surprising, therefore, if the rather good agreement of calculations making use of potential energy curves were somewhat accidental. It was thought of interest, therefore, to look into the possibility of describing the ^1S interaction on a basis intermediate between that of a boundary condition at zero distance and a potential energy curve. It was found that if one assumes that the logarithmic derivative of the wave function has a value independent of the energy at a suitable distance then the agreement with experiment is about as good as for a potential energy well. The distance at which this is the case is about 0.47 e^2/mc^2 and the value of the homogeneous logarithmic derivative Y is about 0.08. The function of which the derivative is taken is the distance times the radial function so that the homogeneous logarithmic derivative of the radial function is -0.92. It is supposed in this consideration that the radial function is continued from large distances down to the point at which Y is calculated without any interaction potential except for the Coulomb energy. If such an explanation had a literal significance it would imply that the kinetic energy before collision has no influence on the wave function of the protons just inside the boundary distance. This cannot be true, of course, except as an approximation, and it is not desired to emphasize the rather good fit with experiment which one obtains except insofar as it shows that the extreme requirement of energy independence of Y is in no worse agreement with experiment than the requirement of an energy independent potential energy curve. If the boundary condition under discussion were to be taken literally, one would have to conclude that the kinetic energy before collision becomes subdivided among mesons and nucleons inside the sphere of interaction to such a degree

[4] G. Breit, E. U. Condon, and R. D. Present, Phys. Rev. **50**, 825 (1936), referred to as BCP. G. Breit, H. M. Thaxton, and L. Eisenbud, Phys. Rev. **55**, 1018 (1939), referred to as BTE. L. E. Hoisington, S. S. Share, and G. Breit, Phys. Rev. **56**, 884 (1939), referred to as HSB. E. Creutz, Phys.Rev. **56**, 893 (1939). Leslie L. Foldy, Phys. Rev. **72**, 125, 731 (1947). R. E. Peierls and M. A. Preston, Phys. Rev. **72**, 250 (1947). G. Breit, A. A. Broyles, and M. H. Hull, Phys. Rev. **73**, 869 (1948), referred to as BBH.

M. A. Tuve, N. P. Heydenburg, and L. R. Hafstad, Phys. Rev. **50**, 806 (1936). Herb, Kerst, Parkinson, and Plain, Phys. Rev. **55**, 247 (1939), referred to as HKPP. L. R. Hafstad, N. P. Heydenburg, and M. A. Tuve, Phys. Rev. **56**, 1078 (1939), referred to as HHT. G. L. Ragan, W. R. Kanne, and R. F. Tashek, Phys. Rev. **60**, 628 (1941), referred to as RKT. A. N. May and C. F. Powell, Proc. Roy. Soc. **A190**, 170 (1947). R. R. Wilson and E. C. Creutz, Phys. Rev. **71**, 339 (1947). Robert R. Wilson, Phys. Rev. **71**, 384 (1947). Wilson, Lofgren, Richardson, Wright, and Shankland, Phys. Rev. **72**, 1131 (1947). I. H. Dearnley, C. L. Oxley, and J. E. Perry, Jr., Phys. Rev. **73**, 1290 (1948). Blair, Freier, Lampi, Sleator, Jr., and Williams, Phys. Rev. **74**, 553 (1948), referred to as BFLSW.

[5] H. Yukawa, Proc. Phys. Math. Soc. Japan **17**, 48 (1935).

that the emerging protons, although possessing the original kinetic energy, have shared it with other particles almost up to the instant of emergence. The logarithmic derivative could conceivably be determined under such conditions mainly by other factors than the external kinetic energy. In this connection it is natural to consider the fact that for a single body problem with potential energy there is a connection between the logarithmic derivative and the probability of the particle being inside the boundary. This states that the rate of change of the logarithmic derivative with energy is proportional to the ratio of the probability of finding the particle inside the boundary to the square of the wave function at the boundary. This relation is the last of the three formulas under Eq. (10.3) of BCP and is in the notation of BCP as well as that of the present paper;

$$\frac{\partial}{\partial E'}\left(\frac{\partial \mathfrak{F}}{\mathfrak{F}\partial r}\right)=\frac{1}{\mathfrak{F}^2}\int_0^r \mathfrak{F}^2 dr.$$

If such a connection were to exist in the physical problem, the requirement of energy independence could not be maintained. It is seen, however, that if the lower limit in the above integral were to be a number close to the upper limit the value of the right side could be appreciably decreased. The change in the lower limit corresponds to applying the one-body wave equation to a part of the region inside the boundary and this fits in with the picture of the protons forming themselves in the physical state in which they emerge a short distance inside the boundary sphere. The evidence for the empirical validity of this boundary value description is discussed in Section III. It is pointed out there that a similar description of the proton-neutron interaction in the ^1S state is possible and that slow neutron scattering data point to approximate constancy of the logarithmic derivative at ~ 0.49 e^2/mc^2 and with a value of $Y \sim 0.06$, values not very different from those for proton-proton scattering. The data on scattering of neutrons of ortho- and parahydrogen are not discussed because the scattering in the triplet state is essential here and because it involves the "tensor force" which provides additional parameters. From the point of view of the boundary condition, calculations of nuclear binding energies by means of Hamiltonian functions with customary exchange forces would require formal changes. Since the evidence considered here is concerned only with the ^1S states, it is possible to avoid difficulties with the saturation of nuclear forces by imposing suitable requirements on states with other orbital and spin angular momenta. It would be possible to attempt to form a theory in which interactions of nuclear particles would be primarily in S states.[6] On such a view p scattering for nucleons would be expected to be present for nuclear particles in a subordinate way if at all. While there are claims of appearance of p waves for proton-proton scattering[7] the experiments used are subject to many sources of error and it is possibly premature to draw a definite conclusion. In closing the preliminary discussion of the energy independent logarithmic derivative, it appears appropriate to emphasize three negative points: (a) There is no evidence so far that the new description is better than the potential energy point of view. (b) It is not probable from the viewpoint of field theories that it can be more than an idealization of a more involved condition. (c) It is possible to fit experiment on the assumption of an energy independent $\partial Y/\partial E$ with a wide choice of values of $\partial Y/\partial E$ of which the value zero, especially emphasized here is a special case.

The last of the three negative points is connected with the fact that $\partial Y/\partial E$ is also nearly energy independent at the boundary of a compact potential well. Some of the consequences of the linear dependence of Y on E are looked into in Section V. The behavior of the energy dependent function f of BCP is considered and its deviation from linearity is worked out. There is a connection between the formulas worked out here and the claims of Landau and Smorodinsky.[8] These writers take a critical and condemning attitude towards the work on proton-proton scattering by one of the authors of the present paper and his former collaborators. In the interests of clarity of the relationship between supposedly different treatments of the same subject as well as a matter of justice to the people involved it becomes necessary to point out that:

(1) Landau and Smorodinsky set up the problem of calculating the dependence of the phase shift on energy on the assumption that the range of force is very small and discuss the limiting case when the range is zero. They find that agreement with experiment is not obtained but they also find that agreement can be secured if the "constant" which they call α_{pp} is made to vary linearly with E. They make a qualitative statement indicating that they believe it reasonable for α_{pp} to vary linearly. They fail to state, however, that the dependence of the phase shift K_0 on E for the case of zero range has been printed as Eq. (7.9) of BCP and that curves corresponding to this relation have been sketched in Figs. 8, 9, 10, 11 of BCP. They also do not state that the quantity with which they deal is within a constant the energy dependent but distance inde-

[6] This general point of view has been considered independently for some years by E. Fermi and E. P. Wigner (private considerations).

[7] R. E. Peierls and M. A. Preston, Phys. Rev. **72**, 250 (1947).

[8] L. Landau and J. Smorodinsky, J. Phys. Acad. Sci. U.S.S.R. **8**, 155 (1944).

pendent quantity f of BCP by means of which phase shifts have been calculated by BCP for the smaller ranges of force.

(2) The requirement of linearity of α_{pp} of Landau and Smorodinsky is identical in consequences with approximating the logarithmic derivative occurring on the left side of Eq. (7.5) of BCP by a linear function of the energy and retaining in the four power series which occur on the right side of this equation only terms of zero and first power in the energy. As has been stated by BCP this Eq. (7.5) is especially suitable for the discussion of conditions for short ranges of force. It is obvious from it that with the approximations just mentioned the quantity f should vary linearly with E, as was realized in the course of the calculations of BCP. The numerical conversion factors have been stated explicitly by BCP, and it was presumed that any interested reader would estimate the relative importance of the terms.

In view of the above it appears fair to say that the relation which Landau and Smorodinsky state to be an improvement on all previous work is in reality an old relation and that its limitations have been stated in the form of equations and numbers in the work to which Landau and Smorodinsky take exception.

(3) It is not true as Landau and Smorodinsky state that the series of papers which they criticize fails to bring out the fact that potential energy curves of different shapes can give the same phase shift in the energy region 0.6 to 2.4 Mev within or close to the accuracy of available data. This was verified by numerical calculation for the square well, Gauss Error, Exponential, and Meson Potentials. In the first column of p. 844 of BCP it was brought out how one can compare potential curves of different shapes making use of the connection between the logarithmic derivative Y and the phase shift and of the general connection between $\partial Y/\partial E$ and the "form factor" of the wave function which is shown by the last of the three formulas in Eq. (10.3) of BCP. The features which matter for the equivalence of potentials of different shapes have been studied in HSB where curves showing approximate sensitivity of phase shift to potential changes at different distances have been given. By means of these one can estimate to what degree potential energy curves of different shapes are equivalent. This work is also stated by L and S as being inadequate. In point of fact it goes farther than the conclusions of L and S because by means of it one can discuss the limitations of equivalence of potential wells.

(4) It is stated by L and S on p. 154 that in the series of papers to which they take exception: "The fundamental conclusion about the approximate identity of the specific forces acting between a proton and a neutron, on the one hand, and a proton and a proton, on the other, is based on the application of the rectangular wells." It is relevant to remark that: (*a*) On p. 845 of the paper by BCP the comparison is made for the Gauss Error potential and in a related paper Breit and Stehn have also used the Gauss Error potential. Similarly BTE made comparisons for the Gauss Error potential. (*b*) Comparisons of proton-proton and proton-neutron interactions have been made for the meson well by HSB.

(5) It is also not true as is implied on p. 224 in a sequel paper by Smorodinsky[9] that the importance of the assumption of equality of range of force for drawing conclusions about equality of depth of potentials has been overlooked in the work preceding the paper of Landau and Smorodinsky. Thus on p. 14 of reference 1 in the section on the "Comparison with the Proton-Neutron Potential" it is stated that the range of force for the proton-neutron interaction is not known directly and that "Assuming that in the $^1S_0$ states the proton-proton and proton-neutron interactions have the same range their magnitudes can be compared." The sentence starting on the second line of the next page brings out the dependence of the estimate on "the assumed range of the triplet interaction." It is hard to see how there could be doubt about the conditional and speculative attitude towards the hypothesis of approximate charge independence of non-Coulombian nuclear forces. In Table XIV, p. 842 of BCP the large sensitivity of the depth of the potential to assumed range is clearly seen and the purpose of this section of BCP's paper has been stated in the first column of p. 842: "It appears of interest to see whether it is possible to consider the proton-proton and proton-neutron interactions to be the same in the ^1S states." The strong sensitivity of depth to range was so well known to the majority of nuclear physicists that it appeared absurd to elaborate the point in the Physical Review. It was brought out more explicitly, however, in the University of Pennsylvania paper[1] which is quoted by L and S. It is also necessary to state that the important part of approximate resonance of the potentials to zero energy for the comparison of the two potentials which Landau and Smorodinsky bring out and apparently consider as meeting the situation is precisely the starting point of Section 5, p. 842 of BCP. In later papers of the ill spoken of series the closely related viewpoint of the virtual level has been used.

(6) It is not the desire to produce the impression that the series of papers under discussion is considered to be correct in all respects or to contain the best point of view. The papers were reports on

[9] J. Smorodinsky, J. Phys. Acad. Sci. U.S.S.R. **8**, 219 (1944).

work done with the usual purpose of making it easier for others to further progress with the aid of information supplied. They were not written in the spirit of patent applications and well known or obvious facts were, therefore, left unemphasized. It is in this perhaps that the misunderstanding lies.

The discussion of the linearity of f is carried out in Section V of the present paper and the accuracy of linear and parabolic approximations is examined. It is found that the linear approximation is good to within ~ 0.13 percent of the scattering cross section at a scattering angle of 45° at 600 kev but that the accuracy decreases as the bombarding energy is increased. Equivalence of range of potential wells having different shapes is discussed more systematically than previously in connection with Eqs. (7.46), (7.47). The determination of range and depth of a square well by means of values of f and df/dE at $E=0$ is discussed in connection with Eqs. (8.6), (8.61), (8.63) by means of which the potential well parameters can be determined by a rapidly converging method of successive approximations. The equivalent parameters for scattering at low energies by other than square well potentials can then be found by means of Eqs. (7.46), (7.47). The expansions of Coulomb functions in powers of the energy involving Bessel functions which have been worked out by Yost, Wheeler, and Breit[10] can be used to work out terms of higher powers in E if desired.

In Section IV the adjustment of the range of force is discussed by the simple method of plotting the logarithmic derivative at the boundary of a square well with approximately the correct range and comparing the graph with the computed value for the square well. Formulas are given for making the final adjustment by means of the graph. This method is more directly related to the properties of the potential well than that of the f function. For potential curves of different shapes the equivalent parameters can be obtained by means of Eqs. (7.44), (7.47).

The possibility of describing the interaction by a boundary condition at $r=0$ is considered in Section II. Characteristic differences between the proton-proton and proton-neutron problem are taken up and reasons for the failure of appearance of simple boundary conditions at $r=0$ are brought out in connection with differences in asymptotic form of the irregular solution in the two cases. The considerations of this section do not exclude the possibility of describing the measured phase shifts by a statement concerning the asymptotic form of the solution of the Schroedinger equation in a Coulomb field extrapolated to $r=0$. This can obviously be done by simple continuation of the function. The discussion indicates, however, that a logarithmic infinity masks the features of the wave function which matter for scattering. The logarithmic term in Eq. (2.5) is the trouble maker and as one sees by means of Eq. (2.4) the logarithmic derivate $d\mathfrak{F}/\mathfrak{F}dr$ is directly related to the quantity f_1 rather than to f. The conclusions of Section II would have to be reversed if one could find a reason for requiring a simple behavior of the limit of $d\mathfrak{F}/\mathfrak{F}dr-(2/a)\times\ln(2r/a)$ rather than of $d\mathfrak{F}/\mathfrak{F}dr$. Equation (7.5) of BCP and the more detailed considerations of Section V of the present paper show to be sure that on the assumption of the potential well explanation the above quantity should vary approximately linearly with E. This restatement of conditions is, however, only another way of describing scattering by means of a potential energy curve and does not bring in a new physical picture. The evidence seems to point to the logarithmic derivative at distances of the order of 1.4×10^{-13} cm rather than zero as being the more relevant.

[10] F. L. Yost, J. A. Wheeler, and G. Breit, Phys. Rev. **49**, 174 (1936).

Notation

M = mass of proton.
$\mu = M/2$ = reduced mass in the collision of two protons.
v = relative velocity of the two protons before the collision.
$E=\frac{1}{2}Mv^2$.
E' = energy in frame of center of gravity $=E/2$.
$\Lambda=h/\mu v$ = de Broglie wave-length.
$k=2\pi/\Lambda$.
$a=\hbar^2/\mu e^2$ = length analogous to Bohr radius.
$\eta=1/ka=(e^2/\hbar c)c/v$.
r = distance between protons.
$\rho=kr$.
$L\hbar$ = angular momentum.
F_L, G_L = regular and irregular solutions of differential equation for $r\times$radial function; the signs and normalization are as in YWB, BCP.
K_L = phase shift for angular momentum $L\hbar$ for proton-proton scattering.
$F=F_0$ when the specification of L is not needed.
$G=G_0$ when the specification of L is not needed.
$K=K_0$ when the specification of L is not needed.
$\gamma=0.5772\cdots$ = Euler's constant.
$\Gamma(x)$ = gamma-function of x.
$\mathfrak{F}$ = solution of wave equation for $r\times$radial function normalized in the same convention as F and G with a phase equal to phase of F plus K.
F_i = internal value of $\mathfrak{F}$.
b = value of r for boundary of proton-proton nuclear well; in some cases r is used for b.
$z=[(M/\hbar^2)(D+E/2)]^{\frac{1}{2}}r=0.4371(rmc^2/e^2)\times[D_{\text{Mev}}+(E/2)_{\text{Mev}}]^{\frac{1}{2}}$.
δ_0 = phase shift in ^1S state for proton-neutron scattering.

TABLE I. Contributions to logarithmic derivative at very small distances.

$\ln(2r/a)$	-3	-4	-5	-6	-7	-8
rmc^2/e^2	0.51	0.187	0.069	0.0254	0.0093	0.00343
Linear term of Eq. (2.4)	0.12	0.025	0.0024	−0.0016	−0.0015	−0.0009
$rd\mathfrak{F}/\mathfrak{F}dr$	0.09	0.021	0.002	−0.0017	−0.0015	−0.0009
Correction for kinetic energy	0.012	0.0017	0.00023	0.00003	0.00000	0.00000
Coulomb correction	−0.041	−0.015	−0.0055	−0.0020	−0.0007	−0.0003
$[rdF_i/F_idr]_{E=0,\,\epsilon=0}$	0.06	0.007	−0.003	−0.004	−0.002	−0.001

r_0 = radius of square well for proton-neutron interaction.
$C_0=[2\pi\eta/(-1+\exp 2\pi\eta)]^{\frac{1}{2}}$.
$\Phi_0=F_0/C_0\rho$.
$\Phi_0^*==dF_0/C_0d\rho$.
$\Theta_0=C_0G_0$, $\Theta_0^*=C_0\rho dG_0/d\rho$.
$\Psi_0=\Theta_0-\rho(2\eta\ln 2\rho+q)\Phi_0$.
$\Psi_0^*=\Theta_0^*-\rho(2\eta\ln 2\rho+q)\Phi_0^*-2\rho\eta\Phi_0$.

Where no confusion can arise the six functions just defined will be written as Φ, Φ^*, Θ, Θ^*, Ψ, Ψ^*, respectively.

$Y=rd\mathfrak{F}/\mathfrak{F}dr$.
$y=d\mathfrak{F}/\mathfrak{F}dr$.
$r_e=rmc^2/e^2$.

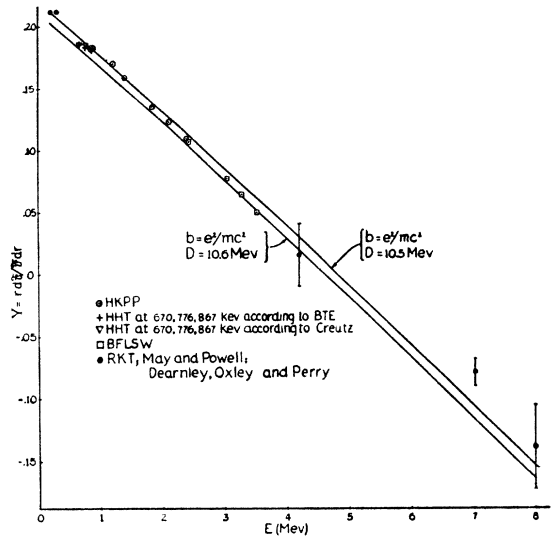

FIG. 1. Homogeneous logarithmic derivative Y at $r=e^2/mc^2$ plotted against energy. Circles, crosses, triangles, and squares correspond, respectively, to the experiments (see reference 4) of HKPP, HHT with values from BTE, HHT with values according to Creutz and BFLSW. The values at energies above 4 Mev and those of RKT are indicated by dots. The same notation is used in Figs. 2, 3, 4. The comparison curves correspond to assumed square well potentials of radius e^2/mc^2 and depths 10.5 and 10.6 Mev. These depths are in the convention of no Coulomb energy for $r<$radius of square well. Treatment of data is discussed in Appendix I.

II. BOUNDARY CONDITIONS AT VERY SMALL DISTANCES

For the analogous neutron-proton scattering problem it has proved useful to deal not only with the phase shift but also with the real and virtual levels in the singlet and triplet states as well as with Fermi's characteristic scattering length. The latter has the significance of an intercept on the axis of r formed by the tangent to the plot of $\mathfrak{F}$ against r. For thermal energy neutrons the characteristic lengths are constant and in this case the scattering can be described by a boundary condition at $r=0$. This boundary condition is simply a statement to the effect that the length $\mathfrak{F}dr/d\mathfrak{F}$ has a certain constant value. There is some evidence that for an energy range of several Mev the intercepts cannot perhaps be assumed to be constant because experimental data can be fitted by potential energy assumptions which imply a variation of the intercepts with energy. The variation in the values of the intercepts with energy is sufficiently mild to make it questionable whether the picture of an interaction potential has more significance than the description of scattering by means of the intercepts. Such a description amounts to the specification of a boundary condition having a mild energy dependence. In the present section the corresponding question is looked into for proton-proton scattering. Specifically the conditions on

$$[Y/r]_{r=0}=[d\mathfrak{F}/\mathfrak{F}dr]_{r=0}.$$

are dealt with.

The value of the logarithmic derivative of the function $\mathfrak{F}$ continued in towards small r from ∞ is conveniently obtainable from Eq. (7.5) of BCP[4] which is in the notation of the present paper.

$$Y=(r/a)\frac{X+[2\ln(2r/a)+f]\Phi^*}{\Psi+(r/a)[2\ln(2r/a)+f]\Phi}, \tag{2}$$

where

$$f=(C_0^2/\eta)\cot K_0+q/\eta-2\ln\eta;\quad C_0^2=2\pi\eta/(e^{2\pi\eta}-1), \tag{2.1}$$

$$X=[\Psi^*+2(r/a)\Phi]/(r/a), \tag{2.2}$$

$$q/2\eta=-1+2\gamma+\text{R.P.}[\Gamma'(i\eta)/\Gamma(i\eta)]. \tag{2.3}$$

In this formula the quantities X, Φ^*, Ψ, Φ are power series in r and in $1/\eta^2$ i.e., in the energy E the first few terms of which are found in Eq. (7.7) of BCP. Expanding the right side of Eq. (2) in powers of (r/a) one obtains

$$Y=(r/a)\{2+f_1+(-4-1/\eta^2-f_1^2)(r/a) +[2+1/\eta^2+(6+1/\eta^2)f_1-f_1^2+f_1^3] \times(r/a)^2+\cdots\}, \quad (2.4)$$

where

$$f_1=f+2\ln(2r/a). \quad (2.5)$$

If r is made to approach zero for a fixed energy, the quantity in the $\{\ \}$ in Eq. (2.4) approaches aymptotically the term $2+f_1$. One may also look at Eq. (2.4) without any reference to an interaction potential. It determines the value of the logarithmic derivative of the continuation of $\mathfrak{F}=F\cos K+G\sin K$ into the region of small r. The continuation is made by means of the radial equation for $\mathfrak{F}$ employing in that equation only the Coulomb energy and no specifically nuclear interaction energy. From this point of view Eq. (2.4) tells one how one should start the radial function $\mathfrak{F}/r$ at small values of r so as to reproduce a preassigned value of the phase shift K_0.

If r goes to zero, Y goes to zero also. On the other hand

$$Y/r\sim(2+f_1)/a \quad (2.6)$$

and this quantity approaches $-\infty$ for small r. If one wishes to account for Y/r by means of an interaction potential between zero and b then one has to use an interaction potential which makes

$$\lim(dF_i/F_i dr)_{r=b,\,b=0}=-\infty, \quad (2.7)$$

where F_i is the wave function in the region between zero and b. This condition as well as Eq. (2.6) imply that the intercept a_1 is such that

$$\lim 1/(b+a_1)=-\infty \quad (2.8)$$

and this means that $b+a_1$ approaches zero from the direction of negative values. A straight line drawn tangentially to the curve representing F_i as a function of r must cut the axis of r at a point for which

$$r=-a_1\sim b+\frac{a}{2|\ln(2b/a)|}. \quad (3)$$

As b approaches zero this point approaches the point $r=b$. This does not mean, however, that $\mathfrak{F}$ has a node at a point approximately located at $r=b$. In fact

$$-a_1/b\sim 1+[(2b/a)\ln(2b/a)]^{-1}. \quad (3.1)$$

The right side of the last equation approaches infinity as b/a approaches zero. With b at a *unit of length* the intercept assumes a position infinitely far to the right of the joining point b. The curve of $\mathfrak{F}$ against r/b when drawn on a scale of b as unit length is thus practically horizontal at the point $r=b$ and is only slightly inclined toward the axis of r/b. This behavior of $\mathfrak{F}$ is readily understood if it is remembered that

$$\mathfrak{F}=F\cos K+G\sin K,$$

and that $G\gg F$ for small r. The dominant term in G is

$$G\sim[1+(2r/a)\ln(2r/a)]/C_0, \quad (3.2)$$

and this means that

$$dG/Gdr\sim(2/a)\ln(2r/a), \quad (3.3)$$

which is just the behavior of $d\mathfrak{F}/\mathfrak{F}dr$ which follows from Eqs. (2.5), (2.6). The nearly horizontal direction of $\mathfrak{F}$ and the slightness of its inclination towards the r axis are obvious from Eq. (3.2). The properties of the boundary condition are thus direct consequences of properties of the irregular solution G.

The logarithmic derivative is seen in Eq. (3.3) to be dominated by a logarithmically infinite negative term. The interaction potential needed to reproduce this value by means of $dF_i/F_i dr$ is such, therefore, that it would correspond in the absence of the Coulombian potential to a real level at an energy E_1 given by

$$E_1\cong-(\hbar^2/M)(4/a^2)[\ln(2b/a)]^2 =-Mc^2(e^2/\hbar c)^2]\ln(2b/a)]^2. \quad (3.4)$$

The coefficient multiplying the square of the logarithm is of the order $mc^2/10.2=50$ kev. The quantity a is approximately 20.5 e^2/mc^2, the value of $\ln(2b/a)$ is roughly -5 when $2b$ is $e^{-2}=1/7.4$ of e^2/mc^2 and the level is then at ~-1.2 Mev. It would be preposterous to attach a literal significance to these numbers but it is true that the level under discussion is reasonably close to zero except for small distances b. For small b a non-relativistic approach is doubtful and the "level" has only a formal significance. This is so also for another reason. The logarithmic derivative has the value $-[M(-E_1)/\hbar^2]^{\frac{1}{2}}$ only if one neglects Coulomb repulsion in the stable state. The quantity referred to in Eq. (3.4) is the value which the energy of the

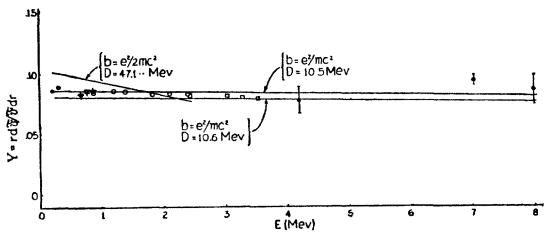

FIG. 2. Values of Y at $r=e^2/2mc^2$ obtained by continuation of Coulomb function with phase shift i.e., by means of $\mathfrak{F}=F\cos K+G\sin K$. The two nearly horizontal lines correspond to phase shifts computed for the square wells used in Fig. 1. The oblique line corresponds to square well with $b=e^2/2mc^2$, $D=47.14$ Mev. The notation of Fig. 1 is used.

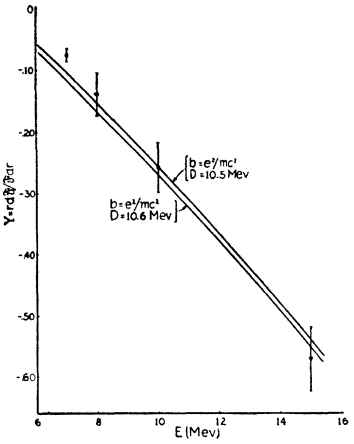

FIG. 3. Values of Y at $r=e^2/mc^2$ plotted against energy. The notation of Fig. 1 is used. This figure is an extension of Fig. 1 to higher energies.

stable level would have if the Coulomb interaction were removed in the region $r>b$. On account of the Coulomb interaction $|y|$ has to be larger than $[M(-E_1)/\hbar^2]^{\frac{1}{2}}$ and the above estimates have no direct relation to the existence of a stable level of two protons in each other's field. It is nevertheless convenient to deal with the concept of a level as employed here because by means of it one can describe by a single quantity some of the properties of the potential well, and because by defining the level in terms of y one can compare the proton-proton interaction directly with corresponding proton-neutron and neutron-neutron interactions. It may be pertinent to recall that in discussions of nuclear binding energies it is customary to characterize the interactions by means of an equivalent potential energy with the convention of considering it as acting in addition to the Coulomb effect and it is convenient for this reason to make the comparison in terms of a characteristic property of the potential well which is not affected by interactions outside the well radius b.

The considerations presented so far as too crude in the following respects:

(a) Only the terms which are dominant in Eq. (2.4) at small distances have been considered.

(b) Corrections for the effect of the kinetic energy of relative motion on y have not been made.

(c) Corrections for the effect of the Coulomb interaction inside the potential well also have to be taken into account.

The corrections mentioned as (b) and (c) will be made here only to the first order. The technique for making them can be found in BCP and in more detail and refinement in BTE.

The effect of these corrections can be put the form

$$[rdF_i/F_idr]_{E=0,\,e=0,\,r=b} = \{Y+0.0489(E'/mc^2)(rmc^2/e^2)^2\xi - 0.0804(rmc^2/e^2)\xi'\}_{r=b}. \quad (4)$$

Here Y is obtainable accurately as the right side of Eq. (2) or as a series expansion in the form of Eq. (2.4). The logarithmic derivative on the left side is the value which follows from the radial wave equation if one starts its solution at $r=0$ subject to usual requirements of regularity and continues the solution to $r=b$. It is supposed that in this process the shape of the potential well has been fixed and that the depth has been adjusted to reproduce the value of Y required by experiment. In this adjustment the kinetic energy of relative motion inside the well and the term e^2/r are not neglected. After the depth of the well has been determined the radial equation is supposed to be solved once more and this time E and e are set equal to zero while the potential well is left unchanged. The function F_i so obtained is used in Eq. (4). If the same potential well acounts for experimental results at all energies then the left side of Eq. (4) is independent of energy. But for an arbitrary shape of the well and an arbitrary range the potential well depth depends on the energy and the left side of Eq. (4) is also in general dependent on the energy.

The first of the two correction terms in Eq. (4) corresponds to the removal of the kinetic energy term in the radial equation. The first three factors in it correspond to a "square well" and to a vanishing slope of F_i at $r=b$. The factor ξ takes into account geometrical differences produced by varying the well shape and also the effect of the slight angle between the direction of the curve representing F as a function of r/b and the horizontal. This correction depends only on

$$F_i^{-2}\int_0^b F_i^2 dr \quad (4.1)$$

and is insensitive to the shape of the well because the function F_i is approximated rather closely by an arc of a sine curve. For small b the value of F_i at $r=b$ is close to the maximum and the effect of deviation from the horizontal direction is slight.

The second correction term in Eq. (4) consists of three factors. The first two factors correspond to the correction under the same conditions as the first three factors of the kinetic energy correction. The last factor ξ' brings in the effects of differences in shape of the potential well and of the deviation of the direction of the plot of F_i against r/b from a horizontal direction. In this case the correction depends on

$$F_i^{-2}\int_0^b (F_i^2/r)dr \quad (4.2)$$

and is again not sensitive to the effects incorporated in ξ'. Both ξ and ξ' are nearly unity.

A numerical example will be helpful in showing

the relative importance of terms contributing to the right side of Eq. (4). For $E=1$ Mev the experimental interpolated value of K_0 is close to 32.4° and $\tan K_0=0.6346$. The other pertinent quantities are $\eta=0.1581$, $C_0^2=0.5842$, $q/\eta=-0.7868$, $f=8.725$. Neglecting terms beyond the one in $(r/a)^3$ of Eq. (2.4) one has values as in Table I. It is seen that the linear term $(r/a)(2+f_1)$ of Eq. (2.4) accounts for the trend of the logarithmic derivative in a qualitatively correct manner. For the smaller r in the table only the linear term and the Coulomb correction matter. At still smaller r the Coulomb correction becomes much smaller than $(r/a)(2+f_1)$ on account of the dominance of the logarithmic term in f_1. Taking into account the difference between the $2+f_1$ and the logarithmic term, one obtains as an improved approximation to the energy value of the quasi-stable level defined in the same way as for Eq. (3.4),

$$-E_1\cong Mc^2(e^2/\hbar c)^2 \times[1+(f/2)+\ln(2r/a)-0.82\xi]^2. \quad (4.3)$$

For the numbers considered in Table I the quantity $1+(f/2)\cong 5.4$ and contributes more to the position of the level than the Coulomb correction. The level lies higher than according to Eq. (3.4) by an amount which increases with $|\ln(2r/a)|$. Experimental values enter only through f and are masked for small r by the logarithmic term. The level moves to $-\infty$ as r approaches zero and the phase shift enters by raising the level by an amount which is infinite but negligible in comparison with the value of $|E_1|$. This differs from the simple conditions for the proton-neutron case. If one is to characterize the function in terms of its behavior at very small distances, then the logarithmic derivative itself would offer a simpler criterion. But either $dF_i/F_i dr$ or $rdF_i/F_i dr$ still contains as a dominant part the term in $\ln r$ which swamps all other effects. The considerations made above appear to indicate that it is improbable that the value of $dF_i/F_i dr$ at $r=0$ can be useful because the reason for its being infinite follows from properties of the irregular Coulomb function at short distances.

III. THE LOGARITHMIC DERIVATIVE AT MODERATE DISTANCE

It is expected from Eqs. (10.3) of BCP that the logarithmic derivative Y of the internal wave function varies linearly with the energy. This fact does not depend on the shape of the potential well as long as the well is "compact" *i.e.*, as long as it does not have a shallow and wide part extending to a large distance. Equations (10.3) of BCP give the rate of change of Y/r with energy in terms of the form factor listed in Eq. (4.1) of the present paper. This quantity is nearly independent of the incident kinetic energy at the boundary of a deep and narrow potential well because of the shape of the function F_i is nearly independent of the incident energy. The slope of the plot of $dF_i/F_i dr$ plotted against E is, therefore, nearly constant and this plot can be represented by a straight line to a good approximation. Two simple circumstances contribute to its goodness. In the first place the incident energy matters in determining the local wave-length by combining itself with the depth of the well under a square root. Secondly, only half of the incident energy counts because half goes into the kinetic energy of the center of mass.

Since the experimentally determined phase shift can be represented approximately by means of a potential well model, one expects that a plot of Y/r against E to be approximately a straight line provided r is made equal to the radius of a "well" that fits experiment. It turns out to be also true that Y/r varies linearly with the energy for other values of r and that the slope of the line can be made nearly zero through a wide range of energies by taking $r\cong e^2/2mc^2$.

The logarithmic derivative cannot be an exactly linear function of the energy through a range of distances r. In fact in terms of

$$y=Y/r, \quad (5)$$

then

$$dy/dr+y^2+(M/\hbar^2)[(E/2)-e^2/r]=0,$$

and if

$$y=a+bE+cE^2+\cdots, \quad (5.1)$$

then

$$\begin{aligned} da/dr+a^2-(M/\hbar^2)e^2/r&=0,\\ db/dr+2ab+M/2\hbar^2&=0,\\ dc/dr+b^2+2ac&=0. \end{aligned} \quad (5.2)$$

The first two of these equations give functions of r for a and b which do not vanish identically. The third equation cannot be satisfied unless c also does not vanish identically. It is thus impossible to have a strictly linear dependence of y on E at different distances. This does not mean however that this condition cannot be a good approximation. In particular, if by a suitable choice of r one makes $b=0$, then for this r a necessary condition for both $c=0$ and $dc/dr=0$ is satisfied.

In Fig. 1 is shown a plot of Y against E in the energy range 0 to 8 Mev for $r=e^2/mc^2$. The two

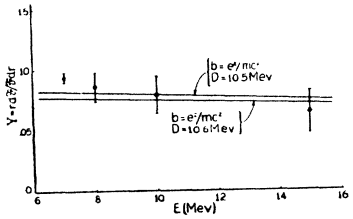

FIG. 4. Values of Y at $r=e^2/2mc^2$ plotted against energy. The notation of Fig. 1 is used. This figure is an extension of Fig. 2 to higher energies.

TABLE II. Values of denominator of Eq. (6.1) for use in finding width of well.

E (Mev)	0.2	0.8	1.4	2.4	3.4	6.0	8.0
z	1.424	1.445	1.464	1.496	1.528	1.606	1.665
(− denominator)	0.41	0.42	0.43	0.45	0.47	0.52	0.56

lines which are nearly straight correspond to the calculated values for square wells with an assumed radius e^2/mc^2 and depths 10.5, 10.6 Mev, respectively, for the upper and lower lines. The experimental values of the phase shifts were used to compute the values of Y indicated by circles, squares, triangles, etc. as explained in the legend. No Coulomb energy was assumed to be acting inside the wells.

In Fig. 2 the values of Y computed for $r=e^2/2mc^2$ are plotted against E. There are in the figure two nearly straight lines between which most of the experimental points lie. These have been obtained by taking $\mathfrak{F}=F\cos K_0+G\sin K_0$ where K_0 corresponds to square wells with radius e^2/mc^2 and depths 10.5, 10.6 Mev, respectively, for the upper and lower lines. The value of Y corresponding to the $\mathfrak{F}$ obtained in the manner described was taken at $r=e^2/2mc^2$. There is besides in Fig. 2 a nearly straight line which cuts obliquely across the other lines. This corresponds to the value of Y at $r=e^2/2mc^2$ which is obtained for a "square well" with radius $e^2/2mc^2$ and depth 47.143 Mev. It is shown for comparison with the approximate fit of the 10.5, 10.6 Mev wells to experiment. The comparison shows the superiority of the larger radius in fitting experiment. The experimental points fall on a practically horizontal straight line. From an empirical point of view one can describe the observations about as well by requiring the logarithmic derivative to be constant at this distance as by dealing with a potential energy curve.

The experimental values of the phase shift which have been derived from the Wisconsin and Minnesota experiments fall approximately on a straight line which in Fig. 2 drops by approximately 0.004 in the value of Y as the energy changes from 0.2 to 3 Mev. The change in r which must be made in order to make the line horizontal will now be estimated. The equation satisfied by $\mathfrak{F}$ is

$$d^2\mathfrak{F}/dr^2+(M/\hbar^2)(E/2-e^2/r)\mathfrak{F}=0.$$

It follows from this equation that

$$rdY/dr=Y-Y^2-(Mr^2/\hbar^2)(E/2-e^2/r), \quad (5.3)$$

where the notation

$$Y=rd\mathfrak{F}/\mathfrak{F}dr \quad (5.4)$$

is used also with the modified meaning of $\mathfrak{F}$. Expressing E in Mev and r in e^2/mc^2 this relation becomes

$$rdY/dr=Y-Y^2-0.0955(r^2E-1.023r). \quad (5.5)$$

It can be verified that the values of dY/dr for the lines marked (10.5, 1) in Figs. 1 and 2 at 0.2 and 3 Mev give to a good approximation the mean rate of change of Y between $r=e^2/2mc^2$ and $r=e^2/mc^2$. This fact indicates that Y varies approximately linearly between these values of r and gives one confidence in Eq. (5.5) as a way of estimating changes in Y caused by changes in r. Interpreting Fig. 2 in the sense that experiment indicates $Y=0.089$ at $E=0.2$ Mev and $Y=0.085$ at $E=3$ Mev for $r=\frac{1}{2}$ one finds by means of Eq. (5.5) that

$$(dY/dr)_{E=0.2}=0.250, \quad (dY/dr)_{E=3}=0.110. \quad (5.6)$$

The line of Y will be made horizontal if the drop of $0.089-0.085=0.004$ is removed by a change from $r=0.5$ to $r=0.5+\delta r$. The value of δr is then such that

$$(0.110-0.250)\delta r=0.004,$$

which gives

$$\delta r=-0.02(9)$$

or

$$r=0.47.$$

The change in Y produced by this change in r is of the order -0.003 at $E=3$ Mev and -0.007 at $E=0.2$ Mev. This is an effect of roughly 10 percent in Y at 0.2 Mev. The slightly oblique line of experimental points can be made horizontal by decreasing the radius ~6 percent and lowering the line by ~10 percent at the left.

Figures 3 and 4 correspond to Figs 1 and 2 for the region 6 to 16 Mev, respectively. The experimental material is less certain in this region. It appears not to contradict either the potential well or the "boundary condition" at $r=0.47\,e^2/mc^2=1.32\times10^{-13}$ cm type of fit. The limits of experimental error indicated in the figures are approximate only.[11]

For proton-neutron scattering a similar boundary condition for the ^1S interaction can be used. This case differs from proton-proton scattering through the absence of the Coulomb field, but it may be expected that there will be no great change because the effect of the Coulomb field is relatively small. If, for example, one assumes that the scattering is represented by a square well of depth D of radius r_0,

[11] The abbreviations HHT, HKPP, etc. of footnote 2 are used in the legends to and in Figs. 1, 2, 3, 4. These refer to the origin of experimental data. The points of BFLSW have been obtained by a somewhat crude analysis by the present authors. Since making the analysis and while the present paper was typed the authors received the results of a more careful study of the Minnesota material by Professor C. L. Critchfield which agrees closely with that used here. The authors should like to acknowledge their indebtedness to Professor Critchfield and to Messrs. Blair, Freier, Lamp, Sleator, and Williams for the communication of their results before publication. An account of the treatment of other experimental data is given in Appendix I.

then an extrapolation of the field free wave function to smaller distances gives the following expansion in E for the homogeneous logarithmic derivate Y^N

$$Y^N(r)=Y_0{}^N(r)+Y_1{}^N(r)E+Y_2{}^N(r)E^2+\cdots. \quad (5.7)$$

Here the coefficients of E are determined by their values at r_0 by the following relations:

$$Y_0{}^N(r)=r/(r+r_0\beta), \qquad \beta=[1-Y_0{}^N(r_0)]/Y_0{}^N(r_0), \quad (5.71)$$

$$Y_1{}^N(r)=-(Ar_0{}^2/3)(1+\beta)+C/[r_0(1+\beta)^2], \quad (5.72)$$

$$Y_2{}^N(r)=-(A^2/45)r(r+r_0\beta)^2 +ACr/3+C^2r/(r+r_0\beta)^3+C_1r(r+r_0\beta)^2, \quad (5.73)$$

where the constant A has the value

$$A=(M/2m)(e^2/\hbar c)^2(1\text{ Mev}/mc^2)=0.0955(3) \quad (5.74)$$

for r and r_0 in e^2/mc^2 and E in Mev. The constants C, C_1 are determined by the known values of $Y_1{}^N(r_0)$ and $Y_2{}^N(r_0)$ for which explicit formulas are listed as Eqs. (8.2), (8.3) later in this paper. It is also possible to compute $Y^N(r)$ directly since

$$Y^N(r)=\rho\cot(\rho+\delta_0), \quad (5.75)$$

where δ_0 is the phase shift for proton-neutron ^1S scattering. This can be determined by means of

$$Y^N(r_0)=z_0\cot z_0, \quad (5.76)$$

where z_0 is as defined in the list of notation. If one assumes that scattering is approximately represented by the square well parameters $D=11.75$ Mev, $r_0=e^2/mc^2$, then one obtains

$$Y^N(e^2/2mc^2)=0.05754-0.00027E -0.00004E^2+\cdots, \quad (5.8)$$

where E is in Mev. The linear term in E can be made to disappear by changing r to $\sim 0.490\ e^2/mc^2$ which gives $Y^N=0.0566$ at $E=0$. The value of r which makes the average slope zero between 1 and 10 Mev is $\sim 0.488\ e^2/mc^2$ which is practically the same as $0.490\ e^2/mc^2$ which makes the slope zero at $E=0$. If one wishes to reproduce the scattering expected for the square well parameter $D=11.33$ Mev, $r_0=e^2/mc^2$, then one obtains

$$Y^N(e^2/2mc^2)=0.07927-0.00038E -0.00004E^2, \quad (5.81)$$

and the linear term in E vanishes in this case for $r=0.492\ e^2/mc^2$. The reason for choosing the values 11.75 Mev, 11.33 Mev in the above examples is that the smaller value corresponds to the depth of the proton-proton well when corrected for Coulomb repulsion inside the well while the larger corresponds to the rather large value of 80×10^{-24} cm^2 for the scattering of thermal neutrons by bound protons. The two values are intended to cover approximately the range of possibilities.

The calculations of coefficients of powers of E for Eqs. (5.8), (5.81) have been checked by comparison with values of $Y^N(r)$ obtained by means of Eq. (5.75). The expansions in powers of E were found to reproduce the directly calculated values with errors not exceeding -0.00002 up to $E=3$ Mev, -0.00006 up to 10 Mev, -0.0015 up to 14 Mev. Powers of E higher than the second are seen not to be important in the energy range considered here. No direct significance is attached to the precise values of the coefficients in Eqs. (5.8), (5.81) or to the values of r for which $Y_1{}^N=0$. They are reproduced here as an illustration of the relative stability of conclusions to adjustments in the interpretation of experimental material. No attempt is being made here to cover the question of variation of the observed proton-neutron scattering cross section with energy since this is concerned also with the triplet state of the two particles. It is only intended to bring out that the contributions to the total cross section which arise from the singlet state can be considered as having closely equal values on the boundary condition and the potential energy points of view. The scattering of neutrons by ortho- and parahydrogen or the coherent scattering by crystals is not considered here because the spin-orbit-spin tensor interaction and the unknown range of the triplet force complicate the situation by introducing additional parameters. As in the case of proton-proton scattering the most that can be claimed for the boundary condition point of view is that it might deserve consideration alongside with the potential energy curve explanation.

It may also be of interest to mention that the constant value of $Y\sim 0.08$ for proton-proton scattering is approximately the same as the absolute value of the correction to Y for internal Coulomb effect for $r=e^2/mc^2$. If one were to make this correction simply as in Eq. (4) one would obtain a much smaller Y which would correspond to $d\mathfrak{F}/dr\cong 0$ at $r=0.47\ e^2/mc^2$. One could hardly justify the inclusion of the whole last term of Eq. (4) for $r=e^2/mc^2$ and the above coincidence of numbers appears to have no direct significance. The fact that the value of Y can be expected to become somewhat smaller when corrected for the internal Coulomb effect appears to be less open to criticism and seems to be connected with the fact that $Y_{E=0}{}^N\sim 0.06$ of the proton-neutron interaction is a smaller number than $Y_{E=0}\sim 0.09$ for protons interacting with protons. On this view there is a possibility of regarding the proton-proton and proton-neutron interactions as basically the same.

In concluding this section it is desired to emphasize the speculative character of the boundary condition, the fact that it is closely related to the velocity dependent potentials of BTE and also that no special significance is attached to exact

constancy rather than slow variability with energy of the logarithmic derivative.

IV. ADJUSTMENTS OF RANGE OF FORCE FOR SQUARE WELLS

The figures described in the preceding section have been prepared without making a correction for the effect of Coulomb energy acting between 0 and e^2/mc^2 on y. The lines marked as 10.5 Mev and 10.6 Mev correspond to interaction potentials which are constant between $r=0$ and $r=e^2/mc^2$. The practical considerations for preferring these lines to other possible lines is the simplicity of calculation for square wells without internal Coulomb energy and the fact that most of the more accurate experimental points fall between them. The estimate of the distance r at which y is energy independent is not affected by whether one takes the internal Coulomb energy into account or not. This estimate makes use of the 10.5 Mev, 10.6 Mev lines only as convenient computational devices. The estimates of range of equivalent potentials depend slightly on whether one assumes an internal Coulomb potential or not. This fact has been recognized in the work of BCP and BTE. Table XVI and Eqs. (11.2), (11.3) of the latter reference make it easy to compute the effect. The internal phase z used in Table XVI of BTE has the values 1.416, 1.482, 1.545, 1.635 at $E=0$, 2, 4, 7 Mev, respectively. It varies nearly linearly with energy. Table XVI of BTE and the numerical example preceding it show that only the first-order correction is important. The second line of the table shows that as z changes from 1.4 to 1.6 the internal Coulomb effect correction to the equivalent depth of the square well increases fractionally in the ratio $1.655/1.613=1.026$. The whole correction is of the order of 0.83 Mev, and the effect of taking into account the internal Coulomb energy is, therefore, to increase the effective depth by $0.83\times0.026=0.021$ Mev for a change of internal phase z of 0.2. The change in z from $E=0$ to $E=4$ Mev is $1.545-1.416=0.129$. The equivalent square well depth with internal Coulomb potential required to reproduce the phase shifts for a square well without Coulomb potential increases, therefore, approximately linearly by $(0.129/0.2)0.021=0.013$ Mev from $E=0$ to $E=4$ Mev. This change is about 1/8 of the difference in equivalent square well depth between the two lines marked (10.5, 1), (10.6, 1) in the figures. The experimental points form a line which appears to cut across the (10.5, 1), (10.6, 1) reference lines between $E=0$ and $E=4$ Mev. If the change in the range of force from e^2/mc^2 is as large as this cross over would call for, then it is about 8 times larger than the change from a nominal range of e^2/mc^2 which results from bringing in the internal Coulomb energy. The same experimental material calls for a slightly shorter range of force with internal Coulomb potential than without. The numerical value of this correction can be given more conveniently after the discussion of changes produced by newer trends in experimental values.

The approximate equivalence between a progressive depth change between two energies and a change in the range of force can be represented by a simple formula which can be derived by means of Eq. (10.3) of BCP. The derivation is straightforward and is omitted. The formula is:

$$\delta b/b=(\Delta D/2D)/\{[zs^2/(z-sc)]_2-[zs^2/(z-sc)]_1\}, \quad (6)$$

where

$$s=\sin z, \quad c=\cos z.$$

Here it is supposed that for an assumed value of the range b experiment requires a potential depth D at $E=E_1$ and a depth $D+\Delta D$ at $E=E_2$. The range is supposed to be changed to $b+\delta b$ and the depth to $D+\delta D$ in such a way that the same range and depth account for experiment at both E_1 and E_2. All changes are supposed small enough to justify first-order calculations. The change in depth δD is eliminated and Eq. (6) results. The subscripts 1, 2 in this equation indicate that quantities are evaluated for $E=E_1$, E_2, respectively. The denominator consists of a difference of two such quantities and can be approximated by the result of differentiating the quantity in brackets with respect to E. It is found in this way that

$$\frac{\delta b}{b}\cong\frac{(\Delta D/D)[D+(E/2)]/(E_2-E_1)}{[zs^2/(z-sc)][\frac{1}{2}+zc/s-zs^2/(z-sc)]}, \quad (6.1)$$

where z, s, c are to be evaluated at a suitably chosen mean energy. The denominator in the above Eq. (6.1) has the values listed in Table II as a function of energy. The table gives also the corresponding values of z. If the experimental curve is intrepreted as crossing over from the (10.5, 1) line at $E=1.4$ Mev to the (10.6, 1) line at $E=3.4$ Mev then Eq. (6.1) and Table II give

$$\delta b/b=(1/10.5)[(10.5+1.2)/2]/(-0.45)=-0.12$$

indicating $b\cong0.88e^2/mc^2=2.47\times10^{-13}$ cm. If the cross over takes place from $E=0.2$ Mev to $E=3.4$ Mev a similar estimate gives $\delta b/b=-0.07(7)$ indicating $b=2.59\times10^{-13}$ cm. If the cross over takes place from $E=0.2$ Mev to $E=8$ Mev then $\delta b/b=-0.032$ and $b=2.72\times10^{-13}$ cm. It is difficult to decide which of these values is the better.

The discussion of the effect of the thermal Coulomb potential indicated that it is approximately equivalent to an increase in well depth by 0.013 Mev from $E=0$ to $E=4$ Mev. Equation (6.1) and Table II indicate a corresponding $\delta b/b=-0.008$ which corresponds to an additional 0.8 percent

shortening of the range. This effect is smaller than the apparent uncertainties of available experimental material.

Changes in equivalent square well range obtained above can be translated into changes of the range parameter of other potential energy wells by means of Eq. (7.47) in a later section of this paper. It should be pointed out, however, that according to the work of HSB and of BBH the equivalence between potential wells is only approximate and that the equivalence in the sense of Eq. (7.47) has the meaning of giving the same results for scattering under a somewhat idealized condition of zero scattering energy. The adjustment of range outlined above is independent of questions of degree of linearity of the logarithmic derivative with energy and can be made as an over-all adjustment between any two energies.

V. EXPANSIONS IN TERMS OF ENERGY; NATURE OF APPROXIMATIONS

Solving Eq. (2) for f one has

$$f+2\ln(2r/a)=\frac{X-(a/r)Y\Psi}{Y\Phi-\Phi^*}, \tag{7}$$

where the quantities X, Φ, Φ^*, Ψ, f are as defined in Eqs. (2.1), (2.2), (2.3) and corresponding formulas of BCP. The quantity Y is the homogeneous logarithmic derivative of Eq. (5.4). Whether the potential energy is assumed to be a square well or some other potential curve which does not extend to large distances, Y varies nearly linearly with energy as discussed in Section III with reference to Eq. (10.3) of BCP. The quantities[12] X, Ψ, Φ, Φ^* also vary approximately linearly with E. One expects,[13] therefore, that the left side of Eq. (7) will be an approximately linear function of E.

If the potential acting inside the well is assumed to have the form

$$V=-A\mathfrak{v}(r/\beta), \tag{7.1}$$

where $\mathfrak{v}$ is an otherwise unspecified but regular function one has the depth parameter A and the range parameter β available for adjustment to observation. For an assumed β one can fit experiment at one energy say $E=E_1$ by adjusting the depth parameter A. The pair of values A, β will not fit in general the experimental values at another energy $E=E_2$. Whether it does or not depends only on how Y varies with E because V matters only through Y. If all quantities are approximated by linear functions of E then $\partial Y/\partial E$ depends only on the expression in Eq. (4.1). This expression will be called the range-energy form factor. The range parameter β enters the form factor implicitly through the internal function F_i. For different forms of the shape function $\mathfrak{v}$ the form factors can be made to have the same value by suitably adjusting β. Values of β obtained in this manner can be said to be equivalent for range adjustment. In making the comparison of form factors for potential energy curves having different shapes the larger of the two nuclear well radii b has to be used so as to be able to have X, Ψ, Φ, Φ^* the same. It is clear that it is possible to introduce an equivalent square well range for a potential of any shape and that the equality of the range energy form factor for equivalent ranges is a convenient criterion. Potentials with different shapes can have the same equivalent range at one energy but different equivalent ranges at other energies on account of the presence of terms in E^2 and E^3 in Y. This situation will now be described quantitatively.

Expansions convenient for Φ, Φ^* can be obtained through the work of YWB.[10] A power series for the irregular function has been set up in that paper also. This has been more firmly established in connection with the present work and its accuracy has been tested by comparison with other methods of calculation. The series does not show rapid convergence for the irregular Coulomb function for energies of 10 Mev or higher. Nevertheless it gives a convenient representation of X and Ψ with satisfactory accuracy for $r=e^2/mc^2$ without terms in E of higher degree than the second. In the notation of YWB the expansions and other relevant relations are:

$$x=(8\rho\eta)^{\frac{1}{2}}=(8r/a)^{\frac{1}{2}};\qquad \rho=kr=Mv/2\hbar;\quad \eta=e^2/\hbar v', \tag{7.2}$$

$$v=\text{relative velocity}, \tag{7.21}$$

$$F_0=C_0\rho\Phi_0, \tag{7.22}$$

$$G_0=\Theta_0/C_0, \tag{7.23}$$

$$\Phi_0=\varphi_0+\varphi_1/\eta^2+\varphi_2/\eta^4+\cdots, \tag{7.24}$$

$$\Phi_0^*=\varphi_0^*+\varphi_1^*/\eta^2+\varphi_2^*/\eta^4+\cdots, \tag{7.25}$$

$$-2\Theta_0/x^2=\theta_0+\theta_1/\eta^2+\theta_2/\eta^4+\cdots, \tag{7.26}$$

$$-\partial\Theta_0/x\partial x=\theta_0^*+\theta_1^*/\eta^2+\theta_2^*/\eta^4+\cdots, \tag{7.27}$$

$J_\nu(ix)=$ Bessel function of order ν of argument ix.

$$I_\nu=I_\nu(x)=i^{-\nu}J_\nu(ix), \tag{7.28}$$

$$\mathrm{K}_\nu=\mathrm{K}_\nu(x)=\lim(\pi/2)[I_{-\nu}(x)-I_\nu(x)]\cot(\pi\nu)$$
$$=\lim\cos(\pi\nu)[\text{Basset's } K_\nu]. \tag{7.29}$$

For integral ν the quantity of which the limit is

[12] BCP, footnote 2. Additional terms of the series are given in Eqs. (7.5) to (7.94) of the present paper.

[13] This is an obvious consequence of Eqs. (7.5), (7.6), (7.7) of BCP and has been known since that paper. The reason for introducing the quantity X and f was that of isolating the exponential dependence of η contained in C_0^2 from the quantities varying slowly with energy such as X, Ψ, Φ, Φ^*. The same step has been made by Landau and Smorodinsky (see reference 8) afterwards in a less complete form.

TABLE III. Values of $\Phi_0^* - Y\Phi_0$ and its approximation by quadratic in energy.

E (Mev)	Exact value	Quadratic approximation
1	0.8684	0.8685
2.8	0.8683	0.8683
6	0.8689	0.8692
9	0.8710	0.8719

taken becomes indeterminate. The notation is the same as that of Whittaker and Watson, *Modern Analysis*. Watson's Bessel functions denotes by K_ν a function which for integral ν is $(-)^\nu$ times the K_ν used here and by YWB. This choice of K_ν makes the recurrence relations for the K_ν have the same form as for the I_ν.

$$\varphi_0=(2/x)I_1, \quad \varphi_1=(x/2)^3(I_3-I_1)/24; \tag{7.3}$$

$$\varphi_2=[(x/2)^5+(x/2)^3]I_3/240-(x/2)^5I_1/1440. \tag{7.31}$$

$$\varphi_0^*\equiv I_0, \quad \varphi_1^*=(x/2)^3(I_3-3I_1)/24. \tag{7.32}$$

$$\varphi_2^*=\{(5/2)(x/2)^7(I_1-I_3)+[9(x/2)^5+6(x/2)^3]I_3-(x/2)^5I_1\}/1440. \tag{7.33}$$

If

$$\varphi_j=\textstyle\sum_{i,\nu} a_{ji}{}^\nu x^i I_\nu, \quad \varphi_j^*=\sum_{i,\nu} b_{ji}{}^\nu x^i I_\nu, \tag{7.34}$$

then

$$\theta_j=\textstyle\sum_{i,\nu} a_{ji}{}^\nu x^i \mathrm{K}_\nu, \quad \theta_j^*=\sum_{i,\nu} b_{ji}{}^\nu x^i \mathrm{K}_\nu, \tag{7.35}$$

Replacement of letter I by K in formulas changes φ_j into θ_j, φ_j^* into θ_j^*. The remaining relations needed here are:

$$\Psi=-(x^2/2)\{\theta_0+\theta_1/\eta^2+\theta_2/\eta^4+\cdots+[\ln(x/2)+\tfrac{1}{2}S(\eta)]\times(\varphi_0+\varphi_1/\eta^2+\varphi_2/\eta^4+\cdots)\}, \tag{7.36}$$

$$S(\eta)=-1+2\gamma+1/12\eta^2+1/120\eta^4+\cdots=-1+2\gamma-\ln\eta+\mathrm{R.P.}\,\Gamma'(i\eta)/\Gamma(i\eta), \tag{7.37}$$

where $\gamma=0.5772\cdots=$Euler constant.

$$X=-4\{\theta_0^*+\theta_1^*/\eta^2+\theta_2^*/\eta^4+\cdots+[\ln(x/2)+\tfrac{1}{2}S(\eta)]\times(\varphi_0^*+\varphi_1^*/\eta^2+\varphi_2^*/\eta^4+\cdots)\}. \tag{7.38}$$

These expansions can be used, of course, not only for the proton-proton scattering problem. For the latter one has the special relation

$$1/\eta^2=40.00(5)E/\mathrm{Mev}. \tag{7.39}$$

The expansions in terms of the I_ν, K_ν will be discussed in more detail in another publication. Detailed understanding of reasons for similarity of form of different expressions is not essential for the present discussion. It suffices to know that X, Ψ, Φ, Φ^* are power series both in the energy and in the radius. The expansions in Bessel functions simply give a way of writing down the coefficient of any power of r which occurs with the first or the second power of the energy. Otherwise Eqs. (7.2) . . . (7.38) do not contain more information than Eqs. (7.7) of BCP. In the latter the expansion is arranged in powers of r/a with coefficients which are polynomials in the quantity $1/\eta^2$ which is proportional to the energy E. In the present paper the same series are rearranged in powers of $1/\eta^2$ with coefficients which are functions of the quantity x and are power series in r/a as a consequence of the definition of x given in Eq. (7.2).

In order to use Eq. (7) one has to know the value of Y for different energies. This quantity depends on the shape of the potential well, i.e., on the nature of the function $\mathfrak{v}$ of Eq. (7.1). For small energy changes and for compact potential wells, however, one can speak of the equivalence of potential wells of different shapes. It follows, for instance, from the work of BCP, BTE, and HSB that the square well, the Gauss error well, the exponential well, and the meson well give very similar variations of the phase shift K_0 with E. The reason can be explained in terms of the three relations listed under Eq. (10.3) in BCP where the usefulness of the relations for the present purpose has been indicated but not elaborated. By means of these relations the quantity Y can be expressed by a first-order perturbation calculation making use of the wave function for a condition in which the depth A and range β have been adjusted to give $Y=0$, for $E=0$ at the boundary and, therefore, also outside the potential well. Let it be supposed that there are two potentials to be compared with each other and that they are represented by Eq. (7.1) and by

$$V'=-A'\mathfrak{v}'(r/\beta'). \tag{7.4}$$

Here $\mathfrak{v}'$ is not the derivative of $\mathfrak{v}$ but a different function. Thus, for example, $\mathfrak{v}$ can be the function appropriate to the square well while $\mathfrak{v}'$ can apply to the Gauss error potential. The values of A and A' which correspond to the resonance depth for zero external kinetic energy will be denoted by A_0 and A_0'. They are connected with the range parameters β, β' by the relations

$$A_0\beta_0{}^2=\lambda_0\hbar^2/M, \quad A_0'\beta_0'{}^2=\lambda_0'\hbar^2/M, \tag{7.41}$$

where λ_0 is defined as the eigenvalue of the problem

$$d^2\mathfrak{F}/d\xi^2+\lambda_0\mathfrak{v}(\xi)\mathfrak{F}=0, \quad \xi=r/\beta \tag{7.42}$$

subject to usual conditions of regularity at $r=0$ and of $dF/d\xi=0$ at $\xi=\infty$. The definition of λ_0' is similar to that of λ_0 with a change from $\mathfrak{v}_0$ to $\mathfrak{v}_0'$. In comparing two potentials in their effects on

scattering, it will be convenient to choose a distance $r=R$, such that both potentials are negligible for $r>R$. One has then from the relations in BCP, on expanding Y in powers of E and $A-A_0$ and keeping only the first-order terms, the following conditions for the equivalence of the potentials:

$$[(A-A_0)/\mathfrak{F}^2]\int_0^\infty \mathfrak{F}^2\mathfrak{v}(r/\beta)dr$$

$$=[(A'-A_0')/\mathfrak{F}'^2]\int^\infty \mathfrak{F}'^2\mathfrak{v}'(r/\beta')dr, \quad (7.43)$$

$$(1/\mathfrak{F}^2)\int_0^R \mathfrak{F}^2dr=(1/\mathfrak{F}'^2)\int_0^R \mathfrak{F}'^2dr. \quad (7.44)$$

The first of these relations makes the two potentials give the same value of Y for $E=0$. The second makes the variation of Y with E the same at the two energies. In Eqs. (7.43), (7.44) the functions $\mathfrak{F}$, $\mathfrak{F}'$ are, respectively, the solutions of Eq. (7.42) and the equation obtained from the latter by changing $\mathfrak{v}$ into $\mathfrak{v}'$ and β into β'. In other words $\mathfrak{F}$, $\mathfrak{F}'$ are solutions of the radial equation for the reference states of zero energy having zero slope at $r=\infty$. The second of the two equivalence relations just considered does not involve the depth of the wells and gives, therefore, a convenient criterion for the equivalence of range. According to this Eq. (7.44) if one were to normalize the functions in such a way as to have the same amplitude in the region

$$R<r<\infty, \quad (7.45)$$

then

$$\int_0^R \mathfrak{F}^2dr=\int_0^R \mathfrak{F}'^2dr. \quad (7.46)$$

This condition means qualitatively that two potential wells have equivalent ranges for the resonant states at $E=0$ if their respective wave functions come up to their asymptotic values equally rapidly. It is perhaps useful to state the condition in another form *viz.*

$$\int_0^\infty [-\mathfrak{F}^2+\mathfrak{F}^2(\infty)]dr=\int_0^\infty [-\mathfrak{F}'^2+\mathfrak{F}'^2(\infty)]dr,$$
$$\mathfrak{F}^2(\infty)=\mathfrak{F}'^2(\infty), \quad (7.47)$$

which expresses equivalence of range in terms of equality of areas between the curves for $\mathfrak{F}^2$, $\mathfrak{F}'^2$ and their constant asymptotes at $r=\infty$.

While the existence of relations given here as Eqs. (7.46), (7.47) was realized at the time of writing the paper referred to here as BCP, it was also realized that there are some limitations as well as practical disadvantages to this method of comparing ranges. The comparison was, therefore, usually stated in terms of direct calculation. The reasons were as follows:

(a) Range adjustment by means Eqs. (7.46), (7.47), and (7.48) is subject to errors caused by neglect of quadratic terms in E and in $A-A_0$. These increase with energy and it is somewhat better, therefore, to make the fit by direct calculation at energies equal or close to those for which measurements are available. (b) In order to make a comparison of ranges by any method one needs reasonably exact wave function for energy. There is little difference in the work required to obtain a solution of Eq. (7.42) or at an energy which is not zero. Also the amount of work per numerical integration of a differential equation is smaller if a number of them are performed for neighboring values of parameters. (c) Potentials such as the Exponential and the Meson are not definitely compact. (d) At the time of the work of BCP the effective range of nuclear force was not known. It was necessary, therefore, to include calculations for longer ranges of force such as $2e^2/mc^2$ for which the linearity of Y as a function of E is poorer than for the shorter ranges.

TABLE IV. Effect on scattering at 45° and phase shift of quadratic terms in E.

E (Mev)	δf	$-\delta K_0$ (degrees)	—change in scattering (percent)	Power series value for preceding column
0.2	0.00023	0.00024	−0.003	
0.6	0.0020	0.0069	+0.13	
0.8	0.0035	0.014	0.16	
1.0	0.0056	0.025	0.21	0.21
2.2	0.0265	0.12	0.50	
3	0.049	0.21	0.72	0.79
6	0.19	0.56	1.6	1.8
9	0.42	0.96	2.7	
10	0.49	1.04	2.9	3.1

The expansions of Φ_0, Φ_0^*, Ψ, X in terms of Bessel functions of imaginary argument which are described by Eqs. (7.25)–(7.39) can be arranged as power series in x or in r. They appear then as follows:

$$\varphi_0=\sum_0^\infty \frac{(x/2)^{2s}}{s!(s+1)!}=\sum_0^\infty \frac{(2r/a)^s}{s!(s+1)!}, \quad (7.5)$$

$$\varphi_1=\frac{(x/2)^4}{24}\left[-1-2\sum_0^\infty \frac{(x/2)^{2(s+1)}}{(s+1)!(s+3!)}\right]$$

$$=\frac{1}{6}\left(\frac{r}{a}\right)^2\left[-1-2\sum_0^\infty \frac{(2r/a)^{s+1}}{(s+1)!(s+3)!}\right]$$

$$=-\frac{1}{6}\left(\frac{r}{a}\right)^2\left[1+\frac{2}{3}\frac{r}{a}+\frac{1}{6}\left(\frac{r}{a}\right)^2+\frac{1}{45}\left(\frac{r}{a}\right)^3+\cdots\right].$$

$$\varphi_2=\frac{(x/2)^3}{1440}\sum_0^\infty\frac{(5s+18)(x/2)^{2s}}{s!(s+4)!}$$
$$=\frac{1}{90}\left(\frac{r}{a}\right)^4\sum_0^\infty\frac{(5s+18)}{s!(s+4)!}\left(\frac{2r}{a}\right)^s$$
$$=\left(\frac{r}{a}\right)^4\left(\frac{1}{120}+\frac{23}{5400}\frac{r}{a}+\cdots\right). \quad (7.52)$$

$$\varphi_3=\left(\frac{r}{a}\right)^6\left(-\frac{1}{5040}+\cdots\right). \quad (7.52')$$

$$\varphi_0{}^*=\sum_0^\infty\frac{(2r/a)^s}{(s!)^2}. \quad (7.53)$$

$$\varphi_1{}^*=-\left(\frac{r}{a}\right)^2\left[\frac{1}{2}+\frac{1}{3}\sum_0^\infty\frac{(s+4)(2r/a)^{s+1}}{(s+1)!(s+3)!}\right]$$
$$=-\left(\frac{r}{a}\right)^2\left[\frac{1}{2}+\frac{4}{9}\left(\frac{r}{a}\right)+\frac{5}{36}\left(\frac{r}{a}\right)^2\right.$$
$$\left.+\frac{1}{45}\left(\frac{r}{a}\right)^3+\cdots\right]. \quad (7.54)$$

$$\varphi_2{}^*=\frac{1}{90}\left(\frac{r}{a}\right)^4\sum_0^\infty\frac{(s+5)(5s+18)}{s!(s+4)!}\left(\frac{2r}{a}\right)^s$$
$$=\left(\frac{r}{a}\right)^4\left[\frac{1}{24}+\frac{23}{900}\frac{r}{a}+\cdots\right]. \quad (7.55)$$

$$\varphi_3{}^*=\left(\frac{r}{a}\right)^6\left(-\frac{1}{720}+\cdots\right). \quad (7.55')$$

It is useful to remember that in order to obtain the coefficient of r^s in a $\varphi_j{}^*$ one has to multiply the coefficient of r^s in φ_j by $s+1$. While the occurrence of the K_ν in $\theta_j, \theta_j{}^*$ is simply expressed by means of Eqs. (7.34), (7.35) in terms of the corresponding relations involving I_ν, φ_j, $\varphi_j{}^*$, an arithmetical calculation of Ψ or X takes place through Eqs. (7.36), (7.38). In these the functions K_ν occur in the combinations

$$\mathfrak{K}=K_\nu+[\ln(x/2)+\gamma-(1/2)]I_\nu. \quad (7.6)$$

These functions are the following power series in x:

$$\mathfrak{K}_1=-\frac{1}{x}+\frac{1}{2}\sum_0^\infty\frac{(x/2)^{2s+1}}{s!(s+1)!}$$
$$\times\left[-1+\left(1+\frac{1}{2}+\cdots+\frac{1}{s}\right)\right.$$
$$\left.+\left(1+\frac{1}{2}+\cdots+\frac{1}{s+1}\right)\right]; \quad (7.7)$$

$$\mathfrak{K}_3=-\frac{1}{2}\left[2\left(\frac{2}{x}\right)^3-\frac{2}{x}+\frac{1}{2}\frac{x}{2}\right]$$
$$+\frac{1}{2}\sum_0^\infty\left[-1+\left(1+\frac{1}{2}+\cdots+\frac{1}{s}\right)\right.$$
$$\left.+\left(1+\frac{1}{2}+\cdots+\frac{1}{s+3}\right)\right]\frac{(x/2)^{2s+3}}{s!(s+3)!}. \quad (7.8)$$

It will be noticed that these functions do not contain the $\ln x$ term and that the dependence of Ψ and X on $\ln x$ which might be surmised from Eqs. (7.36), (7.38) is only apparent. It is useful to have on hand the formula

$$\Psi=-x\mathfrak{K}_1-\frac{1}{12\eta^2}\left[\left(\frac{x}{2}\right)^5(\mathfrak{K}_3-\mathfrak{K}_1)+\left(\frac{x}{2}\right)\varphi_0\right]$$
$$-\frac{(x/2)^2}{\eta^4}\left[\frac{(x/2)^5+(x/2)^3}{120}\mathfrak{K}_3\right.$$
$$\left.-\frac{(x/2)^5}{720}\mathfrak{K}_1+\frac{\varphi_0}{120}+\frac{\varphi_1}{12}\right]+\cdots \quad (7.9)$$

which gives the coefficient of any power of x in the linear and quadratic terms in the energy E. From Ψ one readily finds Ψ^* by means of Eqs. (19), (25) of YWB. These equations are needed here only to show that the series for Ψ^* when arranged in powers of r has coefficients of r^j which are equal to j times the coefficient of r^j in Ψ. By means of Eq. (2.2) of the present paper and the series for Φ one has available terms of any power in r in the coefficients of $1/\eta^2$ and $1/\eta^4$ in the quantity X. The recurrence formulae for the coefficients of which are dealt with by YWB are not needed if use is made of the formulae just described. The accuracy required for proton-proton scattering problems is sufficiently moderate to make it possible to neglect terms in E^3 in most of the applications in the present paper. The following series often suffice:

$$X=2-4(r/a)-4(r/a)^2$$
$$-(32/27)(r/a)^3-(19/108)(r/a)^4$$
$$-(107/6750)(r/a)^5-\cdots$$
$$+\frac{(r/a)}{\eta^2}\left[-1+\frac{37}{27}\left(\frac{r}{a}\right)^2+\frac{71}{108}\left(\frac{r}{a}\right)^3\right.$$
$$\left.+\frac{353}{2700}\left(\frac{r}{a}\right)^4+\cdots\right]$$
$$+\frac{(r/a)^3}{\eta^4}\left[\frac{1}{6}-\frac{r/a}{36}-\frac{1261}{13500}\left(\frac{r}{a}\right)^2+\cdots\right]$$
$$+\frac{(r/a)^5}{\eta^6}\left[-\frac{1}{120}+\cdots\right]+\cdots. \quad (7.91)$$

$$X_{E=0}=-4\mathfrak{R}_0(x)=-4\Big[-\frac{1}{2}+\frac{2r/a}{(1!)^2}(1-\tfrac{1}{2})+\frac{(2r/a)^2}{(2!)^2}(1+\tfrac{1}{2}-\tfrac{1}{2})+\frac{(2r/a)^3}{(3!)^2}\times(1+\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{2})+\cdots\Big]. \quad (7.92)$$

$$\Psi=1-3(r/a)^2-(14/9)(r/a)^3-(35/108)(r/a)^4-(101/2700)(r/a)^5-\cdots+\frac{1}{\eta^2}\Big[-\frac{(r/a)^2}{2}+\frac{(r/a)^3}{9}+\frac{43}{108}(r/a)^4+\frac{77}{540}(r/a)^5+\cdots\Big]+\frac{1}{\eta^4}\Big[\frac{(r/a)^4}{24}-\frac{2(r/a)^5}{225}-\frac{172(r/a)^6}{10125}+\cdots\Big]+\frac{1}{\eta^6}\Big[-\frac{(r/a)^6}{720}+\cdots\Big]+\cdots. \quad (7.93)$$

$$\Psi_{E=0}=-x\mathfrak{R}_1(x)=1+\frac{(2r/a)^2}{1!2!}[1-1-(1+\tfrac{1}{2})]+\frac{(2r/a)^3}{2!3!}[1-(1+\tfrac{1}{2})-(1+\tfrac{1}{2}+\tfrac{1}{3})]+\cdots. \quad (7.94)$$

The relative importance of successive powers of E will now be illustrated. For a square well potential the expansion of Y as a power series in the energy is

$$Y=Y_0+Y_1E+Y_2E^2+\cdots, \quad (8)$$

where

$$Y_0=z_0\cot z_0, \quad z_0=z_{E=0}, \quad (8.1)$$

$$Y_1=(1/4D)[Y_0-(z_0/\sin z_0)^2], \quad (8.2)$$

$$Y_2=(1/32D^2)\{2[(z_0/\sin z_0)^2-1]Y_0-(z_0/\sin z_0)^2\}. \quad (8.3)$$

Experiment is fitted reasonably well by $D=10.5$ Mev, $r=b=e^2/mc^2$. These values give

$$Y=0.2205-0.04367E-0.0003881E^2+\cdots, \quad (8.4)$$

where E is in Mev.

The conversion factors in this illustration are as follows:

$$\rho\eta=r/a=0.04886(rmc^2/e^2), \quad \eta=0.1581/E_{\text{Mev}}{}^{\frac{1}{2}}.$$

The numbers in Eq. (8.4) and in the following Eqs. (8.41), (8.42) are carried out to a larger number of digits than is warranted by the accuracy with which the conversion factors are known. The object in listing the extra digits is that of showing how the terms combine and how some of them nearly cancel. This will enable the reader to form his own judgment about the applicability of some of the approximations.

The Coulomb function expansions yield

$$\begin{aligned}\Phi_0&=1.049662-0.016443E+0.00007793E^2+\cdots,\\ \Phi_0{}^*&=1.100133-0.049860E+0.00039160E^2+\cdots,\\ \Psi&=0.992655-0.047144E+0.00037574E^2+\cdots,\\ X&=1.794872-1.94820E+0.030810E^2+\cdots.\end{aligned} \quad (8.41)$$

Calculation of f_1 by means of Eq. (7) gives

$$f_1=3.090\frac{1+0.3160E+0.00191E^2+\cdots}{1-0.00045E+0.0000733E^2+\cdots}=3.090[1+0.3164E+0.00198E^2+\cdots]. \quad (8.42)$$

Here E is again in Mev. It will be noted that:

(a) Most of the dependence on E is contained in the numerator of Eq. (8.42). It arises as a result of the dependence on E of the quantity $X-(a/r)Y\Psi$ of Eq. (7). The denominator of Eq. (7) is on the other hand nearly constant being equal in the present case to

$$-0.8687+0.00039E-0.0000637E^2+\cdots.$$

Comparing this expression with $\Phi_0{}^*$ in the form of Eq. (8.41) it is clear that $\Phi_0{}^*-Y\Phi_0$ depends on the energy relatively much less than $\Phi_0{}^*$.

(b) The energy dependence of $\Phi_0{}^*$ is appreciably reduced by the subtraction of $Y\Phi_0$ not only in the linear but also in the quadratic terms in E.

(c) The cancellation of terms is more pronounced for $\Phi_0{}^*-Y\Phi_0$ than for $\Phi_0{}^*/\Phi_0-Y$. The latter quantity can be expected to be approximately energy independent because Φ^*/Φ is the homogeneous logarithmic derivative of the regular Coulomb function while Y is the homogeneous logarithmic derivative of the internal function and because neither of these varies with energy especially rapidly and also because the two quantities vary in the same direction. The cancellation under discussion is a more precise one, however.

In view of the fact that this cancellation makes it possible to approximate the relation of the phase shift to the energy in a simple way it appears appropriate to explain why it occurs. In terms of the regular Coulomb function F_0 denoted here by F and the energy-dependent but distance-independent parameter C_0 one has

$$\Phi_0{}^*-\Phi_0Y=[dF/d\rho-Fd\mathfrak{F}/\mathfrak{F}d\rho]/C_0=(\sin K_0)/\mathfrak{F}C_0, \quad (8.5)$$

TABLE V. Values of quantities for range determination by means of Eqs. (8.6), (8.61).

r_e	$K_0(x)/I_0(x)$	$I_0(x)$	$(2/x)I_1(x)$
0.75	0.79 95	1.07 46	1.03 71
0.80	0.77 17	1.07 97	1.03 96
0.85	0.74 58	1.08 48	1.04 21
0.90	0.72 17	1.08 99	1.04 46
0.95	0.69 90	1.09 50	1.04 71
1.00	0.67 77	1.10 01	1.04 97
1.05	0.65 77	1.10 53	1.05 22
1.10	0.63 87	1.11 04	1.05 47

use having been made in the last step of the constancy of the Wronskian and of the asymptotic forms of $\mathfrak{F}$, F for the last step. It follows, on the other hand from the differential equations satisfied by F, $\mathfrak{F}$ that

$$\sin K_0 = -(M/k^2\hbar^2)\int_0^b VF\mathfrak{F}d\rho, \qquad (8.51)$$

as has been pointed out by Ramsey.[14] Combining Eqs. (8.5), (8.51) one has

$$\Phi_0^* - \Phi_0 Y = -(M/\hbar^2\mathfrak{F}_b)\int_0^b V(r\Phi_0)\mathfrak{F}dr. \qquad (8.52)$$

Here V is the potential energy in excess of the Coulombian and $\mathfrak{F}_b$ is the value of $\mathfrak{F}$ at the nuclear boundary. The energy dependent quantities on the right side are seen to be $r\Phi_0$ and $\mathfrak{F}/\mathfrak{F}_b$. The energy dependent part of $r\Phi_0$ is to a first approximation the same as though there were no Coulomb interaction as may be seen from the expansions given above. For example, the approximation

$$\Phi_0 \cong \sin\rho/\rho = 1 - 0.0159E(r/mc^2/e^2)^2 + \cdots$$

agrees with the first two terms for Φ_0 in the first of the four Eqs. (8.41) at $rmc^2/e^2 = 1$ which give $1.04966[1 - 0.0157E + \cdots]$ for the same quantity. The good agreement of the variation of Φ_0 with energy and distance in the Coulombian and field free cases suggests that the high degree of energy independence of the quantity $\Phi_0^* - \Phi_0 Y$ can be explained making use of the field free approximation. This can be done by means of Eq. (8.52) the right side of which contains the energy only in the quantity Φ_0 and the ratio $\mathfrak{F}/\mathfrak{F}_b$. The former has been discussed. The latter has the same energy dependence for a compact shape of V as in the absence of a Coulomb field even to a higher degree than $r\Phi_0$ because the effective wave-length for $\mathfrak{F}$ is largely determined by V rather than e^2/r. Equation (8.52) shows, therefore, that the variability of $\Phi_0^* - \Phi_0 Y$ with energy can be discussed to a good approximation by calculating this quantity in the absence of the Coulomb field. One finds by a straightforward calculation for this case that

$$\Phi_0^* - \Phi_0 Y \cong 1 - z_0\cot z_0 + \frac{E}{2D}\left[\left(\frac{z_0^3}{6} - \frac{z_0}{2}\right)\cot z_0 + \frac{z_0^2}{2}\cot^2 z_0\right] \qquad (8.53)$$

with

$$z_0 = [MD/\hbar^2]^{\frac{1}{2}}\cdots, \qquad (8.54)$$

where a specialization to the case of a square well has been made. The correction term of first order in E vanishes for $z_0 = \pi/2$ which is approximately the case. Setting

$$z_0 = \pi/2 - \epsilon \qquad (8.55)$$

and keeping only terms of first order in ϵ there results

$$[\Phi_0^* - \Phi_0 Y]/[\Phi_0^* - \Phi_0 Y]_{E=0} \cong 1 - 0.07\epsilon E/D, \qquad (8.56)$$

where 0.07 arose as the digital equivalent of $\pi/8 - \pi^3/96$. In making a numerical estimate it is proper to employ for D the value which corresponds to an assumed action of the Coulomb field inside the potential well because in Eq. (8.52) the potential energy V is that acting in addition to the Coulombian. A fair value of this D, which has been designated as D^c in BCP, for an assumed range e^2/mc^2 is 11.3 Mev and the value of ϵ which corresponds to this well depth is 0.10. Substitution of these numbers into Eq. (8.56) gives

$$1 - 0.0006E \qquad (8.57)$$

for the right side of Eq. (8.56). This estimate agrees approximately with the denominator of Eq. (8.42). A precise agreement cannot be expected on account of the relative crudeness of the approximations made in arriving at Eq. (8.56). The existence of the agreement shows that the compensation of effects does not have anything special to do with the Coulomb field and that the approximate resonance at $E = 0$ is a strong contributing factor. It is also seen from the form of the right side of Eq. (8.52) that this quantity will behave similarly for potential wells of different shapes because $\mathfrak{F}/\mathfrak{F}_b$ behaves similarly. This is also expected on the grounds of equivalence of different shapes of potential wells for effects on Y which are linear in the energy E.

The constancy of $\Phi_0^* - \Phi_0 Y$ with E is not only a property of the approximation which neglects higher powers of E than E^2 but is practically as good including all powers of E. The condition is shown in Table III. The error in f caused by neglecting the quadratic terms in E in the numerator and denominator of Eq. (8.42) is small at small energies but not negligible at several Mev. This situation and the effect on scattering at a scattering angle of 45° in the laboratory system is illustrated in Table IV. In this table an alternative convention for recording the error of quadratic terms based on the form of f_1 obtained by expanding

[14] W. H. Ramsey, Proc. Camb. Phil. Soc. **44**, 87 (1948); *cf.* also L. Hulthén, K. Fysiogr, Sällsk. Lund Forhandl. **14**, Nrs. 8, 21 (1944).

it into a power series in E is given as well. The values of the phase shift corresponding to a square well with a range e^2/mc^2 and a depth $D=10.5$ Mev were used as an approximation to the experimental values of the phase shifts in preparing the table. In the first column of Table IV are listed the values of the energy. In the second column the corresponding values of the quantity δf are given. This quantity is the value of the fraction representing f_1 in Eq. (8.42) minus the value of the fraction obtained on neglecting the terms in E^2 in the numerator and denominator. In the third column are listed the negatives of corresponding differences of the phase shift K_0 expressed in degrees. In the fourth column are the negatives of corresponding differences in theoretically expected scattering for a scattering angle of 45° in the laboratory system. By "theoretically expected" scattering in the preceding sentence is meant the scattering which one would compute if one were to use either the quadratic or the linear approximation to the numerator and denominator of the fraction representing f_1. The fifth column of Table IV differs from the fourth only in that the effect of the quadratic term in the power series obtained by dividing the numerator by the denominator is dealt with. It will be noted from Table IV that: (a) there is no marked difference between the values in the last two columns of the table. (b) At energies below 0.6 Mev the effect of quadratic terms is smaller than the accuracy of most experiments. (c) At 2.2 Mev it is preferable to include the effect of quadratic terms because the experimental error is not many times greater than 0.5 percent. From 3 Mev on the approximation of neglecting quadratic terms in E appears to be definitely undesirable because systematic errors between 1 and 3 percent in scattering result. If one adopted a policy, common in some fields, of having the theoretical calculation ten times better than the experimental error and if the experimental error were one percent, one would hesitate to apply the linear approximation at bombarding energies above 0.5 Mev. (d) The importance of quadratic terms decreases as the range of force b is decreased. If one were sure of the range being shorter than e^2/mc^2, in square well equivalents, then it would be possible to apply the linear approximation at higher energies with the same utilization of experimental accuracy. The effect of a possible change in range while appreciable is not a very pronounced one as may be seen from the following estimate. An appreciable part of the quadratic term in X is canceled by terms in $(a/r)Y\Psi$ when one substitutes into Eq. (7) to obtain Eq. (8.42). The term in X which is quadratic in E is approximately proportional to r^3. It appears fair to estimate the dependence of quadratic terms on b as being not more critical than b^4, therefore. A change in the radius b from

TABLE VI. Values of quantities determining linear and quadratic dependence on E.

r_e	X_0	$r_e^{-1}X_1$	$r_e^{-3}X_2$	$(\Psi_0)_0$	$r_e^{-2}(\Psi_0)_1$	$r_e^{-4}(\Psi_0)_2$
0.75	1.8480	−1.9511	0.03090	0.99589	−0.047313	0.000377
0.80	1.8375	−1.9506	0.03089	0.99532	−0.047281	0.000377
0.85	1.8269	−1.9500	0.03087	0.99472	−0.047247	0.000376
0.90	1.8163	−1.9495	0.03085	0.99407	−0.047213	0.000376
0.95	1.8056	−1.9488	0.03084	0.99338	−0.047179	0.000376
1.00	1.7949	−1.9482	0.03082	0.99265	−0.047144	0.000376
1.05	1.7841	−1.9475	0.03080	0.99189	−0.047108	0.000376
1.10	1.7733	−1.9468	0.03079	0.99109	−0.047071	0.000375
r_e	$(\Phi_0)_0$	$r_e^{-2}(\Phi_0)_1$	$r_e^{-4}(\Phi_0)_2$	$(\Phi_0^*)_0$	$r_e^{-2}(\Phi_0^*)_1$	$r_e^{-4}(\Phi_0^*)_2$
0.75	1.0371	−0.01631	0.0000774	1.0746	−0.04933	0.000389
0.80	1.0396	−0.01634	0.0000775	1.0797	−0.04943	0.000389
0.85	1.0421	−0.01636	0.0000776	1.0848	−0.04954	0.000390
0.90	1.0446	−0.01639	0.0000777	1.0899	−0.04965	0.000390
0.95	1.0471	−0.01642	0.0000778	1.0950	−0.04975	0.000391
1.00	1.0497	−0.01644	0.0000779	1.1001	−0.04986	0.000392
1.05	1.0522	−0.01647	0.0000780	1.1053	−0.04997	0.000392
1.10	1.0547	−0.01650	0.0000781	1.1104	−0.05007	0.000393

2.81×10^{-13} cm to 2.47×10^{-13} cm which was discussed in connection with Table II corresponds to a factor 0.88 in b and 0.6 in b^4. Such a factor would indeed extend appreciably the range of applicability of the linear approximation. If one were to be satisfied with 0.5 percent accuracy in theoretical prediction the region below $E=3.5$ Mev rather than the 2.2 Mev of Table IV would be satisfactory. But if b were 2.65×10^{-13} cm which is the mean of the other two values considered after Table II, the error would decrease in comparison with that for $b=e^2/mc^2$ by the factor 0.8 which makes little difference in comparison with Table IV.

It is useful to consider the relative magnitudes of contributions to f by its component parts. At 1 Mev the total of 8.7 is made up of 5.82 for $(C_0^2/\eta)\cot K_0$, −0.79 for q/η and 3.69 for $-2\ln\eta$. At 6 Mev the total of 13.7 is made up of contributions 9.1, −0.84, 5.48 in the same order. At 9 Mev the contributions are $11.9-0.84+5.9=16.9$. The numbers just given show that only part of f contains the phase shift in the form of $\cot K_0$ and that at 1 Mev a contribution of 2.9 out of a total of 8.7 comes from $(q/\eta)-2\ln\eta$ which does not contain the phase shift. The way in which f varies with E is conditioned by several factors and the increase of $(C_0^2/\eta)\cot K_0$ with E is not directly related to the increase of K_0 with E but rather to the behavior of C_0^2/η which more than compensates the decrease in $\cot K_0$. The term $-2\ln\eta$ is zero at $0.1581^2=0.025$ Mev. For smaller E it becomes negative and approaches $-\infty$ at $E=0$. The combination $q/\eta-2\ln\eta$ is finite, however, at $E=0$ and so is $(C_0^2/\eta)\cot K_0$, therefore. For small values of E one can approximate

$$q/\eta-2\ln\eta\cong2(2\gamma-1)+1/6\eta^2\cong0.309+20E/3, \quad (8.58)$$

where E is in Mev. This approximation is fair at $E=0.2$ and lower energies but very crude at higher energies. It is seen that from the point of view of seeing simply how errors in the experi-

mental values affect conclusions regarding the interaction potential, the quantity f is not the easiest to deal with. Also, while it owes its linearity in E to the linearity of the logarithmic derivative Y, it is related to the latter through the quantities X, Ψ, Φ, Φ^*. It may also be mentioned that if the quantity f is considered as a function of K_0 and η, then its values for the Gauss error and meson potentials show deviations from linearity and from each other which are of the same order as those for a square well. This matter will be discussed in more detail in another publication. For the limited energy region considered by BTE the form of f suggested plotting $(\eta/C_0^2)\tan K_0$ which is approximately constant in that energy region.

The plot of f against E has advantages at low energies because here the linearity is good. The slope of f as a function of E is related to the range of force of the equivalent square well. By means of Eq. (7.47) this range determines also the range parameter for other compact potentials. It is, therefore, of interest to examine df/dE and f for $E=0$. The value of f at $E=0$ will be referred to as the intercept of f. It can be found by extrapolating the experimentally determined line for f to the axis $E=0$ and taking the intercept on that axis. The value of the intercept will be seen to be sensitive to Y_0 and hence to the depth of the potential well for an assumed range.

The value of the intercept is obtained from Eqs. (7), (7.3), (7.32), (7.92), (7.94) as

$$f_{E=0}=4\frac{\mathfrak{K}_0-(2/x)\mathfrak{K}_1Y_0}{I_0-(2/x)I_1Y_0}-2\ln(2r/a).$$

The relation

$$-I_0\mathrm{K}_1+I_1\mathrm{K}_0=1/x$$

has as a consequence

$$-I_0\mathfrak{K}_1+I_1\mathfrak{K}_0=1/x,$$

and hence, making use of the definition of $\mathfrak{K}$ in Eq. (7.6)

$$f_{E=0}=4\left[\frac{\mathrm{K}_0}{I_0}+\frac{2Y_0/(x^2I_0)}{I_0-(2/x)I_1Y_0}\right]+4\gamma-2. \quad (8.6)$$

In the determination of the slope df/dE the approximate constancy of the denominator in Eq. (7) will be made use of. By neglecting the energy dependence of the denominator the theoretically expected slope will be only ~ 0.15 percent too low for a range of e^2/mc^2 as is apparent from Eq. (8.42). For a square well one obtains by means of Eqs. (7.91), (7.93), (8), (8.1), (8.2)

$$(df/dE)_{E=0}\cong\{(r/a)(1-Y_0)(1+Y_0/z_0^2)/2 +(r/a)^2Y_0/9+\cdots\} \times(d\eta^{-2}/dE)/[I_0-(2/x)I_1Y_0], \quad (8.61)$$

where the result has been ordered in powers of r/a. Here Y_1 has been expressed in terms of Y_0 and for E in Mev

$$d\eta^{-2}/dE=40.0. \quad (8.62)$$

The phase z_0 is the value of the phase z for $E=0$. A convenient approximation is obtained by using Eq. (8.55) and neglecting powers of ϵ higher than the first.

In this approximation

$$[I_0-(2/x)I_1Y_0](df/dE)_{E=0} \cong 40.0\{(\tfrac{1}{2}-0.467\epsilon)(r/a)+0.174\epsilon(r/a)^2\}, \quad (8.63)$$

where the numbers 0.467, 0.174 arose as $(\pi/4)-1/\pi$, $\pi/18$, respectively. For $r=e^2/mc^2$ the above formula contains a quadratic term in r which is ~ 0.3 percent of the linear term. The slope of a plot of f against E is, therefore, nearly proportional to the square well range b which in the remainder of the paper will be also called r. For $b=e^2/mc^2$, $D=10.5$ Mev the right side of Eq. (8.63) is ~ 0.838 while the value of the left side is ~ 0.848. The error made in introducing the approximations between Eq. (8.61) and Eq. (8.63) amounts to ~ 1.2 percent and is in the direction of overestimating the range.

Equations (8.6), (8.61) are suitable for the determination of the range b and Y_0 by successive approximations. The essential reason for the rapid convergence of the process is that Eq. (8.6) is equivalent to Eq. (2) for the determination of Y_0. According to Eq. (2)

$$Y_0\cong(2+f_1^0)(r/a), \quad (8.7)$$

where f_1^0 is the value of f_1 of Eq. (2.5) for $E=0$. Since f_1^0 is approximately 3, the above equation determines Y_0 as $\sim 5r/a$ which is $\sim 1/4$ for the value of r/a indicated by experiment. On the other hand $Y_0\cong\pi\epsilon/2$ and hence ϵ is so small that its precise value matters little in the determination of r/a by means of Eq. (8.63). The value of r/a so determined is nearly independent of the initial guess. For a small error $\delta\zeta$ in $\zeta=r/a$ the solution of Eq. (2) for Y_0 gives an error δY_0 in Y_0 which to within the first two powers of ζ is

$$\delta Y_0=\{(f_1^0+4)/[\zeta(f_1^0+2)] -4(f_1^0-2)/(f_1^0+2)^2 +2f_1^0/(f_1^0+2)-(f_1^0+2)\}Y_0\delta\zeta. \quad (8.8)$$

To a good approximation the employment of $\zeta+\delta\zeta$ in $I_0-(2/x)I_1Y_0=B$ together with the use of $Y_0+\delta Y_0$ instead of Y_0 in that expression produces an error

$$\delta B\cong[2(1+\zeta)-(1+2\zeta/3)Y_0]\delta\zeta-(1+\zeta)\delta Y_0. \quad (8.81)$$

The quantity which multiplies r/a in Eq. (8.61) is

$$A=(1-Y_0)(1+Y_0/z_0^2)/2. \quad (8.82)$$

The error in it caused by the error δY_0 is

$$\delta A = -(1/2)\{1-z_0^{-2}+2Y_0z_0^{-2}[1-(1-Y_0^2)z_0^{-2}] \times[1-Y_0(1-Y_0)z_0^{-2}]^{-1}\}\delta Y_0. \quad (8.83)$$

The coefficient of $40.0E/B$ in the right side of Eq. (8.61) is a quadratic expression in (r/a). The error in its value caused by δB, and the errors in its coefficients caused by δY_0 result in an error

$$\frac{\delta(r/a)}{(r/a)} = \frac{\delta B}{B} - \frac{\delta A}{A} - \frac{\zeta}{9A}\left[\delta Y_0 + Y_0\left(\frac{\delta B}{B} - 2\frac{\delta B}{A}\right)\right]. \quad (8.84)$$

In addition to the approximation of neglecting terms in $(\delta r)^2$ this formula takes into account the term in $Y_0(r/a)^2/9$ on the right side of Eq. (8.61) to the first order also. The latter approximation is a good one. The first two terms in Eq. (8.84) give the error which would exist if the term in $(r/a)^2/9$ were absent. Assuming, for purposes of illustration, that f, df/dE have values corresponding to $b=e^2/mc^2$, $D=10.5$ Mev and that b, D are to be determined by successive approximations Eq. (8.84) indicates a value of $\delta(r/a)/(r/a)$ about 1/40 of the initial error $\delta\zeta/\zeta$. The first two terms account for $\sim$80 percent of the right side of Eq. (8.84). Calculations carried out in detail for an assumed initial error of 10 percent in ζ indicate that the ratio of improvement is more nearly 1/35 than 1/40 and that Eq. (8.8) underestimates δY_0 by about 2 percent. The method of successive approximations is seen to converge very rapidly and to be very practical for range determination. Values of quantities involving Bessel functions occurring in Eqs. (8.6), (8.61) are listed in Table V so as to facilitate range determination by means of these formulas.

In Table V the quantities are listed so as to agree with conversion factors in the following relations:

$$\zeta = r/a = \rho\eta = 0.04886r_e, \quad x^2 = 8\zeta = 0.39088rmc^2/e^2, \quad r_e = rmc^2/e^2.$$

The number of significant figures in the table and in the conversion factors is greater than that corresponding to the accuracy with which fundamental constants are known. The object in including the extra digits is to make it possible to carry out self-consistent calculations for assumed values of fundamental constants which matter here only in the conversion factor 0.04886. To change to another choice of the constants it suffices to change the numbers in the first column of the table.

In Table VI are listed values of quantities which are needed in order to express X, Ψ_0, Φ_0, Φ_0^* as power series in E. The unit of energy is 1 Mev. The columns are labeled in terms of X_0, X_1, $\cdots$, $(\Phi_0^*)_2$ defined by:

$$\Psi_0 = (\Psi_0) + (\Psi_0)_1E + (\Psi_0)_2E^2 \quad (8.86)$$

and similarly for the other quantities. In order to facilitate interpolation the quantities tabulated are the coefficients of E divided by a suitable power of r_e. The object of including the extra digits is the same as for Table V. The relation $\eta^{-2}=40.00(7)E$ was used. If it is desired to change the choice of fundamental constants so that η^{-2}/E is changed then one has to multiply the columns for linear effects in E by $(\eta^{-2}/E)/40.007$ and the columns for quadratic effects by the square of that quantity. Changes in fundamental constants resulting in a change of ζ/r_e result in a change in the first column as for Table V.

The entrance of the range of force for the important quantities X, Ψ for coefficients of different powers of E can be seen in Eqs. (7.91), (7.93). The principal dependence of Y_0 on the assumed range of force is given by Eq. (8.7) and Y_1 is approximately proportional to the square of the range. The main contributions to the coefficient of E in f come from X and Y_1. While the above discussion refers mainly to properties of square wells, Eq. (7.47) makes it possible to transfer conclusions to wells of other shapes for terms in f varying linearly with E. Calculations made in collaboration with Messrs. Hatcher and Arfken indicate that quadratic terms in E depend on the shape of the potential well.

The writers wish to express their indebtedness to Professors Critchfield and Williams and other workers at the University of Minnesota,[11] to Professor Feshbach of M.I.T. for critical remarks concerning the boundary condition view, and to Messrs. Arfken and Hatcher for permission to quote their work before publication.

APPENDIX I

Experimental errors are not indicated for the points of HHT, HKPP, BFLSW, and RKT because the statistical counting errors are not significant in these cases and the internal consistency at one energy is not a completely satisfactory criterion. In most measurements of this group the estimated errors lead to uncertainties small compared with the distance between the $D=10.5$, 10.6 Mev lines. Comparison of points obtained by different experimental groups with each other is probably the best criterion of accuracy and allows some judgment of the certainty of conclusions from this group of observations. The phase shifts corresponding to the work of RKT were obtained by observing that their higher scattering angle data form a self consistent picture at 250 and 300 kev according to their Figs. 10, 11 and that the possibility of multiple scattering might be responsible for the apparent inconsistencies at the smaller angles. The relative position of the experimental values with respect to the theoretical curves in their Figs. 10, 11 indicated $K_0=9.0°$ for $E=250$ kev and $K_0=10.9°$ at $E=300$ kev. The points on the graphs were also located by comparison of the theoretical values of the logarithmic derivative y expected for $b=e^2/mc^2$, $D=10.5$, 10.6 Mev and $b=0.75\ e^2/mc^2$, $D=19.6905$ Mev. The data of Heydenburg, Hafstad and Tuve in the region 220-640

kev were not used in the present analysis partly because, as remarked by J. K. Lubansky and C. De Jager,[15] these observations do not agree very well with other measurements and primarily because these data were taken mainly in order to confirm the smallness of scattering at 90° in the center of mass system rather than for quantitative purposes.

The values and experimental uncertainties for $E=4.2$, 7.03 Mev have been used in accordance with published statements of May and Powell and of Dearnley, Oxley, and Perry. The data of R. R. Wilson and E. C. Creutz[4] at 8 Mev, R. R. Wilson[4] at 10 Mev and of Wilson, Lofgren, Richardson, Wright, and Shankland were treated in the following manner. The values of the phase shift and of y, Y at 8 Mev were determined from the value $\sigma=1.7\pm0.1\times10^{-25}$ cm^2 for the scattering cross section in the laboratory system of Wilson and Creutz. This was compared with a theoretical value of 1.75×10^{-25} cm^2 which corresponds to $K_0=53.6°$, which is the value of Hoisington and Thaxton[4] at $E=8$ Mev for $b=e^2/mc^2$, $D=10.5$ Mev. By means of the relation

$$\delta Y=-\rho\mathfrak{Y}^{-2}\delta K_0, \qquad \text{(I, 1)}$$

which applies to small changes in the logarithmic derivative and phase shift and the approximate first-order relation:

$$\delta K_0\cong\tan K_0(\delta\sigma/2\sigma) \qquad \text{(I, 2)}$$

gave $\delta Y=0.017$. This is the estimated difference between Y for the experimental cross section and the Y corresponding to $b=e^2/mc^2$, $D=10.5$ Mev. The limits of experimental uncertainty were drawn in so as to correspond to the uncertainty $\pm0.1\times10^{-25}$ cm^2 in σ.

At $E=14.5$ Mev it was assumed that the value of σ for a scattering angle of 45° in the laboratory system is about 0.0125×10^{-25} cm^2 higher than that expected for a square well with $b=e^2/mc^2$, $D=10.5$ Mev. The reason for this assumption is that the experimental point for the above mentioned scattering angle is about 0.025×10^{-25} cm^2 higher than the S wave curve in Fig. 3 of Wilson *et al.* It appears that a shift of the S wave curve up by about the assumed amount would fit experiment reasonably well. On this admittedly rough and somewhat arbitrary interpretation it was calculated that, $\delta K_0=1.2(8)°$ and by means of (I, 1) it followed that $\delta Y=-0.031$. The approximation (I, 2) agrees with this estimate within about 10 percent. Since δY is used as an addition to the Y of the square well the accuracy required is moderate. The experimental uncertainty was made to correspond to an uncertainty of $\pm0.2\times10^{-26}$ cm^2 in $\sigma_{\rm cm}$ the cross section in center of mass system. Rough checks were made on the curve marked S in Wilson *et al.*

Since for 10 Mev absolute values of the cross section are not available the experimental error was taken to be about the same as for 8 Mev and 14.5 Mev. The relative values of the cross section at different angles do not determine K_0 with good accuracy even on the assumption that other phase shifts are absent. It is difficult to do much with the value at 10 Mev. The value of Y for this energy was somewhat arbitrarily taken to be on the $D=10.5$ Mev line. This position fits in which the points at $E=8$, 14.5 Mev but the point at $E=10$ Mev has very little weight in comparison with the others.

The experimental uncertainties in the 8 Mev–14.5 Mev region are so large that accurate calculations appeared out of place. It was assumed throughout that only S wave scattering enters close to $\Theta=45°$. This assumption can be doubted. It was made only because it is impossible to evaluate the complicated effect of higher phase shifts and because there is no decisive experimental evidence indicating their presence.

[15] J. K. Lubansky and C. De Jager, Physica **XIV**, 8 (1948).

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Separation of a Gas Mixture Flowing through a Long Tube at Low Pressure*

R. D. PRESENT
University of Tennessee, Knoxville, Tennessee
AND
A. J. DEBETHUNE
Boston College, Chestnut Hill, Massachusetts

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The separation of a binary gas mixture by diffusion through a capillary of radius r depends on the fact that the molecules have different masses m_i and mean speeds v_i. When the inlet pressure is so low that the mean free path λ is much greater than r, the flow is diffusive and the separation factor (at zero outlet pressure) has its maximum value $(m_2/m_1)^{\frac{1}{2}}$. At high pressures ($\lambda\ll r$) no separation occurs. This paper treats the intermediate case ($\lambda\approx r$) where the transfer of forward momentum from light to heavy molecules in unlike collisions equalizes the transport velocities and decreases the separation factor. As the inlet pressure rises, this effect makes the flow non-separative before it becomes viscous. Flow equations are derived by equating the momentum acquired by the light component from the pressure gradient to the momentum lost to the wall plus that transferred to the other component. The viscous effects are treated as a small additive perturbation on the flow. The integrated flow equations express the separation factor as a function of the inlet and outlet pressures.

INTRODUCTION

FOR purposes of orientation, we consider first the effusion of a gas mixture through a circular orifice. The nature of the flow depends on the comparative magnitude of the mean free path λ and the radius r of the opening. When the opening is large ($r\gg\lambda$), many collisions occur in the vicinity of the orifice and, if two kinds of molecules are present, there is a continual transfer of momentum from the lighter, faster molecules to the heavier, slower molecules with the result that both kinds of

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