PHYSICAL REVIEW VOLUME 74, NUMBER 8 OCTOBER 15, 1948

# On the Theory of Diffraction by an Aperture in an Infinite Plane Screen. I

HAROLD LEVINE AND JULIAN SCHWINGER
*Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts*
(Received June 9, 1948)

The diffraction of a scalar plane wave by an aperture in an infinite plane screen is examined theoretically. The wave function at an arbitrary point in space is expressed in terms of its values in the aperture, and constructed so as to vanish on the screen, in accordance with the assumed boundary condition. An integral equation to determine the aperture field is obtained from the continuity requirement for the normal derivative of the wave function on traversing the plane of the aperture. Utilizing the integral equation (whose solution is generally unobtainable), the amplitude of the diffracted spherical wave at large distances from the aperture is exhibited in a form which is stationary with respect to small variations (relative to the correct values) of the aperture fields arising from a pair of incident waves. This expression is independent of the scale of the aperture fields. The transmission cross section of the aperture for a plane wave is found to be simply related to the diffracted amplitude observed in the direction of incidence. The variational formulation is applied in detail for a wave incident normally on a circular aperture. By comparison with the exact results available for this problem, it appears that the use of suitable trial aperture fields in the variational formulation yields approximate, yet accurate, expressions for the diffracted amplitude and transmission cross section over a wide range of frequencies.

## 1. INTRODUCTION

THE steady-state problem of diffraction by an aperture in an infinite plane screen has attracted attention for many years. Exact solutions are restricted to a few cases where the aperture is of simple geometric shape and may be conveniently described in a coordinate system in which the wave equation is separable; the available theoretical methods for approximating these solutions, and those of other cases, are valid for only a limited range of frequencies or wave-lengths.

The diffraction of waves in a scalar field is considered in treatises on physical optics, after requiring that the wave function vanish on the (perfectly conducting) screen, as befits a rectangular, tangential component of the electric field intensity.[1] A brief description of the approximate and exact theoretical methods follows; these have all been applied in detailed calculations for a circular aperture.

A well-known approximate solution is due to Kirchoff, in which the aperture field is identified with the incident field, and the normal derivative of the wave function is assumed to vanish on the back side of the screen. This procedure is not self-consistent, for the transmitted field so determined from the incident field in the aperture does not vanish on the screen. The results possess a measure of validity only when the aperture dimension is large compared to the wave-length of the vibrations, since then the back side of the screen lies in the shadow and the diffracted field is relatively small; such a condition is amply realized for light diffraction by apertures of macroscopic dimensions.

Another approximate solution is due to Lord Rayleigh,[2] whose results in the case of plane waves incident normally on the screen are applicable when the aperture dimension is small in comparison with the wave-length. The procedure stems from the observation that in the neighborhood of the aperture (at distances from it large in comparison with its dimension, yet small in comparison with the wave-length), the conditions are essentially static, or the same as if the wave-length were infinite. These conditions are described by reference to known solutions for the steady flow of incompressible fluids. The nature of the field at large distances from the aperture is readily determined from the aperture field.

An integral equation formulation of the related mathematical problem of normal acoustic diffraction is described by King.[3] An indication is

[1] According to a form of the Babinet principle (see reference 5), this wave function, appearing in the role of a velocity potential, also describes the diffraction of sound by a rigid disk in the form of the aperture, as well as the radiation of sound by the freely vibrating disk.

[2] Lord Rayleigh, Phil. Mag. **43**, 259 (1897); Sci. Pap. IV, p. 283.

[3] L. V. King, Proc. Roy. Soc. **A153**, 1 (1935).

given of its solution by a method of successive approximations, based on the known solution of the corresponding equations in potential theory. Due to the increasing difficulty of computation in higher approximations, King furnishes only the numerical coefficients of the first few terms in a development of the diffracted amplitude in ascending powers of the characteristic parameter, radius of disk/wave length. Sommerfeld[4] gives a more elegant and detailed mathematical discussion of the integral equation, involving an expansion in characteristic functions and the approximate determination of some of the coefficients; the results also apply for small values of the parameter.

Bouwkamp[5] presents an exact theoretical analysis of the problem of diffraction of a scalar plane wave by a circular aperture, based on the construction of normal solutions of the wave equation, which is separable in oblate spheroidal coordinates. The wave function on the far side of the screen, with the character of a diverging spherical wave at large distances from the aperture, is expanded in an infinite series of these normal solutions (which have the proper symmetry with respect to the plane of the screen, so that the individual terms fulfill the boundary condition on it). The expansion coefficients are determined in the course of satisfying a boundary condition in the aperture. From the amplitude of the asymptotic spherical wave, the energy passing through the aperture is obtained, and then the transmission coefficient, $t$ (transmission cross section/area of aperture), on division by the incident energy flux through the same area. To simplify the numerical calculations, Bouwkamp considers only the case of normal incidence, in which the entire field possesses rotational symmetry. The transmission coefficient is evaluated for a number of wave-lengths in the range $\infty > \lambda/a > 0.6$, where $a$ is the radius of the aperture; at shorter wave-lengths the computations are progressively more difficult, owing to the slow convergence of the series involved.

It is clear from this survey that a general formulation which permits accurate numerical evaluation of the diffracted amplitude and transmission cross section for a wide range of frequencies, is of considerable interest. The purpose of this and a companion paper is to illustrate the utility of variational principles for such calculations. The present paper describes a reformulation of the scalar diffraction problem for an arbitrary aperture in terms of a first variational principle. An important role is assumed by the amplitude of the diffracted spherical wave at large distances from the aperture, expressed in terms of the aperture field. For plane-wave excitation, this quantity is a function of the directions of propagation of the incident wave and of observation for the diffracted wave.

When a plane wave is incident on the aperture from direction (1) and the diffracted wave observed from direction (2), the amplitude obtained is equal to that of a reverse situation, in which the wave is incident in direction (2) and observation made from direction (1). Using this reciprocity relation, the amplitudes are exhibited in a form which is stationary with respect to small variations (relative to the correct values) of the aperture fields arising from the two incident waves. In addition, this expression is independent of the scale of the aperture fields, and is therefore suitable for a first approximation with simple forms of the aperture fields.

From a consideration based on the fact that the energy transmitted through the aperture is the same as that appearing at any remote surface which intersects the plane of the screen, the transmission cross section is recast in appropriate limiting forms at low and high frequencies. From the first of these, the (wave-length)$^{-4}$ proportionality is obtained (as in Rayleigh's theory of diffraction by obstacles small compared to the wave-length), independently of the assumed aperture field, provided the boundary condition at the rim of the screen is satisfied. The second form, on identification of aperture and incident fields, leads to the geometrical optics result, where the cross section is simply the area of the aperture projected on a plane normal to the direction of the incident wave.

Owing to the restricted class of trial aperture fields, in virtue of the imposed boundary condition, the first variational principle is most useful in the low frequency range. Here the

[4] A. Sommerfeld, Ann. d. Physik **42**, 389 (1942).

[5] C. J. Bouwkamp, *Theoretische En Numerieke Behandeling van de Buiging Door Een Ronde Opening* (Dissertation, University of Groningen, 1941); R. D. Spence, J. Acous. Soc. Am. **20**, 380 (1948).

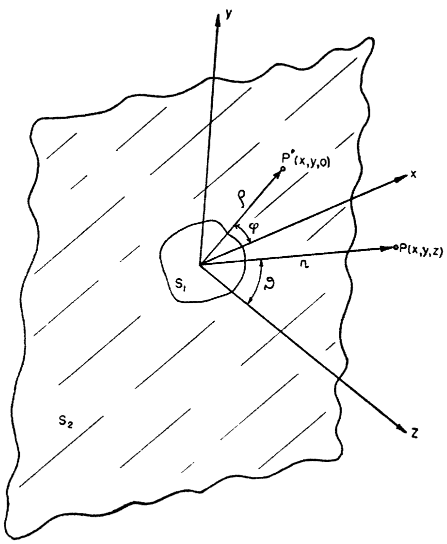


FIG. 1. Diffracting aperture in a plane screen.

qualitative information concerning static aperture fields allows accurate calculations of amplitude and cross section at higher frequencies than are feasible with the existing approximation procedures. Corrections to the geometrical optics result are more difficult to achieve with this formulation. To improve the accuracy of high frequency calculations, a second variational principle for the diffracted amplitude is devised, with details relegated to a separate paper. In this, the discontinuity in normal derivative of the field at the screen (akin to the surface current in the optical problem) replaces the aperture field, although there are additional modifications arising from the infinite extent of the screen. The trial functions adopted for the discontinuity in normal derivative are unrestricted with regard to a boundary condition at the rim of the screen. In particular, the approach to the geometrical optics result with increasing frequency can be well approximated if this function is calculated from the incident field on the illuminated face of the screen.

The distinction between these variational principles in the choice of trial functions is not uniquely related to the boundary condition assumed for the wave function. For a boundary condition which requires that the wave function have vanishing normal derivative at the screen, a pair of analogous variational principles exist, based on (i) the normal derivative of the field in the aperture and (ii) the discontinuity of the field at the screen. Only in the second of these is it necessary to impose a boundary condition on the trial field. Thus it appears that derivatives of the field variable are not restricted by conditions at the boundary of their domain, whereas this is so for the field variables themselves. The reason is that Green's functions and their derivatives, respectively, are factors which naturally accompany these quantities in the formulation, and non-integrable singularities arise in the latter case unless the boundary conditions are satisfied.

To obtain a practical test of the degree of approximation afforded by the variational principles, detailed application is made in this and a companion paper to the evaluation of the transmission coefficient for normal incidence on a circular aperture. Numerical results of the variational and other calculations are compared with the exact values obtained for this problem by Bouwkamp. In addition, the extent to which the two variational principles complement (and agree with) each other for the entire range of frequencies provides a general estimate of their accuracy.

## 2. INTEGRAL EQUATION FORMULATION FOR AN APERTURE OF ARBITRARY OPENING

We consider an infinitesimally thin plane screen $S_2$, of infinite extent, which is perforated by an aperture $S_1$. A rectangular coordinate system is chosen with origin at some point of the aperture, and oriented so that the screen lies in the $x,y$ plane (Fig. 1).

A plane wave is incident upon the aperture in the half-space $z<0$, and it is desired to investigate the diffracted field. The incident wave, propagating in the direction $\vartheta'$, $\varphi'$ ($\vartheta'$ measured from the positive direction of the $z$ axis, and $\varphi'$ from the positive direction of the $x$ axis in the $x,y$ plane) is described by the scalar wave function

$$\phi^{\text{inc}}(\mathbf{r})=\exp(ik\mathbf{n}'\cdot\mathbf{r})=\exp[ik(x\sin\vartheta'\cos\varphi'+y\sin\vartheta'\sin\varphi'+z\cos\vartheta')],\quad(2.1)$$

where $\mathbf{n}'$ is a unit vector in the direction of propagation, $k=2\pi/\lambda$ is the free space propagation constant, and $\lambda$ the corresponding wave-length. The harmonic time dependence, $\exp(-ikct)$, with

$c$ the velocity of wave propagation, is omitted throughout.

The wave function describing the complete (incident+diffracted) field satisfies the wave equation

$$(\nabla^2+k^2)\phi(\mathbf{r})=0 \qquad (2.2)$$

at all points of space, and is subject to the prescribed boundary condition

$$\phi(\mathbf{r})=0, \quad \mathbf{r} \text{ on } S_2; \qquad (2.3)$$

in addition, the wave function and its normal (i.e., $z$) derivative vary continuously on passing through the aperture.

In one method of formulating this boundary value problem, it is convenient to classify the wave functions (2), (3) according to their symmetry in the $z$ coordinate. A symmetric function has vanishing normal derivative in the aperture, whereas an antisymmetric function has the same form as though the aperture were absent. The wave functions of opposite symmetry, appropriate to the incident wave (1), are obtained from the solution of individual boundary value problems in the half-space $z\leqslant 0$; with these we construct, by suitable combination, the wave function in the problem of physical interest.

A solution of the wave equation (2) at any point in the half-space $z\leqslant 0$, which assumes (as yet) arbitrary values in the aperture and satisfies the boundary condition (3) on the remainder of the plane, is given in terms of a Fourier integral,

$$\phi(\mathbf{r})=2i\sin(kz\cos\vartheta')\exp(ik\mathbf{n}'\cdot\boldsymbol{\rho})$$
$$+\int_{S_1}\int_{k_x,k_y=-\infty}^{\infty}\phi(\mathbf{r}')_{z'=0}\frac{\partial}{\partial z'}\left\{\frac{\exp[i\{k_x(x-x')+k_y(y-y')+(k^2-k_x^2-k_y^2)^{\frac{1}{2}}(z'-z)\}]}{4\pi^2 i(k^2-k_x^2-k_y^2)^{\frac{1}{2}}}\right\}_{z'=0}dk_xdk_ydS', \qquad (2.4)$$

where $\boldsymbol{\rho}$ denotes a position vector in the $x,y$ plane, and $dS'$ an element of area in the $x',y'$ plane. It is readily verified that (4) is a solution of the wave equation; furthermore, on performing the indicated differentiation, we find

$$\phi(\mathbf{r})=2i\sin(kz\cos\vartheta')\exp(ik\mathbf{n}'\cdot\boldsymbol{\rho})$$
$$+\frac{1}{4\pi^2}\int_{S_1}\int_{k_x,k_y=-\infty}^{\infty}\phi(\mathbf{r}')_{z'=0}\exp[i\{k_x(x-x')+k_y(y-y')-(k^2-k_x^2-k_y^2)^{\frac{1}{2}}z\}]dk_xdk_ydS',$$

and thus, using the integral property of the Dirac delta function,

$$\begin{aligned}\phi(\mathbf{r})_{z=0}&=\frac{1}{4\pi^2}\int_{S_1}\int_{-\infty}^{\infty}\phi(\mathbf{r}')_{z'=0}\exp[i\{k_x(x-x')+k_y(y-y')\}]dk_xdk_ydS'\\&=\int_{S_1}\phi(\mathbf{r}')_{z'=0}\delta(x-x')\delta(y-y')dS'\\&=\phi(\mathbf{r})_{z=0}=\phi(\boldsymbol{\rho}), \quad \boldsymbol{\rho} \text{ on } S_1,\\&=0, \quad \boldsymbol{\rho} \text{ on } S_2,\end{aligned}$$

in accord with the boundary values.

Introducing the free-space scalar Green's function,

$$\begin{aligned}G(\mathbf{r},\mathbf{r}')&=\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{4\pi|\mathbf{r}-\mathbf{r}'|}=\frac{1}{8\pi^3}\int_{-\infty}^{\infty}\frac{\exp[i\{k_x(x-x')+k_y(y-y')+k_z(z-z')\}]}{k_x^2+k_y^2+k_z^2-k^2}dk_xdk_ydk_z\\&=\frac{i}{8\pi^2}\int_{-\infty}^{\infty}\frac{\exp[i\{k_x(x-x')+k_y(y-y')+(k^2-k_x^2-k_y^2)^{\frac{1}{2}}|z-z'|\}]}{(k^2-k_x^2-k_y^2)^{\frac{1}{2}}}dk_xdk_y\end{aligned} \qquad (2.5)$$
$$|\mathbf{r}-\mathbf{r}'|=((x-x')^2+(y-y')^2+(z-z')^2)^{\frac{1}{2}},$$

which satisfies the inhomogeneous wave equation,

$$(\nabla^2+k^2)G(\mathbf{r},\mathbf{r}')=-\delta(\mathbf{r}-\mathbf{r}'), \tag{2.6}$$

and the radiation condition,[6] the wave function (4) becomes

$$\phi(\mathbf{r})=2i\sin(kz\cos\vartheta')\exp(ik\mathbf{n}'\cdot\boldsymbol{\varrho})$$
$$-2\int_{S_1}\phi(\boldsymbol{\varrho}')\frac{\partial}{\partial z'}G(\mathbf{r};x',y',0)dS',\quad z\leqslant 0. \tag{2.7}$$

The first term of (7) describes the field in the absence of an aperture, being a superposition of incident and specularly reflected waves, whose phases are adjusted so that the combined wave function vanishes at all points of the screen. We denote this antisymmetric function of $z$ by

$$\phi_{\text{odd}}(\mathbf{r})=2i\sin(kz\cos\vartheta')\exp(ik\mathbf{n}'\cdot\boldsymbol{\varrho}). \tag{2.8}$$

On requiring that the $z$ derivative of the wave function (7) vanish in the aperture, we obtain the integral equation

$$k\cos\vartheta'\exp(ik\mathbf{n}'\cdot\boldsymbol{\varrho})=\int_{S_1}\phi(\boldsymbol{\varrho}')K(\boldsymbol{\varrho},\boldsymbol{\varrho}')dS',$$
$$\boldsymbol{\varrho}\text{ on }S_1, \tag{2.9}$$
$$K(\boldsymbol{\varrho},\boldsymbol{\varrho}')=(\partial/\partial z)(\partial/\partial z')G(x,y,0;x',y',0)$$
$$=K(\boldsymbol{\varrho}',\boldsymbol{\varrho}),$$

for the determination of the aperture field, and the resultant symmetrical function of $z$,

$$\phi_{\text{even}}(\mathbf{r})=-2i\sin(k|z|\cos\vartheta')\exp(ik\mathbf{n}'\cdot\boldsymbol{\varrho})$$
$$\mp 2\int_{S_1}\phi(\boldsymbol{\varrho}')(\partial/\partial z')G(\mathbf{r};x',y',0)dS'. \tag{2.10}$$

The upper or lower sign in (10) is to be used for $z<0$, $z>0$, respectively, since $G$ is an even function of $z-z'$.

To describe the physical situation arising from the incidence of the plane wave (1) upon the aperture, we combine the antisymmetric and symmetric functions (8), (10). The resulting wave function has different forms on opposite sides of the screen, owing to the asymmetry of excitation; these are

$$\phi^{(+)}(\mathbf{r})=\tfrac{1}{2}[\phi_{\text{odd}}(\mathbf{r})+\phi_{\text{even}}(\mathbf{r})]$$
$$=\int_{S_1}\phi(\boldsymbol{\varrho}')(\partial/\partial z')G(\mathbf{r};x',y',0)dS',$$
$$z\geqslant 0 \tag{2.11}$$

and

$$\phi^{(-)}(\mathbf{r})=\tfrac{1}{2}[\phi_{\text{odd}}(\mathbf{r})+\phi_{\text{even}}(\mathbf{r})]$$
$$=2i\sin(kz\cos\vartheta')\exp(ik\mathbf{n}'\cdot\boldsymbol{\varrho})$$
$$-\int_{S_1}\phi(\boldsymbol{\varrho}')\frac{\partial}{\partial z'}G(\mathbf{r};x',y',0)dS',$$
$$z\leqslant 0. \tag{2.12}$$

The wave functions (11), (12) vanish on the respective faces of the screen, and are equal at any point of the aperture

$$\phi^{(-)}(\boldsymbol{\varrho})=\tfrac{1}{2}\phi(\boldsymbol{\varrho})=\phi^{(+)}(\boldsymbol{\varrho}),\quad \boldsymbol{\varrho}\text{ on }S_1; \tag{2.13}$$

furthermore, their $z$ derivatives are continuous across the aperture, in consequence of the integral equation (9). In particular, it may be noted that

$$\frac{\partial}{\partial z}\phi^{(+)}(x,y,0)=\frac{1}{2}\frac{\partial}{\partial z}\phi_{\text{odd}}(x,y,0)$$
$$=\frac{\partial}{\partial z}\phi^{\text{inc}}(x,y,0). \tag{2.14}$$

Thus (9), (11), (12) constitute the formal solution of the diffraction problem.

The preceding formulation involves the identity of the two regions $z<0$, $z>0$. An alternative treatment, which is also applicable when such symmetry does not exist, is based on the representation of the field in each region by an appropriate Green's function. The Green's function for our problem, that for a half-space, can be obtained by the method of images, and is

$$\Gamma(\mathbf{r},\mathbf{r}')=G(\mathbf{r},\mathbf{r}')-G(\mathbf{r},\mathbf{r}'-2\mathbf{n}_z\mathbf{n}_z\cdot\mathbf{r}')$$
$$=\Gamma(\mathbf{r}',\mathbf{r}), \tag{2.15}$$
$$\Gamma(x,y,z;x',y',0)=0,$$

where $\mathbf{n}_z$ is a unit vector in the $z$ direction.

We write

$$\phi(\mathbf{r})=\phi^{(+)}(\mathbf{r}),\quad z\geqslant 0,$$
$$=\phi^{(-)}(\mathbf{r}),\quad z\leqslant 0, \tag{2.16}$$

[6] The radiation condition requires that the Green's function describe outgoing spherical waves; this imposes the restriction $\arg(k^2-k_x^2-k_x^2)^{\frac{1}{2}}\geqslant 0$.

and consider the application of Green's second scalar identity in the form

$$\int[\Gamma(\mathbf{r}', \mathbf{r})(\nabla'^2+k^2)\phi(\mathbf{r}')$$
$$-\phi(\mathbf{r}')(\nabla'^2+k^2)\Gamma(\mathbf{r}', \mathbf{r})]d\tau'$$
$$=\int\left[\Gamma(\mathbf{r}', \mathbf{r})\frac{\partial}{\partial n'}\phi(\mathbf{r}')\right.$$
$$\left.-\phi(\mathbf{r}')\frac{\partial}{\partial n'}\Gamma(\mathbf{r}', \mathbf{r})\right]dS', \quad (2.17)$$

where the derivative in the surface integral is taken in the direction of the outward normal at each point.

In the half-space $z\geqslant 0$, we find with the use of (2), (3), (6), and (15) in (17), that the wave function at any point assumes the form

$$\phi^{(+)}(\mathbf{r})=\int_{S_1}\phi^{(+)}(\mathbf{\varrho}')\frac{\partial}{\partial z'}\Gamma(\mathbf{r}; x', y', 0)dS'$$
$$=\int_{S_1}\phi(\mathbf{\varrho}')\frac{\partial}{\partial z'}G(\mathbf{r}; x', y', 0)dS', \quad (2.18)$$

where $\phi^{(+)}(\mathbf{\varrho})=\frac{1}{2}\phi(\mathbf{\varrho})$. On the remote surface which intersects the plane of the screen, $\phi^{(+)}(\mathbf{r})$ and $\Gamma(\mathbf{r}, \mathbf{r}')$ have the character of diverging spherical waves, and the two terms of the integrand cancel. In the half-space $z\leqslant 0$ of the incident wave, the contribution from the corresponding surface integral supplies just the field in the absence of an aperture; thus

$$\phi^{(-)}(\mathbf{r})=2i\sin(kz\cos\vartheta')\exp(ik\mathbf{n}'\cdot\mathbf{\varrho})$$
$$-\int_{S_1}\phi(\mathbf{\varrho}')\frac{\partial}{\partial z'}G(\mathbf{r}; x', y', 0)dS'. \quad (2.19)$$

The reversal in sign of the integral in (19) as compared to (18) is a consequence of the oppositely directed normal derivatives at the plane of the aperture. Equality of the $z$ derivatives of the functions (18), (19) in the aperture provides an integral equation identical with (9), and completes the formal solution.

We shall confine our attention to the properties of the diffracted field at distances from the aperture large compared to its dimensions and the wave-length, since a rigorous and explicit solution of the integral equation (9) is not in general feasible. For effective use of approximate solutions to the integral equation, we cast the far field amplitude in a form which is stationary with respect to small variations of the aperture fields about their correct values.

On introducing the asymptotic form of the free space Green's function

$$G(\mathbf{r}, \mathbf{r}')\simeq\exp[ik(r-\mathbf{n}\cdot\mathbf{r}')]/4\pi r,$$
$$\mathbf{n}=\mathbf{r}/r, \quad r\rightarrow\infty \quad (2.20)$$

in (11), we obtain the transmitted field in the form of a diverging spherical wave,

$$\phi^{(+)}(\mathbf{r})\simeq-(ik/4\pi)\cos\vartheta(e^{ikr}/r)$$
$$\times\int_{S_1}\phi_{\mathbf{n}'}(\mathbf{\varrho}')\exp(-ik\mathbf{n}\cdot\mathbf{\varrho}')dS', \quad (2.21)$$

with explicit indication that the aperture field is generated by a wave incident in a definite direction $\mathbf{n}'$. It is clear from (11), (12) that a similar spherical wave appears on the other side of the screen. In terms of the amplitude

$$A(\mathbf{n}, \mathbf{n}')=-(ik/4\pi)\cos\vartheta$$
$$\times\int_{S_1}\phi_{\mathbf{n}'}(\mathbf{\varrho})\exp(-ik\mathbf{n}\cdot\mathbf{\varrho})dS, \quad (2.22)$$

of the spherical wave (21), we have

$$\lim_{r\rightarrow\infty} r^2|\phi^{(+)}(\mathbf{r})|^2=|A(\mathbf{n}, \mathbf{n}')|^2. \quad (2.23)$$

We assume as the expression for the time average energy flux per unit area,

$$\mathbf{S}=Re(1/ik)\phi^*(\mathbf{r})\nabla\phi(\mathbf{r}), \quad (2.24)$$

and verify that this corresponds to dissipationless transport of energy by the waves, for

$$\nabla\cdot\mathbf{S}=Re\nabla\cdot((1/ik)\phi^*(\mathbf{r})\nabla\phi(\mathbf{r}))$$
$$=Re(1/ik)(|\nabla\phi(\mathbf{r})|^2-k^2|\phi(\mathbf{r})|^2)=0. \quad (2.25)$$

Since $(\partial/\partial r)\phi^{(+)}(\mathbf{r})\sim ik\phi^{(+)}(\mathbf{r})$, $r\rightarrow\infty$, we find with the use of (23), (24) that the average power transmitted into the solid angle $d\Omega$, about the direction $\mathbf{n}$ is

$$P(\mathbf{n}, \mathbf{n}')d\Omega=|A(\mathbf{n}, \mathbf{n}')|^2d\Omega; \quad (2.26)$$

thus, the ratio of the total transmitted power to that falling on the aperture per unit area normal to the direction of the incident wave (transmission cross section) becomes

$$\sigma(\mathbf{n}')=\int_0^{2\pi}\int_0^{\pi/2}|A(\mathbf{n},\mathbf{n}')|^2\sin\vartheta d\vartheta d\varphi. \quad (2.27)$$

A simpler expression for the transmission cross section can be derived as follows. We note that the power transmitted normally through the aperture is the same as that appearing at any remote surface which intersects the plane of the screen; thus, from (24),

$$\sigma(\mathbf{n}')=Re(1/ik)\int_{S_1}\phi^{(+)*}(\boldsymbol{\varrho})(\partial/\partial z)\times\phi^{(+)}(x,y,0)dS. \quad (2.28)$$

(The same result is obtained on integrating (25) in the volume bounded by this surface and the plane $z=0$, and observing that the wave function and its complex conjugate vanish on the screen.) Using (13), (14) the expression (28) becomes

$$\begin{aligned}\sigma(\mathbf{n}')&=\tfrac{1}{2}Re\cos\vartheta'\int_{S_1}\phi_{\mathbf{n}'}{}^*(\boldsymbol{\varrho})\exp(ik\mathbf{n}'\cdot\boldsymbol{\varrho})dS\\&=-(2\pi/k)Re\, iA(\mathbf{n}',\mathbf{n}')^*\\&=-(2\pi/k)Im\, A(\mathbf{n}',\mathbf{n}'). \quad (2.29)\end{aligned}$$

The transmission cross section is thus proportional to the imaginary part of the amplitude of the transmitted spherical wave observed in the direction of the incident plane wave. A relation of the form (29) holds generally in scattering problems, its physical interpretation being that a decrease in amplitude of the incident wave is consequent to the generation of scattered waves.

## 3. VARIATIONAL PRINCIPLE FOR DIFFRACTED WAVE AMPLITUDE

If we multiply through in the integral equation (2.9) by $\phi_{\mathbf{n}''}(\boldsymbol{\varrho})$ and integrate over the area of the aperture, there results

$$\cos\vartheta'\int_{S_1}\phi_{\mathbf{n}''}(\boldsymbol{\varrho})\exp(ik\mathbf{n}'\cdot\boldsymbol{\varrho})dS=(1/ik)\int_{S_1}\phi_{\mathbf{n}''}(\boldsymbol{\varrho})K(\boldsymbol{\varrho},\boldsymbol{\varrho}')\phi_{\mathbf{n}'}(\boldsymbol{\varrho}')dSdS'. \quad (3.1)$$

Since the right-hand member of this equation is symmetrical in $\mathbf{n}'$, $\mathbf{n}''$ (or the angular coordinates $\vartheta'$, $\varphi'$, and $\vartheta''$, $\varphi''$), division by the left-hand member and a similar term in which $\mathbf{n}'$ and $\mathbf{n}''$ are interchanged, and use of (2.22), yields

$$\frac{1}{A(\mathbf{n}'',\mathbf{n}')}=\frac{1}{A(-\mathbf{n}',-\mathbf{n}'')}=\frac{(4\pi/k^2)\int_{S_1}\phi_{\mathbf{n}'}(\boldsymbol{\varrho})K(\boldsymbol{\varrho},\boldsymbol{\varrho}')\phi_{-\mathbf{n}''}(\boldsymbol{\varrho}')dSdS'}{\cos\vartheta'\cos\vartheta''\int_{S_1}\phi_{\mathbf{n}'}(\boldsymbol{\varrho})\exp(-ik\mathbf{n}''\cdot\boldsymbol{\varrho})dS\int_{S_1}\phi_{-\mathbf{n}''}(\boldsymbol{\varrho})\exp(ik\mathbf{n}'\cdot\boldsymbol{\varrho})dS}. \quad (3.2)$$

Here $\phi_{-\mathbf{n}}(\boldsymbol{\varrho})$ denotes the aperture field generated by a plane wave incident in the direction opposite to $\mathbf{n}$. Equality of the wave amplitudes $A(\mathbf{n}'',\mathbf{n}')$ and $A(-\mathbf{n}',-\mathbf{n}'')$ describes a reciprocity condition for incidence and observation along a pair of directions in space.

The expression (2) is homogeneous in the fields $\phi_{\mathbf{n}'}(\boldsymbol{\varrho})$, $\phi_{-\mathbf{n}''}(\boldsymbol{\varrho})$ and stationary with respect to independent first-order variations about their correct values (determined by integral equations of the form (2.9)). Thus, on performing such a variation, we obtain

$$\begin{aligned}\delta A(\mathbf{n}'',\mathbf{n}')\int_{S_1}\phi_{\mathbf{n}'}(\boldsymbol{\varrho})K(\boldsymbol{\varrho},\boldsymbol{\varrho}')\phi_{-\mathbf{n}''}(\boldsymbol{\varrho}')dSdS'\\=\int_{S_1}\delta\phi_{\mathbf{n}'}(\boldsymbol{\varrho})dS\Big[(k^2/4\pi)\cos\vartheta'\cos\vartheta''\\\times\exp(-ik\mathbf{n}''\cdot\boldsymbol{\varrho})\int_{S_1}\phi_{-\mathbf{n}''}(\boldsymbol{\varrho})\exp(ik\mathbf{n}'\cdot\boldsymbol{\varrho})dS\\-A(\mathbf{n}'',\mathbf{n}')\int_{1}K(\boldsymbol{\varrho},\boldsymbol{\varrho}')\phi_{-\mathbf{n}''}(\boldsymbol{\varrho}')dS'\Big]\\+\int_{S_1}\delta\phi_{-\mathbf{n}''}(\boldsymbol{\varrho})dS\Big[(k^2/4\pi)\cos\vartheta'\cos\vartheta''\\\times\exp(ik\mathbf{n}'\cdot\boldsymbol{\varrho})\int_{S_1}\phi_{\mathbf{n}'}(\boldsymbol{\varrho})\exp(-ik\mathbf{n}''\cdot\boldsymbol{\varrho})dS\\-A(\mathbf{n}'',\mathbf{n}')\int_{S_1}K(\boldsymbol{\varrho},\boldsymbol{\varrho}')\phi_{\mathbf{n}'}(\boldsymbol{\varrho}')dS'\Big]. \quad (3.3)\end{aligned}$$

If $A$ is stationary for arbitrary variations, the quantities within brackets on the right side of (3) must vanish. From the latter of these we find

$$ik\cos\vartheta'\exp(ik\mathbf{n}'\cdot\boldsymbol{\rho})\left[-(ik/4\pi)(\cos\vartheta''/A(\mathbf{n}'',\mathbf{n}'))\int_{S_1}\phi_{\mathbf{n}'}(\boldsymbol{\rho})\exp(-ik\mathbf{n}''\cdot\boldsymbol{\rho})dS\right]=\int_{S_1}K(\boldsymbol{\rho},\boldsymbol{\rho}')\phi_{\mathbf{n}'}(\boldsymbol{\rho}')dS'$$

which, with the exception of the (constant) factor in brackets, is just the integral equation (2.9) for $\phi_{\mathbf{n}'}(\boldsymbol{\rho})$.[7] Similarly, $\phi_{-\mathbf{n}''}(\boldsymbol{\rho})$ obeys a corresponding integral equation.

We next examine the forms assumed by the transmission cross section (Cf. (2.29), (2)),

$$\sigma(\mathbf{n})=-\tfrac{1}{2}k\cos^2\vartheta\, Im\frac{\int_{S_1}\phi_{\mathbf{n}}(\boldsymbol{\rho})\exp(-ik\mathbf{n}\cdot\boldsymbol{\rho})dS\int_{S_1}\phi_{-\mathbf{n}}(\boldsymbol{\rho})\exp(ik\mathbf{n}\cdot\boldsymbol{\rho})dS}{\int_{S_1}\phi_{\mathbf{n}}(\boldsymbol{\rho})K(\boldsymbol{\rho},\boldsymbol{\rho}')\phi_{-\mathbf{n}}(\boldsymbol{\rho}')dSdS'},\qquad(3.4)$$

in the limits of low and high frequencies, respectively. The frequency expansion of the Green's function,

$$\begin{aligned}G(\mathbf{r},\mathbf{r}')&=\exp(ik|\mathbf{r}-\mathbf{r}'|)/4\pi|\mathbf{r}-\mathbf{r}'|\\&=G_s(\mathbf{r},\mathbf{r}')+(ik/4\pi)-(k^2/8\pi)|\mathbf{r}-\mathbf{r}'|\\&\quad-(ik^3/24\pi)|\mathbf{r}-\mathbf{r}'|^2+O(k^4),\end{aligned}$$

where

$$G_s(\mathbf{r},\mathbf{r}')=1/4\pi|\mathbf{r}-\mathbf{r}'|$$

is the static value, yields

$$\begin{aligned}K(\boldsymbol{\rho},\boldsymbol{\rho}')&\doteq K_s(\boldsymbol{\rho},\boldsymbol{\rho}')+(ik^3/12\pi),\\K_s(\boldsymbol{\rho},\boldsymbol{\rho}')&=(1/4\pi)(\partial/\partial z)(\partial/\partial z')(1/|\mathbf{r}-\mathbf{r}'|)_{z,z'=0}\end{aligned}\qquad(3.5)$$

in the lowest order of real and imaginary terms. Thus, in the static limit the integral equation (2.9) becomes

$$ik\cos\vartheta\doteq\int_{S_1}\phi_{\mathbf{n}}(\boldsymbol{\rho}')K_s(\boldsymbol{\rho},\boldsymbol{\rho}')dS',\qquad(3.6)$$

which shows that the corresponding aperture field $\phi_{\mathbf{n}}(\boldsymbol{\rho})$ is wholly imaginary and independent of the incident direction, save for the constant factor $\cos\vartheta$. Introducing (5) in (4) and omitting indices as well as scale factors for the aperture fields, we find as the leading term in the frequency expansion of the transmission cross section,

$$\sigma(\mathbf{n})=\frac{k^4}{24\pi}\cos^2\vartheta\frac{\left(\int_{S_1}\phi(\boldsymbol{\rho})dS\right)^4}{\left(\int_{S_1}\phi(\boldsymbol{\rho})K_s(\boldsymbol{\rho},\boldsymbol{\rho}')\phi(\boldsymbol{\rho}')dSdS'\right)^2},\quad k\rightarrow 0.\qquad(3.7)$$

The frequency (or wave-length) dependence contained in (7) agrees with that of Rayleigh's general theory of diffraction by obstacles small compared to the wave-length. In addition, the correct angular variation of the cross section is secured independently of the form of the aperture field. As remarked in the introduction, the class of admissible trial aperture fields in (7) is restricted to those satisfying the boundary condition (2.3) on the rim of the screen. Additional terms in the frequency expansion of the cross section can be obtained; these, in common with (7), are independent of the scale of the aperture fields.

At high frequencies, we assume as the aperture field

$$\phi_{\mathbf{n}}(\boldsymbol{\rho})=\Phi_{\mathbf{n}}(\boldsymbol{\rho})\phi_{\mathbf{n}}^{\text{inc}}(\boldsymbol{\rho})=\Phi_{\mathbf{n}}(\boldsymbol{\rho})\exp(ik\mathbf{n}\cdot\boldsymbol{\rho}),$$
$$\phi_{-\mathbf{n}}(\boldsymbol{\rho})=\phi_{\mathbf{n}}^*(\boldsymbol{\rho}),\qquad(3.8)$$

where $\Phi_{\mathbf{n}}(\boldsymbol{\rho})$ is a real function which is slowly varying over distances comparable to the wave-length. Inserting this field and its complex con-

[7] The bracketed term is actually equal to unity, by (2.22), so that the integral equations are identical; however, this is not essential to the argument.

jugate in (4), we obtain

$$\sigma(\mathbf{n}) = -\tfrac{1}{2}k\cos^2\vartheta\, Im \frac{\left(\int_{S_1}\Phi_{\mathbf{n}}(\boldsymbol{\varrho})dS\right)^2}{\int_{S_1}\Phi_{\mathbf{n}}(\boldsymbol{\varrho})\exp(ik\mathbf{n}\cdot\boldsymbol{\varrho})K(\boldsymbol{\varrho},\boldsymbol{\varrho}')\Phi_{\mathbf{n}}(\boldsymbol{\varrho}')\exp(-ik\mathbf{n}\cdot\boldsymbol{\varrho}')dSdS'}. \tag{3.9}$$

Employing the Fourier integral representation (2.5), the integral in the denominator of (9) becomes

$$I = \frac{i}{8\pi^2}\int_{S_1}\int_{-\infty}^{\infty}(k^2-k_x^2-k_y^2)^{\frac{1}{2}}\Phi_{\mathbf{n}}(\boldsymbol{\varrho})\Phi_{\mathbf{n}}(\boldsymbol{\varrho}')$$
$$\times\exp[i\{\alpha(x-x')+\beta(y-y')\}]dk_xdk_ydSdS',$$

where

$$\alpha = k\sin\vartheta\cos\varphi + k_x, \quad \beta = k\sin\vartheta\sin\varphi + k_y.$$

As $k\rightarrow\infty$, the phase variations of the exponential factors in the integrand occur with increasing rapidity, so that the range of the variables $x,y$ may be extended over the entire plane $z=0$. Introducing

$$x-x'=\xi, \quad y-y'=\eta$$

and ignoring the variations of $\xi$, $\eta$ in the argument of $\Phi_{\mathbf{n}}(\boldsymbol{\varrho})$, we find that integration with respect to the new variables leads to

$$I \simeq \frac{i}{2}\int_{S_1}\int_{-\infty}^{\infty}(k^2-k_x^2-k_y^2)^{\frac{1}{2}}\Phi_{\mathbf{n}}^2(\boldsymbol{\varrho}')$$
$$\times\delta(k\sin\vartheta\cos\varphi+k_x)$$
$$\times\delta(k\sin\vartheta\sin\varphi+k_y)dk_xdk_ydS'$$
$$= \frac{i}{2}k\cos\vartheta\int_{S_1}\Phi_{\mathbf{n}}^2(\boldsymbol{\varrho})dS,$$

and finally,

$$\sigma(\mathbf{n}) = \cos\vartheta\frac{\left(\int_{S_1}\Phi_{\mathbf{n}}(\boldsymbol{\varrho})dS\right)^2}{\int_{S_1}\Phi_{\mathbf{n}}^2(\boldsymbol{\varrho})dS}, \quad k\rightarrow\infty. \tag{3.10}$$

The expression (10) is stationary for first-order variations of the function $\Phi_{\mathbf{n}}(\boldsymbol{\varrho})$ about a constant value, the cross section being equal to the projected area of the aperture on a plane normal to the direction of the incident wave. This corresponds to the limit of geometrical optics, in which the aperture field is identified with the incident field, and the energy falling on the aperture is transmitted without diffraction effects. By appealing to the Schwarz inequality, it follows that the cross section obtained from (10) with any other form of the function $\Phi_{\mathbf{n}}(\boldsymbol{\varrho})$ is necessarily smaller than the geometrical optics result. The representation of the cross section by an asymptotic series in reciprocal powers of $k$ requires a more elaborate analysis.

## 4. DIFFRACTION BY A CIRCULAR APERTURE

In this section we illustrate the utility of the variational formulation by applying it to the case of normal incidence of a plane wave on a circular aperture. We calculate, in particular, the transmission coefficient $t$ of the aperture as a function of the characteristic parameter,

$$ka = 2\pi(\text{radius of aperture})/(\text{wave-length}).$$

The expression (3.2) for the diffracted amplitude $A$ observed in the direction of incidence ($\vartheta=0$) involves a single aperture field; on introducing polar coordinates $\rho$, $\varphi$ in the plane of the aperture (with origin at its center), and noting that the common aperture field is independent of $\varphi$, we obtain

$$A = \frac{(k^2/4\pi)\left(\int_{S_1}\phi(\rho)\rho d\rho d\varphi\right)^2}{\int_{S_1}\phi(\rho)K(\boldsymbol{\varrho},\boldsymbol{\varrho}')\phi(\rho')\rho d\rho d\varphi\rho' d\rho' d\varphi'}. \tag{4.1}$$

To evaluate $A$ we expand the aperture field in the complete set of functions (see Appendix 1 and reference 4),

$$\phi(\rho) = \sum_{n=1}^{\infty}A_n(1-(\rho^2/a^2))^{n-\frac{1}{2}} \tag{4.2}$$

where the $A_n$ are arbitrary coefficients; the individual terms of this expansion satisfy the boundary condition (2.3) on the rim of the screen. We define

$$B_n=\int_{S_1}(1-(\rho^2/a^2))^{n-\frac{1}{2}}\rho d\rho d\varphi$$
$$=2\pi a^2/(2n+1), \quad (4.3)$$

and

$$C_{mn}=\int_{S_1}(1-(\rho^2/a^2))^{m-\frac{1}{2}}K(\boldsymbol{\varrho},\boldsymbol{\varrho}')(1-(\rho'^2/a^2))^{n-\frac{1}{2}}$$
$$\times\rho d\rho d\varphi\rho' d\rho' d\varphi'=C_{nm}; \quad (4.4)$$

thus (1) becomes

$$A\sum_{m,n=1}^{\infty}A_mA_nC_{mn}=(k^2/4\pi)(\sum_{n=1}^{\infty}A_nB_n)^2. \quad (4.5)$$

On differentiating (5) with respect to $A_m$, it follows, in view of the stationary character of $A$, that[8]

$$A\sum_{n=1}^{\infty}A_nC_{mn}=(k^2/4\pi)B_m\sum_{n=1}^{\infty}A_nB_n,$$
$$m=1,\cdots. \quad (4.6)$$

We next define a set of coefficients $D_n$ by the relation

$$A_n=(k^2/4\pi A)D_n\sum_{n=1}^{\infty}A_nB_n \quad (4.7)$$

and find, on multiplying through by $B_n$, and summing over $n$,

$$A=(k^2/4\pi)\sum_{n=1}^{\infty}B_nD_n. \quad (4.8)$$

Finally, by inserting (7) in (6), we obtain the infinite set of inhomogeneous linear equations to determine the coefficients $D_n$,

$$\sum_{n=1}^{\infty}C_{mn}D_n=B_m, \quad m=1,\cdots \quad (4.9)$$

with which to calculate $A$ from (8).

A rigorous solution of (9) is not attempted; rather, we reduce these to a finite set of linear equations by placing $B_n, D_n=0$, $n>N$. The corresponding approximation to $A$ is given by

$$A^{(N)}=(k^2/4\pi)\sum_{n=1}^{N}B_nD_n, \quad (4.10)$$

where

$$\sum_{n=1}^{N}C_{mn}D_n=B_m, \quad m=1,\cdots,N. \quad (4.11)$$

It will now be demonstrated that a few terms of (10) suffice to give a very accurate approximation to $A$. For this purpose we require explicit knowledge of the quantities $B_n$, $C_{mn}$. $B_n$ is obtained directly from (3); to determine $C_{mn}$ we make use of the integral representation for the free-space Green's function,[9]

$$\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{4\pi|\mathbf{r}-\mathbf{r}'|}=\frac{1}{4\pi}\int_0^\infty J_0(\zeta R)$$
$$\times\frac{\zeta\exp[-(\zeta^2-k^2)^{\frac{1}{2}}(z-z')]}{(\zeta^2-k^2)^{\frac{1}{2}}}d\zeta, \quad z-z'\geqslant 0, \quad (4.12)$$
$$R=(\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi'))^{\frac{1}{2}},$$

where $J_0$ denotes the Bessel function of order zero, and the path of integration avoids the singularity $\zeta=k$ by an indentation below the singular point, and

$$\arg(\zeta^2-k^2)^{\frac{1}{2}}=0, \quad \zeta>k; \quad =-\pi/2, \quad \zeta<k.$$

Inserting (12) in (4), and employing the addition theorem for Bessel functions,

$$J_0(\zeta R)=\sum_{n=0}^{\infty}(2-\delta_{0n})J_n(\zeta\rho)J_n(\zeta\rho')\cos n(\varphi-\varphi'),$$
$$\delta_{pq}=\begin{matrix}0, & p\neq q\\ 1, & p=q\end{matrix} \quad (4.13)$$

we find, on performing the angular integrations,

$$C_{mn}=-\pi\int_0^\infty\zeta(\zeta^2-k^2)^{\frac{1}{2}}d\zeta\int_0^a\rho\left(1-\frac{\rho^2}{a^2}\right)^{m-\frac{1}{2}}$$
$$\times J_0(\zeta\rho)d\rho\int_0^a\rho'\left(1-\frac{\rho'^2}{a^2}\right)^{n-\frac{1}{2}}J_0(\zeta\rho')d\rho'.$$

[8] We use a symmetrical formulation, although it is only the relative amplitudes $A_n/A_1$ that are important for the variational calculation.

[9] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Teddington, 1945), p. 416.

With the change of variables

$$\rho = a\sin\vartheta,\quad \rho' = a\sin\vartheta',$$

and use of Sonine's first finite integral,[10]

$$\int_0^{\pi/2} J_\mu(z\sin\vartheta)\sin^{\mu+1}\vartheta\cos^{2\nu+1}\vartheta d\vartheta$$
$$=2^\nu\Gamma(\nu+1)z^{-(\nu+1)}J_{\mu+\nu+1}(z),\ Re\mu,\ \nu>-1,\quad (4.14)$$

we obtain

$$C_{mn} = -2\pi a(2/ka)^{m+n-2}\Gamma(m+\tfrac{1}{2})\Gamma(n+\tfrac{1}{2})$$
$$\times\int_0^\infty v^{-(m+n)}(v^2-1)^{\frac{1}{2}}J_{m+\frac{1}{2}}(kav)$$
$$\times J_{n+\frac{1}{2}}(kav)dv.\quad (4.15)$$

The integral

$$F_{mn}(\alpha) = \int_0^\infty v^{-(m+n)}(v^2-1)^{\frac{1}{2}}$$
$$\times J_{m+\frac{1}{2}}(\alpha v)J_{n+\frac{1}{2}}(\alpha v)dv\quad (4.16)$$

can be reduced to a suitable form for numerical evaluation. An indication of this procedure, which is the more laborious for large values of $m,n$, is given in Appendix 2.

Resolving $F_{mn}(\alpha)$ into real and imaginary parts,

$$F_{mn}(\alpha) = R_{mn}(\alpha) - iI_{mn}(\alpha),\quad (4.17)$$

where

$$I_{mn}(\alpha) = \int_0^1 v^{-(m+n)}(1-v^2)^{\frac{1}{2}}$$
$$\times J_{m+\frac{1}{2}}(\alpha v)J_{n+\frac{1}{2}}(\alpha v)dv\quad (4.18)$$

and

$$R_{mn}(\alpha) = \int_1^\infty v^{-(m+n)}(v^2-1)^{\frac{1}{2}}$$
$$\times J_{m+\frac{1}{2}}(\alpha v)J_{n+\frac{1}{2}}(\alpha v)dv,\quad (4.19)$$

we find, for the first few values of $m$, $n$,

$$I_{11}(\alpha) = \frac{\alpha}{2\pi} - \frac{1}{4\pi\alpha} + \frac{1}{8\alpha^2}S_0(2\alpha)$$
$$-\frac{1}{16\alpha^3}\int_0^{2\alpha}S_0(t)dt - \frac{1}{4\alpha}\int_0^{2\alpha}t^{-1}S_1(t)dt,$$

$$I_{12}(\alpha) = \frac{\alpha^2}{9\pi} - \frac{1}{4\pi} - \frac{3}{8\pi\alpha^2} + \frac{3}{16\alpha^3}S_0(2\alpha) + \frac{1}{8\alpha^2}S_1(2\alpha)$$
$$+\frac{1}{8\alpha^2}\left(1-\frac{3}{4\alpha^2}\right)\int_0^{2\alpha}S_0(t)dt$$
$$-\frac{1}{4\alpha^2}\int_0^{2\alpha}t^{-1}S_1(t)dt,$$

$$I_{22}(\alpha) = \frac{\alpha^3}{36\pi} - \frac{\alpha}{12\pi} + \frac{3}{32\pi\alpha} - \frac{45}{64\pi\alpha^3}$$
$$-\frac{9}{32\alpha^2}\left(1-\frac{5}{4\alpha^2}\right)S_0(2\alpha)$$
$$+\frac{1}{16\alpha}\left(1+\frac{9}{2\alpha^2}\right)S_1(2\alpha)$$
$$-\frac{1}{16\alpha}\left(1-\frac{21}{4\alpha^2}+\frac{45}{16\alpha^4}\right)\int_0^{2\alpha}S_0(t)dt$$
$$-\frac{27}{64\alpha^3}\int_0^{2\alpha}t^{-1}S_1(t)dt,$$

$$R_{11}(\alpha) = \frac{1}{4\alpha}\left(1+\frac{1}{4\alpha^2}\right)\int_0^{2\alpha}J_0(t)dt$$
$$-\frac{1}{8\alpha^2}J_0(2\alpha) - \frac{1}{4\alpha}J_1(2\alpha),$$

$$R_{12}(\alpha) = \frac{1}{8\alpha^2}\left(1+\frac{3}{4\alpha^2}\right)\int_0^{2\alpha}J_0(t)dt$$
$$-\frac{3}{16\alpha^3}J_0(2\alpha) - \frac{3}{8\alpha^2}J_1(2\alpha),$$

$$R_{22}(\alpha) = \frac{1}{16\alpha}\left(1+\frac{3}{2\alpha^2}+\frac{45}{16\alpha^4}\right)\int_0^{2\alpha}J_0(t)dt$$
$$+\frac{9}{32\alpha^2}\left(1-\frac{5}{4\alpha^2}\right)J_0(2\alpha) - \frac{1}{16\alpha}\left(1+\frac{45}{4\alpha^2}\right)J_1(2\alpha).$$

In these expressions $S_0$, $S_1$ and $J_0$, $J_1$ denote the zero- and first-order Struve and Bessel functions, respectively.

With this information relating to $B_n$, $C_{mn}$, we return to (10), (11) and prepare for the detailed evaluation of the first two approximations to the

[10] See reference 9, p. 373.

transmission coefficient. With $N=2$, we find

$$A^{(2)}=\frac{k^2}{4\pi}\left[\frac{B_1^2}{C_{11}}+\frac{(B_2C_{11}-B_1C_{12})^2}{C_{11}(C_{11}C_{22}-C_{12}^2)}\right]$$
$$=A^{(1)}+\frac{k^2}{4\pi}\frac{(B_2C_{11}-B_1C_{12})^2}{C_{11}(C_{11}C_{22}-C_{12}^2)}; \quad (4.20)$$

thus, using (2.29),

$$t^{(2)}=-\frac{2}{ka^2}Im A^{(2)}$$
$$=-\frac{k}{2\pi a^2}Im\left[\frac{B_1^2}{C_{11}}+\frac{(B_2C_{11}-B_1C_{12})^2}{C_{11}(C_{11}C_{22}-C_{12}^2)}\right]$$
$$=t^{(1)}-\frac{k}{2\pi a^2}Im\left[\frac{(B_2C_{11}-B_1C_{12})^2}{C_{11}(C_{11}C_{22}-C_{12}^2)}\right]. \quad (4.21)$$

Inserting the expressions for $B_1$, $B_2$, $C_{11}$, $C_{12}$ and $C_{22}$ in (21), it follows that

$$t^{(1)}=\frac{4}{9\pi}ka Im\frac{1}{F_{11}(ka)}$$
$$=\frac{4}{9\pi}ka\frac{I_{11}(ka)}{(I_{11}(ka))^2+(R_{11}(ka))^2}, \quad (4.22)$$

and

$$t^{(2)}=\frac{4}{9\pi}ka Im\frac{F_{22}(ka)-\frac{2}{5}kaF_{12}(ka)+(1/25)(ka)^2F_{11}(ka)}{F_{11}(ka)F_{22}(ka)-F_{12}^2(ka)}. \quad (4.23)$$

Numerical values of the transmission coefficients $t^{(1)}$, $t^{(2)}$ are given in Fig. 2, for the interval $0<ka<10$, together with exact values calculated by Bouwkamp. The approximation $t^{(1)}$, based on the aperture field $\phi^{(1)}(\rho)=[1-(\rho^2/a^2)]^{\frac{1}{2}}$ (a constant factor is of no importance here), exhibits considerable accuracy if $0<ka<2.5$. Using the expansions

$$I_{11}(\alpha)=\frac{2\alpha^3}{27\pi}-\frac{4\alpha^5}{675\pi}+\frac{16\alpha^7}{55125\pi}-\cdots,$$
$$R_{11}(\alpha)=\frac{1}{3}-\frac{\alpha^2}{15}+\frac{\alpha^4}{140}-\cdots$$

(see Appendix 2) in (22), we obtain the transmission coefficient in a form appropriate to small values of $ka$:

$$t^{(1)}=(8/27\pi^2)(ka)^4[1+0.32(ka)^2+0.049061(ka)^4+\cdots]. \quad (4.24)$$

The Rayleigh approximation, comprising the first term of (24), is obtained by determining the magnitude of $\phi^{(1)}(\rho)$ in the low frequency limit (see Appendix 1); its restricted range of validity is evident from Fig. 2.

On comparing (24) with Bouwkamp's result,

$$t=(8/27\pi^2)(ka)^4[1+0.32(ka)^2+0.027427(ka)^4-0.004393(ka)^6+\cdots], \quad (4.25)$$

we note that the numerical coefficients of the first two terms in the expansion of $t^{(1)}$ coincide with the exact values.[11]

The approximation $t^{(2)}$, derived from an aperture field of the form

$$\phi^{(2)}(\rho)=A_1(1-\rho^2/a^2)^{\frac{1}{2}}+A_2(1-\rho^2/a^2)^{\frac{3}{2}},$$

holds exactly for values of $ka$ ranging up to 4.5. The expansion of $t^{(2)}$ in powers of $ka$,

$$t^{(2)}=(8/27\pi^2)(ka)^4[1+0.32(ka)^2+0.047823(ka)^4+\cdots], \quad (4.26)$$

differs almost negligibly from $t^{(1)}$ in the numerical coefficient of the $(ka)^4$ term (within brackets), as can be anticipated from Fig. 2. This behavior indicates that successive variational approximations yield improved values for the entire set of coefficients in similar expansions.

It is of interest to examine the behavior of the approximations $t^{(1)}$, $t^{(2)}$ at high frequencies, where the correct transmission coefficient tends to the value unity. Using the asymptotic forms

$$I_{11}(\alpha)\simeq\alpha/2\pi,\quad I_{12}(\alpha)\simeq\alpha^2/9\pi,\quad I_{22}(\alpha)\simeq\alpha^3/36\pi,$$
$$R_{11}(\alpha),\quad R_{12}(\alpha),\quad R_{22}(\alpha)\simeq 0,\quad \alpha\rightarrow\infty$$

(see Appendix 2), in (22), (23) it follows that

$$t^{(1)}\simeq(4/9\pi)(ka/I_{11}(ka))\simeq(8/9),\quad ka\rightarrow\infty \quad (4.27)$$

[11] Equation (36) of King's paper (reference 3) is given incorrectly, and should be replaced by

$$\chi\simeq\frac{2}{3\pi}ika^3\left[1+\left(\frac{1}{5}-\frac{\sin^2\vartheta}{10}\right)(ka)^2-\frac{2i}{9\pi}(ka)^3\right]\cos\vartheta\frac{e^{ikr}}{r};$$

the corresponding asymptotic form of the diffracted wave on the far side of the screen is, to terms of relative order $(ka)^2$, (in our notation)

$$\phi^{(+)}(\mathbf{r})\simeq-(2/3\pi)k^2a^3\left[1+\left(\frac{1}{5}-\frac{\sin^2\vartheta}{10}\right)(ka)^2\right]\cos\vartheta\frac{e^{ikr}}{r}.$$

From this we find

$$t=\frac{2}{a^2}\int_0^{\pi/2}r^2|\phi^{(+)}(\mathbf{r})|^2\sin\vartheta d\vartheta=(8/27\pi^2)(ka)^4\times[1+(8/25)(ka)^2],$$

in agreement with (24), (25).

and

$$t^{(2)} \simeq \frac{4}{9\pi} ka \frac{I_{22}(ka) - (2/5)kaI_{12}(ka) + (1/25)(ka)^2 I_{11}(ka)}{I_{11}(ka)I_{22}(ka) - I_{12}{}^2(ka)} \simeq \frac{24}{25}. \tag{4.28}$$

In general, the asymptotic values of the high frequency approximations to the transmission coefficient are more conveniently obtained by use of (3.10). Recalling the expansion (2), and defining

$$B_n = \int_{S1} (1-\rho^2/a^2)^{n-\frac{1}{2}} \rho d\rho d\varphi = 2\pi a^2/(2n+1),$$

$$C_{mn} = \int_{S1} (1-\rho^2/a^2)^{m+n-1} \rho d\rho d\varphi = \pi a^2/(m+n),$$

we find, as in the derivation of (10),

$$t^{(N)} = 2 \sum_{n=1}^{N} D_n/(2n+1), \quad ka \to \infty \tag{4.29}$$

where

$$\sum_{n=1}^{N} D_n/(m+n) = 2/(2m+1), \quad m = 1, \cdots, N. \tag{4.30}$$

With the latter relations we readily obtain the values of $t^{(1)}$, $t^{(2)}$ given in (27), (28); moreover, it appears that for the $N$th approximation,

$$t^{(N)} \simeq 1 - [1/(2N+1)^2], \quad ka \to \infty. \tag{4.31}$$

For purposes of comparison, the transmission coefficient on the basis of the Kirchoff approximation is also included in Fig. 2. The wave function here is obtained by identifying the aperture field with the (constant) incident field,

$$\phi_K{}^{(+)}(\mathbf{r}) = 2\int_{S1} \frac{\partial}{\partial z'} G(\mathbf{r}; x', y', 0) dS', \tag{4.32}$$

and the transmission coefficient turns out to be

$$\begin{aligned} t_K &= 1 + J_1(2ka)/ka - (1/ka)\int_0^{2ka} J_0(t)dt \\ &\simeq 1, \quad ka \gg 1, \\ &\simeq (ka)^2/6, \quad ka \ll 1. \end{aligned} \tag{4.33}$$

An improvement of the Kirchoff result (33) at low frequencies cannot be obtained from the variational formulation by using a constant aperture field. The reason is that this field violates the boundary condition (2.3) on the rim of the screen, and the multiple integral in (1) involving a product of such functions together with $K(\boldsymbol{\varrho}, \boldsymbol{\varrho}')$ diverges. The correct field at high frequencies is constant over most of the area of the aperture, with variations confined to a distance of the order of a wave-length from the rim; it is only in the limit of vanishing wave-length that the boundary condition for the aperture field is relaxed.

The assistance of Miss J. Klimas and Mrs. D. Abkowitz with numerical calculations is gratefully acknowledged. This work was supported in part by the Office of Naval Research Contract N5ori-76, T.O.I.

## APPENDIX 1.

### Solution of the Integral Equation (2.9) for Normal Incidence on a Circular Aperture

On employing the differential equation (2.6) satisfied by the space Green's function, the integral equation for normal incidence,

$$ik = -\int_{S1} \phi(\rho')(\partial^2/\partial z^2) G(\rho, \varphi, 0; \rho', \varphi', 0)\rho' d\rho' d\varphi' \tag{A.1}$$

may be written

$$\left(\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} + k^2\right)\int_0^a \phi(\rho')\bar{G}(\rho, \rho')\rho' d\rho' = ik, \tag{A.2}$$

where

$$\bar{G}(\rho, \rho') = \int_0^{2\pi} G(\rho, \varphi, 0; \rho', \varphi', 0) d\varphi,$$

$$G(\rho, \varphi, 0; \rho', \varphi', 0) = \frac{\exp[ik(\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi'))^{\frac{1}{2}}]}{4\pi(\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi'))^{\frac{1}{2}}}.$$

Integrating (2) as an inhomogeneous Bessel differential equation, we find

$$\int_0^a \phi(\rho')\bar{G}(\rho, \rho')\rho' d\rho' = (i/k) + AJ_0(k\rho), \tag{A.3}$$

with the linearly independent function $N_0(k\rho)$ absent, by virtue of its singularity for $\rho = 0$. The integration constant $A$ is conveniently obtained from (3) on taking $\rho = 0$; thus

$$A = -(i/k) + \tfrac{1}{2}\int_0^a \phi(\rho)\exp(ik\rho)d\rho. \tag{A.4}$$

In order to solve the integral equation (3), we construct an expansion of $\bar{G}(\rho, \rho')$ in terms of a product of orthogonal functions which form a complete set in the region of the

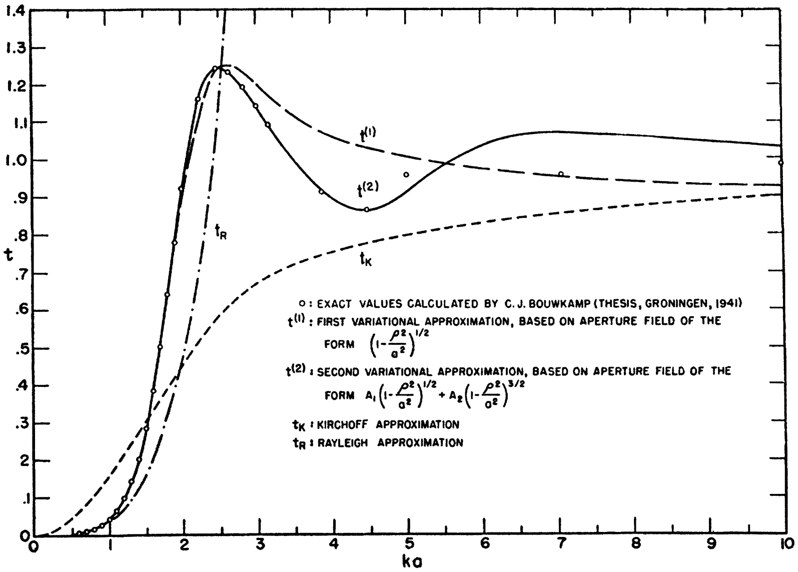


FIG. 2. Transmission coefficient of circular aperture for normally incident plane waves, as a function of the parameter $ka=2\pi$(radius of aperture/wavelength). Scalar wave function assumed to vanish on the screen.

aperture, $0\leqslant\rho\leqslant a$. This representation is facilitated by the change of variable $\rho=a\sin\xi$, whence (3) takes the form

$$\int_0^{\pi/2}\phi(a\sin\xi')\bar{G}(\xi,\xi')\cos\xi'\sin\xi'd\xi' =(i/ka^2)+(A/a^2)J_0(ka\sin\xi). \quad (\text{A}.5)$$

Referring to (2.5) we find, on introducing $k_x=\kappa\cos\psi$, $k_y=\kappa\sin\psi$, along with $x=a\sin\xi\cos\varphi$, $y=a\sin\xi\sin\varphi$ (and similarly for $x'$, $y'$),

$$\frac{\exp[ika(\sin^2\xi+\sin^2\xi'-2\sin\xi\sin\xi'\cos(\varphi-\varphi'))^{\frac{1}{2}}]}{4\pi(\sin^2\xi+\sin^2\xi'-2\sin\xi\sin\xi'\cos(\varphi-\varphi'))^{\frac{1}{2}}} =\frac{i}{8\pi^2}\int_0^{2\pi}d\psi\int_0^{\infty}\frac{\kappa d\kappa}{((ka)^2-\kappa^2)^{\frac{1}{2}}}\exp[i\kappa\sin\xi\cos(\varphi-\psi) -i\kappa\sin\xi'\cos(\varphi'-\psi)].$$

From the plane wave expansion in normalized Legendre functions,

$$\exp(ix\cos\Theta)=\sum_{l=0}^{\infty}i^l(2l+1)j_l(x)P_l(\cos\Theta) =2\sum_{l=0}^{\infty}\sum_{m=-l}^{l}i^lj_l(x)P_l^m(\cos\xi) \times P_l^m(\cos\xi')\exp[im(\varphi-\varphi')],$$

$$\cos\Theta=\cos\xi\cos\xi'+\sin\xi\sin\xi'\cos(\varphi-\varphi'),$$
$$j_l(x)=(\pi/2x)^{\frac{1}{2}}J_{l+\frac{1}{2}}(x)$$
$$\int_0^{\pi}P_l^m(\cos\xi)P_{l'}^m(\cos\xi)\sin\xi d\xi=\delta_{ll'},$$

we obtain, with $\xi'=\pi/2$,

$$\exp[ix\sin\xi\cos(\varphi-\varphi')]=2\sum_{l=0}^{\infty}\sum_{m=-l}^{l}i^lj_l(x)P_l^m(0) \times P_l^m(\cos\xi)\exp[im(\varphi-\varphi')]. \quad (\text{A}.7)$$

Inserting a pair of the latter type expansions in (6), and performing the integration with respect to $\psi$, it follows that

$$\frac{\exp[ika(\sin^2\xi+\sin^2\xi'-2\sin\xi\sin\xi'\cos(\varphi-\varphi'))^{\frac{1}{2}}]}{4\pi(\sin^2\xi-\sin^2\xi'-2\sin\xi\sin\xi'\cos(\varphi-\varphi'))^{\frac{1}{2}}} =\frac{i}{\pi}\sum_{l,l'=0}^{\infty}\sum_{m=-l}^{l}(i)^{l-l'}P_l^m(0)P_{l'}^m(0)P_l^m(\cos\xi) \times P_{l'}^m(\cos\xi')\exp[im(\varphi-\varphi')] \int_0^{\infty}\kappa((ka)^2-\kappa^2)^{-\frac{1}{2}}j_l(\kappa)j_{l'}(\kappa)d\kappa, \quad (\text{A}.8)$$

and finally

$$\bar{G}(\xi,\xi')=\frac{2i}{a}\sum_{l,l'=0}^{\infty}(-)^{l-l'}A_{ll'}P_{2l}(0)P_{2l'}(0) \times P_{2l}(\cos\xi)P_{2l'}(\cos\xi'),$$
$$A_{ll'}=\int_0^{\infty}\kappa((ka)^2-\kappa^2)^{-\frac{1}{2}}j_{2l}(\kappa)j_{2l'}(\kappa)d\kappa. \quad (\text{A}.9)$$

By contour integration, it may be shown that

$$A_{ll'}=\int_0^{ka}\kappa((ka)^2-\kappa^2)^{\frac{1}{2}}j_{2l}(\kappa)h_{2l'}^{(1)}(\kappa)d\kappa,\quad l'\leqslant l,$$
$$h_l^{(1)}(x)=(\pi/2x)^{\frac{1}{2}}(j_l(x)+in_l(x));$$

the latter form is suitable for the development of $A_{ll'}$ in powers of $ka$. In particular, as $ka\rightarrow 0$,

$$A_{ll'}=-[\pi i/2][\delta_{ll'}/(4l+1)],$$

and (9) reduces to a summation with a single index.

The representation (9) in terms of an orthogonal set of Legendre polynomials for the interval $0\leqslant\xi\leqslant\pi/2$,

$$\int_0^{\pi/2}P_{2l}(\cos\xi)P_{2l'}(\cos\xi)\sin\xi d\xi=\tfrac{1}{2}\delta_{ll'} \quad (\text{A}.10)$$

permits the solution of (5) by an expansion in these functions, provided a similar expansion of the inhomogeneous term $J_0(ka\sin\xi)$ is known. Thus, we write

$$\phi(a\sin\xi)\cos\xi=\sum_{l=0}^{\infty}B_lP_{2l}(\cos\xi)/P_{2l}(0), \quad (\text{A}.11)$$

where the relation ($\xi=\pi/2$)

$$\sum_{l=0}^{\infty} B_l=0 \qquad (A.12)$$

eliminates a singularity of the field on the rim of the screen. Placing $\varphi'=0$ in (7), and integrating with respect to $\varphi$,

$$\int_0^{2\pi} \exp[ix\sin\xi\cos\varphi]d\varphi=2\pi J_0(x\sin\xi)$$

$$=4\pi\sum_{l=0}^{\infty}(-)^l j_{2l}(x)P_{2l}(0)P_{2l}(\cos\xi),$$

which provides the desired expansion,

$$J_0(x\sin\xi)=\sum_{l=0}^{\infty} C_lP_{2l}(0)P_{2l}(\cos\xi),\quad C_l=2(-)^l j_{2l}(x). \qquad (A.13)$$

Inserting (9), (11), (13) in (5), and using (10), we obtain

$$\sum_{l,l'=0}^{\infty}(-)^{l-l'}A_{ll'}B_{l'}P_{2l}(0)P_{2l}(\cos\xi)$$

$$=(1/ka)-(iA/a)J_0(ka\sin\xi)$$

$$=\sum_{l=0}^{\infty}[(2/ka)\delta_{l0}-(iA/a)C_l]P_{2l}(0)P_{2l}(\cos\xi),$$

and by identifying the coefficients of $P_{2l}(\cos\xi)$,

$$\sum_{l'=0}^{\infty}(-)^{l-l'}A_{ll'}B_{l'}=(2/ka)\delta_{l0}-(iA/a)C_l. \qquad (A.14)$$

From (4),

$$A=-(i/k)+(a/2)\int_0^{\pi/2}\phi(a\sin\xi)\cos\xi\exp(ika\sin\xi)d\xi,$$

whence, using (11),

$$A=-\frac{i}{k}+a\sum_{l=0}^{\infty}B_lD_l,$$

$$D_l=(1/2P_{2l}(0))\int_0^{\pi/2}P_{2l}(\cos\xi)\exp(ika\sin\xi)d\xi. \qquad (A.15)$$

Substituting (15) in (14), we arrive at the set of inhomogeneous linear equations to determine the expansion coefficients $B_l$ in (11):

$$\sum_{l'=0}^{\infty}B_{l'}[(-)^{l-l'}A_{ll'}+iC_lD_{l'}]=(2/ka)\delta_{l0}-(1/ka)C_l,$$

$$l=0,1,\cdots. \qquad (A.16)$$

Finally, the relation (12) can be used to eliminate $B_0$ from (16), with the result

$$\sum_{l'=1}^{\infty}B_{l'}[(-)^l\{(-)^{l'}A_{ll'}-A_{l0}\}+iC_l\{D_{l'}-D_0\}]$$

$$=(2/ka)\delta_{l0}-(1/ka)C_l,\quad l=0,1,\cdots. \qquad (A.17)$$

The preceding analysis thus furnishes a complete set of functions for the expansion of the aperture field, as in (11), or equivalently, in the original coordinate

$$\phi(\rho)=\sum_{l=1}^{\infty}\bar{B}_l(1-\rho^2/a^2)^{l-\frac{1}{2}}. \qquad (A.18)$$

A study of the coefficients reveals that the leading term in the expansion of $B_l$ is of order $(ka)^{2l-1}$, $l\geqslant 1$. Thus (11), (18) represent expansions of the aperture field in ascending powers of the frequency; clearly, these contain the most suitable trial functions for use in the variational principle.

If we solve (17) for the coefficient $B_1$ (taking $B_2=B_3=\cdots=0$), it turns out that

$$B_1 \doteq i(4ka/3\pi),\quad ka\to 0,$$

and thus, since $B_0=-B_1$, we obtain from (11),

$$\phi^{(1)}(\rho)=-(4ika/\pi)(1-\rho^2/a^2)^{\frac{1}{2}}.$$

Inserting this aperture field in (2.21) and neglecting the phase variations of the exponential factor, we obtain the asymptotic form of the diffracted field,

$$\phi^{(+)}(\mathbf{r})\sim-(2/3\pi)k^2a^3\cos\vartheta(e^{ikr}/r),\quad r\to\infty,\quad ka\to 0 \qquad (A.19)$$

in agreement with Rayleigh's solution.[2] The transmission coefficient computed from (19) is equal to the leading term of (4.24) (see reference 11).

## APPENDIX 2.

## Evaluation of the Integrals $I_{mn}$, $R_{mn}$, Eqs. (4.18), (4.19)

We consider first the integral

$$I_{mn}(\alpha)=\int_0^1 v^{-(m+n)}(1-v^2)^{\frac{1}{2}}J_{m+\frac{1}{2}}(\alpha v)J_{n+\frac{1}{2}}(\alpha v)dv;$$

on introducing the product representation

$$J_\mu(z)J_\nu(z)=(2/\pi)\int_0^{\pi/2}J_{\mu+\nu}(2z\cos\vartheta)\cos(\mu-\nu)\vartheta d\vartheta,$$

$$Re(\mu+\nu)>-1, \qquad (A.20)$$

we find

$$I_{mn}(\alpha)=(2/\pi)\int_0^1 v^{-(m+n)}(1-v^2)^{\frac{1}{2}}dv\int_0^{\pi/2}J_{m+n+1}$$

$$\times(2\alpha v\cos\vartheta)\cos(m-n)\vartheta d\vartheta.$$

Writing $v=\sin\varphi$ and interchanging the orders of integration,

$$I_{mn}(\alpha)=(2/\pi)\int_0^{\pi/2}\cos(m-n)\vartheta d\vartheta\int_0^{\pi/2}J_{m+n+1}$$

$$\times(2\alpha\cos\vartheta\sin\varphi)\sin^{-(m+n)}\varphi\cos^2\varphi d\varphi.$$

The integral with respect to $\varphi$ can be performed by means of the general result[12]

$$\int_0^{\pi/2}J_\alpha(z\sin\varphi)\sin^{1-\alpha}\varphi\cos^{2\beta+1}\varphi d\varphi=2^{1-\alpha}z^{-(1+\beta)}$$

$$\times(1/\Gamma(\alpha))s_{\alpha+\beta,\beta-\alpha+1}(z),\quad Re\beta>-1, \qquad (A.21)$$

where $s_{\mu,\nu}(z)$ is a Lommel function. Taking $\mu=m+n+1$, $\nu=\frac{1}{2}$, it follows that

$$I_{mn}(\alpha)=(2/\pi)2^{-(m+n)}(1/\Gamma(m+n+1))$$

$$\times\int_0^{\pi/2}\cos(m-n)\vartheta\cdot(2\alpha\cos\vartheta)^{-\frac{3}{2}}$$

$$\times s_{m+n+(3/2),\,-(m+n-\frac{1}{2})}(2\alpha\cos\vartheta)d\vartheta.$$

[12] See reference 9, p. 374.

From the recurrence formula

$$s_{\mu+2,\nu}(z)=z^{\mu+1}-[(\mu+1)^2-\nu^2]s_{\mu,\nu}(z),$$

we obtain the more useful expression

$$I_{mn}(\alpha)=2^{1-m-n}(1/\pi\Gamma(m+n+1))\Bigg[\int_0^{\pi/2}(2\alpha\cos\vartheta)^{m+n-1}\cos(m-n)\vartheta d\vartheta-2(m+n)\int_0^{\pi/2}\cos(m-n)\vartheta\cdot(2\alpha\cos\vartheta)^{-\frac{1}{2}}\times s_{m+n-\frac{1}{2},-(m+n-\frac{1}{2})}(2\alpha\cos\vartheta)d\vartheta\Bigg]. \quad (A.22)$$

The Lommel functions with half integral indices, occurring in (22), are defined by

$$s_{\mu,-\mu}(z)=(\pi/2\sin\mu\pi)\bigg[J_\mu(z)\int_0^z z^\mu J_{-\mu}(z)dz-J_{-\mu}(z)\int_0^z z^\mu J_\mu(z)dz\bigg]$$

and are related to the Struve functions, in accordance with the equation

$$s_{\mu,-\mu}(z)=2^\mu\Gamma(1/2)\Gamma(\mu+\tfrac{1}{2})S_\mu(z).$$

Explicit forms for a few of the Lommel functions are as follows

$$s_{1/2,-1/2}(z)=z^{-\frac{1}{2}}[1-\cos z],$$

$$s_{3/2,-3/2}(z)=z^{-\frac{1}{2}}[z-2\sin z+(2/z)(1-\cos z)],$$

$$s_{5/2,-5/2}(z)=z^{-\frac{1}{2}}[z^2+4+8\cos z-(24/z)\sin z+(24/z)(1-\cos z)].$$

$$s_{7/2,-7/2}(z)=z^{-\frac{1}{2}}[z^3+6z+48\sin z+(72/z)+(288/z)\cos z-(720/z^2)\sin z+(720/z^3)(1-\cos z)],$$

Thus, returning to (22), we find, with $m=n=1$,

$$I_{11}(\alpha)=(\alpha/2\pi)-(1/2\pi\alpha^2)\int_0^{\pi/2}[\alpha\cos\vartheta-\sin(2\alpha\cos\vartheta)+(1/\alpha\cos\vartheta)\sin^2(\alpha\cos\vartheta)]\cos^{-2}\vartheta d\vartheta.$$

Writing

$$P(\alpha)=\int_0^{\pi/2}[\alpha\cos\vartheta-\sin(2\alpha\cos\vartheta)+(1/\alpha\cos\vartheta)\sin^2(\alpha\cos\vartheta)]\cos^{-2}\vartheta d\vartheta,$$

we obtain on integration by parts,

$$P(\alpha)=\int_0^{\pi/2}\bigg[\alpha\sin\vartheta-2\alpha\sin\vartheta\cos(2\alpha\cos\vartheta)+(\sin\vartheta/\cos\vartheta)\sin(2\alpha\cos\vartheta)-\frac{\sin\vartheta\sin^2(\alpha\cos\vartheta)}{\alpha\cos^2\vartheta}\bigg]\frac{\sin\vartheta}{\cos\vartheta}d\vartheta,$$

whence

$$2P(\alpha)=\int_0^{\pi/2}\bigg[\alpha\cos\vartheta-\sin(2\alpha\cos\vartheta)+\frac{\sin^2(\alpha\cos\vartheta)}{\alpha\cos\vartheta}\bigg]d\vartheta+2\alpha\int_0^{\pi/2}[1-\cos(2\alpha\cos\vartheta)]\sin^2\vartheta\cos^{-1}\vartheta d\vartheta.$$

In terms of the Struve function $S_\nu(z)$, which has the integral representation

$$S_\nu(z)=\frac{2(z/2)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^{\pi/2}\sin(z\cos\vartheta)\sin^{2\nu}\vartheta d\vartheta,$$

we readily find

$$P(\alpha)=(\alpha/2)-(\pi/4)S_0(2\alpha)+(\pi/8\alpha)\int_0^{2\alpha}S_0(t)dt+(\pi\alpha/2)\int_0^{2\alpha}t^{-1}S_1(t)dt$$

and

$$I_{11}(\alpha)=(\alpha/2\pi)-(1/4\pi\alpha)+(1/8\alpha^2)S_0(2\alpha)-(1/16\alpha^3)\int_0^{2\alpha}S_0(t)dt-(1/4\alpha)\int_0^{2\alpha}t^{-1}S_1(t)dt.$$

The integrals which occur in this expression for $I_{11}$ can be evaluated numerically with the existing tables of Struve functions. Other values of $I_{mn}$ are obtained in similar fashion.

A series expansion for $I_{mn}$ is easily derived from the product representation

$$J_\mu(z)J_\nu(z)=\sum_{p=0}^{\infty}\frac{(-)^p(z/2)^{\mu+\nu+2p}\Gamma(\mu+\nu+2p+1)}{p!\Gamma(\mu+\nu+p+1)\Gamma(\mu+p+1)\Gamma(\nu+p+1)}$$

with the result

$$I_{mn}(\alpha)=(\pi^{\frac{1}{2}}/4)\sum_{p=0}^{\infty}\times\frac{(-)^p(\alpha/2)^{m+n+2p+1}(m+n+2p+1)!}{(m+n+p+1)!\Gamma(m+p+\frac{3}{2})\Gamma(n+p+\frac{3}{2})\Gamma(p+\frac{5}{2})}. \quad (A.23)$$

This expansion has been used to verify the forms of $I_{11}$, $I_{12}$, and $I_{22}$ given in Section 4.

The asymptotic behavior of $I_{mn}$ for large values of the argument is obtained by noting that

$$I_{mn}(\alpha)=\alpha^{m+n-1}\int_0^\alpha(1-z^2/\alpha^2)^{\frac{1}{2}}z^{-(m+n)}J_{m+\frac{1}{2}}(z)J_{n+\frac{1}{2}}(z)dz\sim\alpha^{m+n-1}\int_0^\infty z^{-(m+n)}J_{m+\frac{1}{2}}(z)J_{n+\frac{1}{2}}(z)dz,\quad \alpha\to\infty,$$

and using the result

$$\int_0^\infty t^{-\lambda}J_\mu(t)J_\nu(t)dt=\frac{\Gamma(\lambda)\Gamma\Big(\frac{\mu+\nu-\lambda+1}{2}\Big)}{2^\lambda\Gamma\Big(\frac{\lambda+\nu-\mu+1}{2}\Big)\Gamma\Big(\frac{\lambda+\mu+\nu+1}{2}\Big)\Gamma\Big(\frac{\lambda+\mu-\nu+1}{2}\Big)},$$

whence

$$I_{mn}(\alpha)\backsimeq\frac{\alpha^{m+n-1}}{2^{m+n}(m+n)\Gamma(m+\frac{1}{2})\Gamma(n+\frac{1}{2})},\quad \alpha\to\infty. \quad (A.24)$$

For the integral

$$R_{mn}(\alpha)=\int_1^\infty v^{-(m+n)}(v^2-1)^{\frac{1}{2}}J_{m+\frac{1}{2}}(\alpha v)J_{n+\frac{1}{2}}(\alpha v)dv,$$

the representation (20) leads to

$$R_{mn}(\alpha)=(2/\pi)\int_0^{\pi/2}\cos(m-n)\vartheta d\vartheta\int_1^\infty v^{-(m+n)}\times(v^2-1)^{\frac{1}{2}}J_{m+n+1}(2\alpha v\cos\vartheta)dv.$$

Using the integral[13]

$$\int_z^\infty\frac{J_\nu(ax)}{x^{\nu-1}}(x^2-z^2)^\mu dx=\frac{2^\mu\Gamma(\mu+1)}{a^{\mu+1}z^{\nu-\mu-1}}J_{\nu-\mu-1}(az),\quad a\geqslant 0,\quad Re(\nu/2-\tfrac{1}{4})>Re\mu>-1, \quad (A.25)$$

[13] See reference 9, p. 417.

we find

$$R_{mn}(\alpha)=(2/\pi)^{\frac{1}{2}}\int_0^{\pi/2}\cos(m-n)\vartheta\cdot(2\alpha\cos\vartheta)^{-\frac{1}{2}}\times J_{m+n-\frac{1}{2}}(2\alpha\cos\vartheta)d\vartheta. \quad (A.26)$$

The Bessel function occurring in (26), whose order is half of an odd integer, is expressible in finite terms by means of algebraic and trigonometric functions of the argument. For example,

$$R_{11}(\alpha)=(1/2\pi\alpha^2)\int_0^{\pi/2}\left[\frac{\sin(2\alpha\cos\vartheta)}{2\alpha\cos\vartheta}-\cos(2\alpha\cos\vartheta)\right]\cos^{-2}\vartheta d\vartheta.$$

Writing

$$P(\beta)=\int_0^{\pi/2}[(1/\beta\cos\vartheta)\sin(\beta\cos\vartheta)-\cos(\beta\cos\vartheta)]\cos^{-2}\vartheta d\vartheta,$$

we find, on integration by parts,

$$P(\beta)=\int_0^{\pi/2}\left[\sin(\beta\cos\vartheta)\beta\sin\vartheta+\frac{\cos(\beta\cos\vartheta)\sin\vartheta}{\cos\vartheta}-\frac{\sin(\beta\cos\vartheta)\sin\vartheta}{\beta\cos^2\vartheta}\right]\sin\vartheta\cos^{-1}\vartheta d\vartheta,$$

whence

$$2P(\beta)=(\beta+1/\beta)\int_0^{\pi/2}\sin(\beta\cos\vartheta)\cos^{-1}\vartheta d\vartheta-\int_0^{\pi/2}\cos(\beta\cos\vartheta)d\vartheta-\beta\int_0^{\pi/2}\sin(\beta\cos\vartheta)\cos\vartheta d\vartheta$$
$$=\pi/2\left[(\beta+1/\beta)\int_0^{\beta}J_0(t)dt-J_0(\beta)-\beta J_1(\beta)\right]$$

and

$$R_{11}(\alpha)=P(2\alpha)/2\pi\alpha^2=\frac{1}{4\alpha}\left(1+\frac{1}{4\alpha^2}\right)\int_0^{2\alpha}J_0(t)dt-\frac{1}{8\alpha^2}J_0(2\alpha)-\frac{1}{4\alpha}J_1(2\alpha).$$

Integrals of the zero-order Bessel function are available in the literature.[14] Additional values of $R_{mn}$ are derived by similar procedures.

An expansion for $R_{mn}$ is obtained by inserting the Bessel function series

$$J_\nu(z)=\sum_{p=0}^{\infty}\frac{(-)^p(z/2)^{\nu+2p}}{p!\Gamma(\nu+p+1)}$$

in (26), and employing the integral

$$\int_0^{\pi/2}\cos^{m+n-2}\vartheta\cos(m-n)\vartheta d\vartheta=\frac{\pi\Gamma(m+n)}{(m+n-1)2^{m+n-1}\Gamma(m)\Gamma(n)},\quad m+n>1;$$

the result is

$$R_{mn}(\alpha)=(\pi^{\frac{1}{2}}/4)\sum_{p=0}^{\infty}\times\frac{(-)^p(\alpha/2)^{m+n+2p-2}(m+n+2p-2)!}{p!(m+p-1)!(n+p-1)!\Gamma(m+n+p+\frac{1}{2})}. \quad (A.27)$$

It follows from the asymptotic behavior of the Bessel function in (26) that $R_{mn}$ vanishes for infinitely large argument.

[14] A. N. Lowan and M. Abramowitz, J. Math. and Phys. **22**, 1 (1943).