PHYSICAL REVIEW VOLUME 74, NUMBER 8 OCTOBER 15, 1948

Phenomenological Quantum-Electrodynamics

J. M. JAUCH AND K. M. WATSON*
Department of Physics, State University of Iowa, Iowa City, Iowa

(Received May 24, 1948)

The quantization of the pure radiation field in a uniformly moving refractive medium is carried through both in the Hamiltonian and in the symmetrical four-dimensional form. The total energy and momentum are diagonalized. For a medium velocity larger than c/n there occur photons with negative energy.

PART I. THE PURE RADIATION FIELD AND ITS QUANTIZATION

1. Introduction

IN the classical description of electromagnetic phenomena in matter two points of view have been found useful. The first, which also is the historically older one, is the so-called phenomenological approach. In this the actual properties of matter are considered only in so far as they can be directly expressed in terms of the electromagnetic field quantities. For isotropic insulators these properties are embodied in the two characteristic constants: the dielectric constant ϵ and the magnetic permeability μ. This theory gives no information about the actual values of ϵ and μ nor of their dependence on the frequency of the radiation or dispersion.

In the second approach matter is described in terms of the fundamental properties of the constituent elementary particles. The field equations of the phenomenological theory appear as equations between certain average values of the field quantities in this theory taken over volumes containing a large number of particles. The actual values of the phenomenological constants ϵ and μ as well as their frequency dependence can be expressed in terms of the fundamental properties of the elementary particles. It is at once evident that the latter approach is more fundamental and far-reaching.

The quantum mechanical formulation of the electromagnetic field equations has so far been applied principally to the second case. That is, the quantization is applied to the field equations for a vacuum only and the interaction of the field with matter is usually introduced as a direct interaction of the field with elementary particles. As far as we are aware, no systematic development of the quantum theory of phenomenological electrodynamics exists. It is clear from the foregoing remark that from such a theory there cannot be expected any new results. That is, if the quantum electrodynamics for a vacuum were completely satisfactory it would contain all the results that could be derived from a phenomenological theory and in addition it would reduce the phenomenological constants to the fundamental parameters in the theory such as the mass m and charge e of the interacting particles, Planck's constant h and the velocity of light c. However there are primarily three reasons why it seemed to us desirable to have this theory developed. First, the theory presents certain interesting aspects from a purely formal point of view. Formally it appears as an extension or generalization of the quantum electrodynamics of a vacuum in which it goes over in the limiting case $\epsilon\rightarrow 1$, $\mu\rightarrow 1$. Second, it seems that certain types of problems can be handled much more easily in the phenomenological than in the atomistic theory. This is the case, for instance, for the discussion of the Čerenkov radiation and the radiation of charged particles passing a discontinuity in ϵ or μ. In the third place, we were interested in such a theory with a view to a possible application of the theory for a vacuum. It is well known that the hole theory predicts a polarizability of the vacuum. So far, this prediction could not be taken seriously because the field-proportional part of the polarization is divergent. A phenomenological approach might give new information as to the possible form which such a vacuum polarization can have.

The relativistic invariance of this theory is guaranteed if we use the four-dimensional tensor

* Now at Institute for Advanced Study, Princeton, New Jersey.

notation. However, it must be kept in mind that such a theory always introduces a preferred coordinate system, usually the system for which the medium is at rest. When applied to a polarizable vacuum the relativity principle in such a theory is thus violated in the sense that it is possible in principle to detect an absolute motion by referring it to the motion of the medium.[1]

2. The Classical Radiation Field

The fundamental equations for an electromagnetic field in a medium were given in the relativistically invariant form by Minkowski.[2] In the absence of charges and currents they are, for the medium at rest,

$$\nabla\times\mathbf{E}=-\dot{\mathbf{B}},\quad \nabla\times\mathbf{H}=\dot{\mathbf{D}}, \quad (1) \qquad \nabla\cdot\mathbf{B}=0,\quad \nabla\cdot\mathbf{D}=0, \quad (2)$$

and

$$\mathbf{D}=\epsilon\mathbf{E},\qquad \mathbf{B}=\mu\mathbf{H}. \quad (3)$$

For the relativistically invariant formulation we use the notation $x^0=-x_0=ct$, $x^k=x_k$, $(k=1, 2, 3)$. Furthermore we shall put $n^2=\epsilon\mu$ for the index of refraction. Equations (1) and (2) may then be written in tensor form by introducing the following two antisymmetrical tensors

$$F_{\lambda\mu}=\begin{Vmatrix} 0 & B_3 & -B_2 & +E_1 \\ -B_3 & 0 & B_1 & +E_2 \\ +B_2 & -B_1 & 0 & +E_3 \\ -E_1 & -E_2 & -E_3 & 0 \end{Vmatrix} \quad (4)$$

and

$$G_{\lambda\mu}=\begin{Vmatrix} 0 & B_3 & -B_2 & +n^2E_1 \\ -B_3 & 0 & +B_1 & +n^2E_2 \\ +B_2 & -B_1 & 0 & +n^2E_3 \\ -n^2E_1 & -n^2E_2 & -n^2E_3 & 0 \end{Vmatrix}. \quad (5)$$

It is easily seen that the field equations (1) and (2) are then identical with the tensor equations:

$$\partial_\lambda F_{\mu\nu}+\partial_\mu F_{\nu\lambda}+\partial_\nu F_{\lambda\mu}=0, \quad (6)$$

and

$$\partial_\mu G^{\lambda\mu}=0. \quad (7)$$

Here $\partial_\lambda\equiv\partial/\partial x^\lambda$.

It follows from Eq. (6), that $F_{\lambda\mu}$ can be derived from a four-vector potential

$$F_{\lambda\mu}=\partial_\lambda\phi_\mu-\partial_\mu\phi_\lambda. \quad (8)$$

With this choice of $F_{\lambda\mu}$, Eq. (6) is identically satisfied. The connections (3) between **E** and **D** on the one hand and **H** and **B** on the other can be written in relativistically covariant form by introducing the four-vector v^μ of the medium velocity. Here $v^\mu=dx^\mu/d\tau$, where τ is the proper time of the medium ($v^\mu v_\mu=-1$). Introducing further the abbreviation $\kappa=n^2-1$ we have,

$$G_{\lambda\mu}=F_{\lambda\mu}+\kappa(F_{\mu\sigma}v^\sigma v_\lambda-F_{\lambda\sigma}v^\sigma v_\mu). \quad (9)$$

Since in the system for which the medium is at rest, $v^k=0$, $v^0=1$, it is easy to verify that (9) reduces in this case to (3).

Equation (9) can be solved for $F_{\lambda\mu}$. We need simply remark that the transformation from G to F is the same as that from F to G with ϵ and μ replaced by $1/\epsilon$ and $1/\mu$ respectively or κ replaced by $-\kappa/(1+\kappa)$. We obtain in this way without further calculation:

$$F_{\lambda\mu}=G_{\lambda\mu}+\frac{\kappa}{1+\kappa}(G_{\lambda\sigma}v^\sigma v_\mu-G_{\mu\sigma}v^\sigma v_\lambda). \quad (9')$$

Equation (9′) can also be verified directly by substituting it back into the right-hand side of (9).

The field equations for the four-vector potential are obtained by introducing (9), together with (8), into (7). Here one can make use of the freedom of choice of the ϕ_λ by imposing on the ϕ_λ the subsidiary condition

$$\chi\equiv\partial^\rho\phi_\rho-\kappa v^\sigma\partial_\sigma v^\rho\phi_\rho=0. \quad (10)$$

We find then from (7)

$$(\partial^\mu\partial_\mu-\kappa\partial_\mu v^\mu\partial_\sigma v^\sigma)\Psi^\lambda=0, \quad (11)$$

where

$$\Psi^\lambda=\phi^\lambda-\kappa\phi^\sigma v_\sigma v^\lambda. \quad (12)$$

Since (12) can be solved for ϕ^λ by the linear transformation

$$\phi^\lambda=\Psi^\lambda+\frac{\kappa}{1+\kappa}\Psi^\sigma v_\sigma v^\lambda, \quad (13)$$

we see that the vector potential ϕ^λ also satisfies

$$(\partial^\mu\partial_\mu-\kappa\partial_\mu v^\mu\partial_\sigma v^\sigma)\phi^\lambda=0. \quad (14)$$

Equations (8) and (14), together with the subsidiary condition (10), may be considered as the fundamental equations of the theory.

[1] An interesting special case where the results are independent of the medium velocity is to be discussed in Part III.

[2] H. Minkowski, Göttinger Nachrichten, p. 53 (1908), Math. Ann. **68**, 472 (1910).

3. The Lagrangian for the Radiation Field

In order to prepare the theory for the quantum formalism we shall first show that the field equations can be derived from a variational principle. For that purpose we consider the Lagrange function

$$L=\int \mathfrak{L} d^3x, \tag{15}$$

with the Lagrange density function defined by

$$\mathfrak{L}=-\tfrac{1}{4}(\partial_\rho\phi_\sigma-\partial_\sigma\phi_\rho)(\partial^\rho\phi^\sigma-\partial^\sigma\phi^\rho)-\tfrac{1}{2}\chi^2 -\tfrac{1}{2}\kappa(\partial^\nu\phi^\rho-\partial^\rho\phi^\nu)(\partial_\rho\phi_\sigma-\partial_\sigma\phi_\rho)v^\sigma v_\nu. \tag{16}$$

The variational principle,

$$\delta\int L dx^0=0, \tag{17}$$

leads then at once to the differential equations

$$\partial^\lambda(\partial/\partial(\partial^\lambda\phi^\mu))=0. \tag{18}$$

Since

$$\frac{\partial\mathfrak{L}}{\partial(\partial^\lambda\phi^\mu)}=-(\partial_\lambda\phi_\mu-\partial_\mu\phi_\lambda)-\kappa[(\partial_\mu\phi_\sigma-\partial_\sigma\phi_\mu)v^\sigma v_\lambda -(\partial_\lambda\phi_\sigma-\partial_\sigma\phi_\lambda)v^\sigma v_\mu] -(g_{\lambda\mu}-\kappa v_\lambda v_\mu)\chi, \tag{19}$$

the Eqs. (18) are equivalent with

$$\partial^\lambda G_{\lambda\mu}+\partial^\lambda(g_{\lambda\mu}-\kappa v_\lambda v_\mu)\chi=0, \tag{20}$$

with $G_{\lambda\mu}$ given by (9).

From this follows, since $G_{\lambda\mu}=-G_{\mu\lambda}$,

$$(\partial^\lambda\partial_\lambda-\kappa\partial^\lambda v_\lambda\partial^\mu v_\mu)\chi=0. \tag{21}$$

Equation (20) is only equivalent to (7) or (14) if we impose the subsidiary condition $\chi=0$. This can be done by requiring that $\chi=0$ and $\dot{\chi}=0$ for all positions at a given time. It follows from (21), which is a partial differential equation of second order in the space and time variables, that $\chi=0$ holds then for all times. From this it is evident that (17) or (18) together with the condition (10) is equivalent to (14).

4. Hamiltonian Formalism

The canonically conjugate variables π_μ are defined by

$$\pi_\mu=\frac{\partial\mathfrak{L}}{\partial(\partial_0\phi^\mu)}$$

From (19) it follows that

$$\pi_\mu=G_\mu{}^0-(g_\mu{}^0-\kappa v^0 v_\mu)\chi, \tag{22}$$

where $g_\mu{}^\lambda$ is the metric tensor and χ is given by (10). In particular for $\mu=0$, we have

$$\pi_0=-(1+\kappa v_0{}^2)\chi, \tag{23}$$

and for $\mu=k$, $(k=1, 2, 3)$,

$$\pi_k=G_\mu{}^0+\kappa v^0 v_k\chi. \tag{24}$$

The Hamiltonian may then be written as

$$H=\int \mathfrak{H} d^3x$$

with

$$\mathfrak{H}=\pi_\mu\dot{\phi}^\mu-\mathfrak{L}. \tag{25}$$

In this expression we must consider the $\dot{\phi}^\mu$ as functions of π^μ, ϕ_λ and their space derivatives. The canonical equations

$$\dot{\pi}_\mu=-\delta H/\delta\phi^\mu,$$

and

$$\dot{\phi}^\mu=\delta H/\delta\pi_\mu, \tag{26}$$

are then equivalent to Eqs. (8), (9), and (20). The explicit expression of H is very complicated. In the following, however, it will not be used since we restrict ourselves to the special case in which the subsidiary condition (10) holds, which corresponds to the Maxwell field. In this case the Hamiltonian can be calculated explicitly. In order to do this we use a three-dimensional vector notation: $\mathbf{v}$, $\boldsymbol{\phi}$, $\boldsymbol{\pi}$, $\mathbf{V}$ and $\mathbf{W}$ are the vectors with components v_k, ϕ_k, π_k, $V_k=F_{0k}$ and $W_k=F_{ij}$ $(i, j, k$ cyclic). On account of (10) we have then $\pi_0=0$ and

$$\pi_k=G_k{}^0.$$

For $\dot{\boldsymbol{\phi}}$ we have in this notation

$$\dot{\boldsymbol{\phi}}=\mathbf{V}+\nabla\phi_0. \tag{27}$$

By separating space and time parts in (9) we obtain

$$\mathbf{V}(1+\kappa v_0{}^2)-\kappa\mathbf{v}(\mathbf{v}\cdot\mathbf{V})=\boldsymbol{\pi}+\kappa v^0(\mathbf{v}\times\mathbf{W}). \tag{28}$$

This equation can be solved for $\mathbf{V}$ in the following form

$$\mathbf{V}=\frac{1}{1+\kappa v_0{}^2}\left\{\boldsymbol{\pi}+\kappa v^0(\mathbf{v}\times\mathbf{W})+\frac{\kappa}{1+\kappa}(\boldsymbol{\pi}\cdot\mathbf{v})\mathbf{v}\right\}. \tag{29}$$

By substituting $\mathbf{V}$ into the right-hand side of

(27) we obtain $\dot{\boldsymbol{\phi}}$ as a function of the space derivatives of $\boldsymbol{\phi}$ and of $\boldsymbol{\pi}$ alone. On account of $\pi_0=0$ we shall not need the expression for $\dot{\phi}_0$. The expressions for $\dot{\phi}^\mu$ may be substituted into (25) and we obtain in this way for the Hamiltonian the following result:

$$\mathcal{H}=\frac{1}{2(1+\kappa v_0{}^2)}\left\{\boldsymbol{\pi}\cdot\boldsymbol{\pi}+2\kappa v^0(\mathbf{v}\times\mathbf{W})\cdot\boldsymbol{\pi}+\frac{\kappa}{1+\kappa}(\boldsymbol{\pi}\cdot\mathbf{v})^2+\kappa(\mathbf{v}\times(\mathbf{v}\times\mathbf{W}))\cdot\mathbf{W}\right\}+\tfrac{1}{2}\mathbf{W}\cdot\mathbf{W}+\boldsymbol{\pi}\cdot\nabla\phi_0. \quad (30)$$

The last term can be removed because it is equal to

$$\boldsymbol{\pi}\cdot\nabla\phi_0=\nabla\cdot(\boldsymbol{\pi}\cdot\phi_0)-(\nabla\cdot\boldsymbol{\pi})\phi_0.$$

The first term is a space divergence and will not contribute anything to the total Hamiltonian. The last term is zero, because from (20) and (22) it follows

$$\partial^\mu\pi_\mu=-2(1+\kappa v_0{}^2)\partial^0\chi+2\kappa v^0(\nabla\cdot\mathbf{v})\chi,$$

or with the help of (23)

$$\nabla\cdot\boldsymbol{\pi}=-(1+\kappa v_0{}^2)\partial^0\chi+2\kappa v^0(\nabla\cdot\mathbf{v})\chi=0,$$

on account of (10). We denote this new Hamiltonian density with $\mathcal{H}_0$:

$$\mathcal{H}_0=\frac{1}{2(1+\kappa v_0{}^2)}\left\{\boldsymbol{\pi}\cdot\boldsymbol{\pi}+2\kappa v^0(\mathbf{v}\times\mathbf{W})\cdot\boldsymbol{\pi}+\frac{\kappa}{1+\kappa}(\boldsymbol{\pi}\cdot\mathbf{v})^2+\kappa(\mathbf{v}\times(\mathbf{v}\times\mathbf{W}))\cdot\mathbf{W}\right\}+\tfrac{1}{2}\mathbf{W}\cdot\mathbf{W}. \quad (31)$$

5. Quantization of the Radiation Field

The quantum theory of the radiation field is obtained by interpreting the field variables as operators which satisfy certain commutation rules. In doing this we can choose either the Schrödinger or the Heisenberg representation. In the former the canonical variables are constant operators depending only on the space variables $\mathbf{x}$ but not explicitly on the time $t=x^0$. The Schrödinger functional $\Omega(q, t)$ is then time independent and satisfies the Schrödinger equation

$$H\Omega(q, t)=i\dot{\Omega}(q, t). \quad (32)$$

In the Heisenberg representation the Schrödinger functional describing the state of a system is constant and the operators depend explicitly on time. The connection between the time-dependent and the time-independent operators is given by[3]

$$\phi_\lambda(x)=e^{iHt}\phi_\lambda(\mathbf{x})e^{-iHt},$$

and

$$\pi_\mu(x)=e^{iHt}\pi_\mu(\mathbf{x})e^{-iHt}. \quad (33)$$

In the canonical formalism we postulate between the time-independent field variables the commutation rules

$$[\phi_\lambda(\mathbf{x}),\ \pi_\mu(\mathbf{x}')]=ig_{\lambda\mu}\delta(\mathbf{x}-\mathbf{x}'), \quad (34)$$

where $\delta(\mathbf{x}-\mathbf{x}')$ represents the three-dimensional δ-function. All other variables commute.

Since there is complete analogy between commutators and Poisson brackets in classical mechanics and since the canonical equations (26) can always be written in terms of Poisson brackets, it follows that the canonical commutation rules lead back to the Eqs. (8), (9), and (20), which now hold as operator equations. Here the time derivative of any operator 0 not depending explicitly on the time (such as $\phi_\lambda(\mathbf{x})$ or $\pi_\mu(\mathbf{x})$) is defined by

$$\dot{0}=i[H, 0]. \quad (35)$$

In particular, it follows again from (20) that the operator χ defined by (10) satisfies the equation (21). This enables us to specialize our field to the Maxwell field by imposing on the Schrödinger functional the subsidiary condition

$$\chi\Omega(q, t)=0. \quad (36)$$

Since Eq. (21) is of second order in t it is obviously sufficient to require only the initial conditions[4]

$$\chi(\mathbf{x})\Omega(q, 0)=0, \quad \text{and} \quad \dot{\chi}(\mathbf{x})\Omega(q, 0)=0. \quad (37)$$

[3] Here and in the following we shall always use the notation $x=(x^0, x^1, x^2, x^3)$ and $\mathbf{x}=(x^1, x^2, x^3)$ to describe four vectors and space vectors.

[4] For a more detailed discussion of the analogous situation in the vacuum case, cf. G. Wentzel, *Einführung in die Quantentheorie der Wellenfelder* (Wien, F. Deuticke 1943), p. 111ff. The notation here used is the obvious generalization of Wentzel's notation.

The relativistic invariance of the canonical quantization procedure is most easily shown by transforming the commutation rules to the symmetrical four-dimensional form. This can be done by introducing the field variables (33) which depend explicitly on the time. We shall show that the commutation rules (34) are a consequence of the commutation rules

$$i[\phi_\lambda(x),\,\phi_\mu(x')]=\Gamma_{\lambda\mu}D(x-x'),\tag{38}$$

with

$$\Gamma_{\lambda\mu}=g_{\lambda\mu}+\frac{\kappa}{1+\kappa}v_\lambda v_\mu.\tag{39}$$

The D-function which occurs here may be defined by the integral

$$D(y)=\pm\frac{1}{(2\pi)^4}\int_{C\mp}d^4k\frac{\exp(ik_\rho y^\rho)}{k_\rho k^\rho-\kappa v^\rho v^\sigma k_\rho k_\sigma},\tag{40}$$

Here the symbol $C_\mp$ below the integral sign indicates the path of integration for the k^0 variable, which is defined in the following way. Let k', k'' denote the two roots for k^0 of the quadratic equation

$$k_\sigma k^\sigma-\kappa v^\sigma v^\rho k_\sigma k_\rho=0.\tag{41}$$

The explicit expressions for these roots are

$$k'=\frac{\kappa(\mathbf{v}\cdot\mathbf{k})v^0+((1+\kappa v_0{}^2)k^2-\kappa(\mathbf{v}\cdot\mathbf{k})^2)^{\frac{1}{2}}}{1+\kappa v_0{}^2},$$

and

$$k''=\frac{\kappa(\mathbf{v}\cdot\mathbf{k})v^0-((1+\kappa v_0{}^2)k^2-\kappa(\mathbf{v}\cdot\mathbf{k})^2)^{\frac{1}{2}}}{1+\kappa v_0{}^2}.\tag{42}$$

The sign of the radicand is always greater than zero, and the roots k', k'' therefore always real, since by Schwartz' inequality

$$(1+\kappa v_0{}^2)k^2-\kappa(\mathbf{v}\cdot\mathbf{k})^2=k^2(1+\kappa)+\kappa(v^2k^2-(\mathbf{v}\cdot\mathbf{k})^2)>0.$$

The integrand in (40), therefore, has two poles on the real axis. The paths indicated by $C_\mp$ are such that they go along the real axis in the k^0-plane from $-\infty$ to $+\infty$ deformed into the negative (C_-) or positive (C_+) imaginary half plane so as to avoid the poles at the points $k^0=k'$ and $k^0=k''$. Which of the two paths is to be chosen is determined by the sign of y^0 as follows: C_+ for $y^0>0$, C_- for $y^0<0$. The sign in front of the integral must be chosen so that either the upper or lower sign applies throughout.

If we carry out the integration over k^0 according to this prescription we find for $y^0>0$

$$D(y)=\frac{1}{(2\pi)^4(1+\kappa v_0{}^2)}\int d^3k e^{i\mathbf{k}\cdot\mathbf{y}}\times\int_{C_+}dk^0\frac{e^{-ik^0y^0}}{(k^0-k')(k^0-k'')}.\tag{43}$$

The integral over k^0 can be evaluated with the residue theorem.

$$\int_{C_+}\frac{e^{-ik^0y^0}}{(k^0-k')(k^0-k'')}=+\frac{2\pi i}{k''-k'}\times\{e^{-ik''y^0}-e^{-ik'y^0}].\tag{44}$$

This gives for (43)

$$D(y)=\frac{i}{(2\pi)^3(1+\kappa v_0{}^2)}\times\int\frac{d^3k}{k''-k'}e^{i\mathbf{k}\mathbf{y}}\{e^{-ik''y^0}-e^{-ik'y^0}\}.$$

Since from (42) we have $k'(-\mathbf{k})=-k''(\mathbf{k})$ we can transform this last integral into

$$D(y)=\frac{1}{(2\pi)^3(1+\kappa v_0{}^2)}\times\int\frac{d^3k}{k'-k''}\{\sin(\mathbf{k}\cdot\mathbf{y}-k''y^0)-\sin(\mathbf{k}\cdot\mathbf{y}-k'y^0)\}.\tag{45}$$

Exactly the same result is obtained for $y^0<0$. From (40) it is obvious that D is an invariant function of its arguments. The form (45) however is more convenient to verify the following relations:

$$D(y)_{y^0=0}\equiv D(\mathbf{y})=0,\tag{46}$$

$$\dot{D}(y)_{y^0=0}=\delta(y)/(1+\kappa v_0{}^2),\tag{47}$$

$$\partial_i D(y)_{y^0=0}=0,\quad(i=1,\,2,\,3)\tag{48}$$

$$(\partial_\rho\partial^\rho-\kappa\partial^\rho v_\rho\partial^\sigma v_\sigma)D=0.\tag{49}$$

It remains to be shown that the commutation rules (34) follow from (38). We notice first that on account of (12) and (13), (38) may also be written

$$i[\Psi_\lambda(x),\,\phi_\mu(x')]=g_{\lambda\mu}D(x-x').\tag{50}$$

Since $\chi=\partial^\lambda\Psi_\lambda$ it follows immediately that

$$i[\chi(x),\,\phi_\mu(x')]=\partial_\mu D(x-x'), \qquad (51)$$

and

$$i[\chi(x),\,F_{\lambda\mu}(x')]=i[\chi(x),\,G_{\lambda\mu}(x')]=0. \qquad (52)$$

Moreover,

$$i[\chi(x),\,\chi(x')] =(\partial^\mu\partial_\mu-\kappa\partial_\rho v^\rho\partial_\sigma v^\sigma)D(x-x')=0, \qquad (53)$$

on account of (49).

From (38) we obtain:

$$i[F_{\lambda\mu}(x),\,\phi_\nu(x')]=(\partial_\lambda\Gamma_{\mu\nu}-\partial_\mu\Gamma_{\lambda\nu})D(x-x'), \qquad (54)$$

and with the help of (9)

$$i[G_{\lambda\mu}(x),\,\phi_\nu(x')]=\{\partial_\lambda g_{\mu\nu}-\partial_\mu g_{\lambda\nu} +\kappa\partial_\sigma v^\sigma(\Gamma_{\lambda\nu}v_\mu-\Gamma_{\mu\nu}v_\lambda)\}D(x-x'). \qquad (55)$$

With the help of (22) and (51) we obtain finally

$$i[\pi_\lambda(x),\,\phi_\nu(x')]=\{\partial_\lambda g_\nu{}^0-\partial^0 g_{\lambda\nu} +\kappa\partial_\sigma v^\sigma(\Gamma_{\lambda\nu}v^0-\Gamma_\nu{}^0 v_\lambda) -\partial_\nu g_\lambda{}^0+\kappa v_\lambda v^0 v_\nu\}D(x-x'). \qquad (56)$$

By specializing this expression for $x_0=x_0'$ and by using the relations (46), (47), (48), this reduces to

$$i[\pi_\lambda(\mathbf{x}),\,\phi_\nu(\mathbf{x}')]=g_{\lambda\nu}\delta(\mathbf{x}-\mathbf{x}'),$$

which is Eq. (34). In order to complete the proof that (38) reduces to the canonical commutation rules it remains to be shown that all the other variables commute. For the components ϕ_λ this is an immediate consequence of (38) and (46). Furthermore from (23), (52) and (53) it follows also that $[\pi_0,\,\pi_i]=0$. A rather lengthy, although straightforward, calculation shows then that the space components π_i commute also among themselves. This completes the proof that (38) are the four-dimensional invariant commutation rules of the radiation field.

6. Transition to Momentum Space

We carry out the transition to momentum space by introducing the new variables $Q_\nu(\mathbf{k})$ and $P_\nu(\mathbf{k})$,

$$\phi_\nu=(2\pi)^{-\frac{3}{2}}\int d^3k Q_\nu(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}},$$

and

$$\pi_\nu=(2\pi)^{-\frac{3}{2}}\int d^3k P_\nu(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}. \qquad (57)$$

The reality conditions for ϕ_ν and π_ν require that

$$Q_\nu^+(\mathbf{k})=Q_\nu(-\mathbf{k}),\quad P_\nu^+(\mathbf{k})=P_\nu(-\mathbf{k}). \qquad (58)$$

The operators Q_ν and P_ν satisfy the canonical commutation rules

$$i[P_\nu(\mathbf{k}),\,Q_\mu(\mathbf{k}')]=g_{\nu\mu}\delta(\mathbf{k}-\mathbf{k}') \qquad (59)$$

while all the other variables commute. Substituting (57) into the Hamiltonian (31) and using the vector notation for the spatial components of Q and P we have

$$H_0=\tfrac{1}{2}\int d^3k\Bigg[\frac{1}{1+\kappa v_0{}^2}\Bigg\{\mathbf{P}^+\cdot\mathbf{P}+2\kappa i v^0(\mathbf{v}\cdot\mathbf{Q})(\mathbf{k}\cdot\mathbf{P}) -2\kappa i v^0(\mathbf{v}\cdot\mathbf{k})(\mathbf{Q}\cdot\mathbf{P})+\frac{\kappa}{1+\kappa}(\mathbf{P}^+\cdot\mathbf{v})(\mathbf{P}\cdot\mathbf{v}) -\kappa(\mathbf{v}\times(\mathbf{k}\times\mathbf{Q}^+))\cdot(\mathbf{v}\times(\mathbf{k}\times\mathbf{Q}))\Bigg\} +\tfrac{1}{2}(\mathbf{k}\times\mathbf{Q}^+)\cdot(\mathbf{k}\times\mathbf{Q})\Bigg]. \qquad (60)$$

This expression can be simplified by using the subsidiary condition (36) and its consequences. It follows then that the second term in (60) vanishes. A further simplification is introduced if we choose for each $\mathbf{k}$ a special coordinate system by putting

$$\mathbf{e}_1=\frac{\mathbf{k}\times\mathbf{v}}{|\mathbf{k}\times\mathbf{v}|};\quad \mathbf{e}_2=\frac{\mathbf{k}\times(\mathbf{k}\times\mathbf{v})}{|\mathbf{k}\times(\mathbf{k}\times\mathbf{v})|};\quad \mathbf{e}_3=\frac{\mathbf{k}}{k}, \qquad (61)$$

and

$$\mathbf{P}=P^{(1)}\mathbf{e}_1+P^{(2)}\mathbf{e}_2+P^{(3)}\mathbf{e}_3,\quad \mathbf{Q}=Q^{(1)}\mathbf{e}_1+Q^{(2)}\mathbf{e}_2+Q^{(3)}\mathbf{e}_3. \qquad (62)$$

This coordinate system is orthogonal and normalized

$$(\mathbf{e}_i\cdot\mathbf{e}_j)=\delta_{ij}. \qquad (63)$$

We obtain thus for

$$P^{(i)}=(\mathbf{P}\cdot\mathbf{e}_i),\quad Q^{(i)}=(\mathbf{Q}\cdot\mathbf{e}_i), \qquad (64)$$

and

$$\mathbf{P}\cdot\mathbf{v}=-vP^{(2)}|\sin\alpha|+vP^{(3)}\cos\alpha \qquad (65)$$

$$(\mathbf{k}\times\mathbf{Q})=k(-Q^{(2)}\mathbf{e}_1+Q^{(1)}\mathbf{e}_2) \qquad (66)$$

$$(\mathbf{v}\times(\mathbf{k}\times\mathbf{Q}))=v\cdot k(-Q^{(1)}\cos\alpha\,\mathbf{e}_1 -Q^{(2)}\cos\alpha\,\mathbf{e}_2-Q^{(2)}|\sin\alpha|\mathbf{e}_3), \qquad (67)$$

where α is the angle between $\mathbf{v}$ and $\mathbf{k}$. Since $\mathbf{e}_1$ and $\mathbf{e}_3$ change sign with $\mathbf{k}$, the reality conditions (58) are now

$$\begin{aligned} Q^{+(1)}(\mathbf{k}) &= -Q^{(1)}(-\mathbf{k}),\\ Q^{+(2)}(\mathbf{k}) &= Q^{(2)}(-\mathbf{k}),\\ Q^{+(3)}(\mathbf{k}) &= -Q^{(3)}(-\mathbf{k}),\\ P^{+(1)}(\mathbf{k}) &= -P^{(1)}(-\mathbf{k}),\\ P^{+(2)}(\mathbf{k}) &= P^{(2)}(-\mathbf{k}),\\ P^{+(3)}(\mathbf{k}) &= -P^{(3)}(-\mathbf{k}). \end{aligned} \tag{58'}$$

In these variables we obtain for (60):

$$\begin{aligned} H_0=\frac{1}{2}\int d^3k\Bigg[&\frac{1}{1+\kappa v_0^2}\Bigg\{P^{(1)+}P^{(1)} +P^{(2)+}P^{(2)}\left(1+\frac{\kappa v^2\sin^2\alpha}{1+\kappa}\right)\Bigg\}\\ &+k^2\left(1-\frac{\kappa v^2\cos^2\alpha}{1+\kappa v_0^2}\right)Q^{(1)+}Q^{(1)}\\ &+k^2\left(1-\frac{\kappa v^2}{1+\kappa v_0^2}\right)Q^{(2)+}Q^{(2)}\\ &-2\kappa i v^0 v\cdot k\cos\alpha\left(Q^{(1)}P^{(1)}+Q^{(2)}P^{(2)}\right)\Bigg]. \end{aligned} \tag{68}$$

The operators $Q^{(i)}$, $P^{(i)}$ satisfy the commutation rules

$$i[Q^{(j)}(\mathbf{k})\, P^{(i)}(\mathbf{k})]=\delta_{ij}\delta(\mathbf{k}-\mathbf{k}'), \tag{69}$$

since the transformation (62) is orthogonal.

It is possible to diagonalize this Hamiltonian by introducing the absorption and emission operators $a(k)$, $a^+(k)$ by setting

$$\begin{aligned} Q^{(1)}(\mathbf{k}) &= \frac{1}{\sqrt{2}}\alpha\left(a_1(\mathbf{k})-a_1^+(-\mathbf{k})\right),\\ P^{(1)}(\mathbf{k}) &= \frac{i}{\sqrt{2}}\frac{1}{\alpha}\left(a_1^+(\mathbf{k})+a_1(-\mathbf{k})\right),\\ Q^{(2)}(\mathbf{k}) &= \frac{1}{\sqrt{2}}\beta\left(a_2(\mathbf{k})+a_2^+(-\mathbf{k})\right), \end{aligned} \tag{70}$$

and

$$P^{(2)}(\mathbf{k}) = \frac{i}{\sqrt{2}}\frac{1}{\beta}\left(a_2^+(\mathbf{k})-a_2(-\mathbf{k})\right).$$

The factors α and β will be determined later.

The inverse formulas are

$$a_1(\mathbf{k})=\frac{1}{\sqrt{2}}\left\{\frac{1}{\alpha}Q^{(1)}(\mathbf{k})+i\alpha P^{+(1)}(\mathbf{k})\right\},$$

and

$$a_2(\mathbf{k})=\frac{1}{\sqrt{2}}\left\{\frac{1}{\beta}Q^{(2)}(\mathbf{k})+i\beta P^{+(2)}(\mathbf{k})\right\}. \tag{71}$$

From these follow the commutation rules for the a's in the form:

$$[a_i(\mathbf{k}),\, a_j^+(\mathbf{k}')]=\delta_{ij}\delta(\mathbf{k}-\mathbf{k}'). \tag{72}$$

Introducing the $a_i(\mathbf{k})$ of (71) into (68) we can make the cross products of the form $a_1(\mathbf{k})a_1(-\mathbf{k})$ and $a_2(\mathbf{k})a_2(-\mathbf{k})$ vanish by choosing α, β according to the following expressions:

$$\left.\begin{aligned} \alpha &= [k^2(1+\kappa v_0^2-\kappa v^2\cos^2\alpha)]^{-\frac{1}{4}}\\ \beta &= \left[\frac{1+\kappa v_0^2-\kappa v^2\cos^2\alpha}{k^2(1+\kappa)^2}\right]^{\frac{1}{4}}. \end{aligned}\right\} \tag{73}$$

The Hamiltonian (68) reduces then to

$$H_0=\tfrac{1}{2}\sum_r\int d^3k\epsilon_{\mathbf{k}}\{a_r(\mathbf{k})a_r^+(\mathbf{k})+a_r^+(\mathbf{k})a_r(\mathbf{k})\}, \tag{74}$$

with

$$\epsilon_k=k'=\frac{\kappa v^0(\mathbf{v}\cdot\mathbf{k})+\sqrt{k^2(1+\kappa v_0^2)-\kappa(\mathbf{v}\cdot\mathbf{k})^2}}{1+\kappa v_0^2}. \tag{75}$$

The operator $N_r(\mathbf{k})$ defined by

$$N_r(\mathbf{k})+\tfrac{1}{2}=\tfrac{1}{2}\{a_r(\mathbf{k})a_r^+(\mathbf{k})+a_r^+(\mathbf{k})a_r(\mathbf{k})\}, \tag{76}$$

is the operator for the photon number associated with the state $(\mathbf{k}, r)$. It has the eigenvalues 0, 1, 2, $\cdots$. The total energy operator is thus diagonal in a representation which makes the $N_r(\mathbf{k})$ diagonal since we have

$$H_0=\sum_r\int (N_r(\mathbf{k})+\tfrac{1}{2})\epsilon_k d^3k. \tag{77}$$

This last result shows that the quantum theory of this field behaves in every respect like the quantum theory of the radiation field in vacuum. There is one important difference, however. In the expression (75) for ϵ_k there occurs only the positive square root. For the vacuum case this corresponds to the fact that only the quanta with positive energy $k^0=k'$ contribute to the total energy. Indeed the expression (75) reduces for $\kappa \rightarrow 0$ to k. However, unlike the vacuum case the expression for ϵ_k is no longer positive under all circumstances, on account of the first term. If the velocity of the medium is sufficiently high it may become negative. The critical velocity for which this may occur is obtained from the equation

$$\kappa v^0 v k = (k^2(1+\kappa v_0^2) - \kappa v^2 k^2)^{\frac{1}{2}},$$

or

$$v^2(1+v^2) = (1+\kappa)/\kappa^2. \quad (78)$$

In terms of the ordinary velocity $u = c(v/(1+v^2)^{\frac{1}{2}})$ this means

$$u = c(1/(1+\kappa)^{\frac{1}{2}}) = c/n. \quad (79)$$

This singularity of the radiation field for a medium velocity larger than the critical one is intimately connected with the occurrence of the Čerenkov radiation. The appearance of negative quanta is necessary in this case because when a particle with velocity greater than c/n is transformed to rest by introducing the coördinate system which moves with the particle, such a particle may spontaneously radiate under emission of quanta corresponding to the Čerenkov radiation. It is well known, however, that for such a process energy and momentum cannot be conserved with positive energy for the quanta. The detailed theory of the Čerenkov radiation will be dealt with in the second part of this paper.

We complete the discussion of the radiation field by calculating also the total momentum of the quanta. An expression for the total momentum may be obtained from a discussion of the canonical energy-momentum tensor

$$T_{\mu\nu} = [\partial\mathfrak{L}/\partial(\partial^\mu\phi^\rho)]\partial_\nu\phi^\rho - g_{\mu\nu}\mathfrak{L}. \quad (80)$$

This expression is not symmetrical in the indices μ and ν. Furthermore, it involves the potentials explicitly. We shall not enter here into the well-known difficulties which arise from the ambiguity of the energy-momentum densities.[5] Here we shall need only the expression for the total energy and the total momentum. We shall assume, moreover, that we are dealing with a Maxwell field where the subsidiary condition $\chi = 0$ holds. For the total momentum we obtain then

$$P_i = \int T_{0i} d^3x = \int \frac{\partial\mathfrak{L}}{\partial(\partial^0\phi^\rho)} \partial_i \phi^\rho d^3x. \quad (81)$$

We may transform this integral into one which involves only the field quantities $F_i{}^\rho$, $G_{0\rho}$ by adding the term

$$-\frac{\partial\mathfrak{L}}{\partial(\partial^0\phi^\rho)}\partial^\rho\phi_i = +\partial^\rho\left(\frac{\partial\mathfrak{L}}{\partial(\partial^\rho\phi^0)}\phi_i\right) + 2\partial^\rho\phi_i(g_{\rho 0} - \kappa v_\rho v_0)\chi. \quad (82)$$

The first term on the right may also be written as

$$\partial^\rho\left(\frac{\partial\mathfrak{L}}{\partial(\partial^\rho\phi^0)}\phi_i\right) = \partial^j\left(\frac{\partial\mathfrak{L}}{\partial(\partial^j\phi^0)}\phi_i\right) - (1+\kappa v_0^2)\dot{\chi}.$$

Thus apart from terms proportional to χ and $\dot{\chi}$ the total momentum is given by

$$P_i = \int \frac{\partial\mathfrak{L}}{\partial(\partial^0\phi^\rho)} F_i{}^\rho d^3x = -\int G_{0\rho} F_i{}^\rho d^3x = -\int \pi_j F_i{}^j d^3x. \quad (83)$$

We note here in passing that the tensor

$$T_{\mu\nu}' = G_{\mu\rho} F_\nu{}^\rho - g_{\mu\nu}\mathfrak{L},$$

is essentially the energy-momentum tensor used by Minkowski[2] in his phenomenological theory of classical electrodynamics.

In momentum space the expression (83) for the total momentum is

$$P_i = \int d^3k k_i (N(\mathbf{k}) + \tfrac{1}{2}). \quad (84)$$

We note in particular that in the rest system the energy of a photon with momentum k is given by $\epsilon_k = k/n$.

[5] For a discussion of these questions see W. Pauli, Relativitätstheorie, Encyklopädie der Math. Wiss. V, 665ff.