A Relativistic Cut-Off for Classical Electrodynamics

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Ordinarily it is assumed that interaction between charges occurs along light cones, that is, only where the four-dimensional interval $s^2=t^2-r^2$ is exactly zero. We discuss the modifications produced if, as in the theory of F. Bopp, substantial interaction is assumed to occur over a narrow range of s^2 around zero. This has no practical effect on the interaction of charges which are distant from one another by several electron radii. The action of a charge on itself is finite and behaves as electromagnetic mass for accelerations which are not excessive. There also results a classical representation of the phenomena of pair production in sufficiently strong fields.

QUANTUM electrodynamics is built from a classical counterpart that already contains many difficulties which remain upon quantization. It has been hoped that if a classical electrodynamics could be devised which would not contain the difficulty of infinite self-energy, and this theory could be quantized, then the problem of a self-consistent quantum electrodynamics would be solved. For this reason many successful attempts have been made to produce such a classical theory. The field equations can be made non-linear,[1] the fields produced by or acting on an electron can be redefined,[2,3] or one may resort to some averaging of the fields over space or time.[4] These theories have, however, met with considerable difficulties when an attempt has been made to quantize them. In this paper a consistent classical theory is described which the author believes can be quantized. Some preliminary results of the quantization of this theory will be discussed in a future paper. Some of the physical ideas of the classical form of the theory are sufficiently interesting in themselves to warrant their discussion first in a separate paper.

The potential at a point in space at a given time depends on the charge at a distance r from the point at a time previous by $t=r$ (taking the speed of light as unity). Speaking relativistically, interaction occurs between events whose four-dimensional interval, s, defined by $s^2=t^2-r^2$, vanishes. There results, however, an infinite action of a point electron on itself. The present theory modifies this idea by assuming that substantial interaction exists as long as the interval s is time-like and less than some small length, a, of order of the electron radius. When t is large since $\Delta(s^2)=2t\cdot\Delta t$ this means a spread in the time of arrival of a signal of amount of order $a^2/2t$. For charges separated by many electron radii there is, therefore, essentially no effect of the modification. For the action of an electron on itself, however, there is a considerable modification. The result is to reduce the infinite self-energy to a finite value. For accelerations which are not extreme, the action of an electron on itself appears simply as an electromagnetic mass. If desired in the classical theory, all the mass of an electron may be represented as electromagnetic. (In the quantum theory this cannot be done in a reasonable way as the electromagnetic mass comes out quite small under reasonable assumptions for a.) We have, therefore, a consistent classical theory which does not disagree with classical experience.

In the remainder of the paper we formulate this idea mathematically, and draw one or two simple consequences. We then discuss a curious

[1] M. Born and L. Infeld, Proc. Roy. Soc. London **A144**, 425 (1935).

[2] P. A. M. Dirac, Proc. Roy. Soc. London **A167**, 148 (1938). An excellent discussion of these matters is given by C. J. Eliezer, Rev. Mod. Phys. **19**, 147 (1947).

[3] J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **17**, 157 (1945).

[4] There are many theories of this nature. The author's theory is essentially that of F. Bopp, Ann d. Physik **42**, 573 (1942). R. Peierls and H. McManus have developed a theory in which the electron is pictured as a rigid distribution of charge in both space and time. The theory can be shown to be exactly equivalent to the present one, at least for a class of f functions. Their physical ideas may offer advantages over the present one in which the function f is not so directly interpretable. I thank Dr. McManus for a copy of his thesis. For a summary of another theory of this type see B. Podolsky and P. Schwed, Rev. Mod. Phys. **20**, 40 (1948). A somewhat different type is that of N. Rosen, Phys. Rev. **72**, 298 (1947).

feature of this theory. It can give a classical representation of the phenomena of pair production in sufficiently strong fields. This is of interest because the physical ideas may possibly be carried over to give a clearer understanding of the hole theory of positrons.

The main result which is to be carried over to quantum problems is this: In any process in which there is no permanent emission of quanta one must assume the field quanta to have a "density" $g(k_4,\mathbf{K})$ in frequency, and wave number space. This replaces the usual assumption that the frequency k_4 equals the magnitude of the wave number, K, and that the density in wave numbers $\mathbf{K}$, is uniform (corresponding to $g(k_4,\mathbf{K})=\delta(k_4^2-K^2)$). The properties $g(k_4,\mathbf{K})$ ought to have are discussed more fully below.

MATHEMATICAL FORMULATION

It is most convenient (but not necessary) to formulate these ideas in the language of action at a distance.[3] Hence a brief summary of that point of view is given here. We start with Fokker's action principle that the action

$$S=\sum_a m_a\int(da_\mu da_\mu)^{\frac{1}{2}}+\sum_{a,b}{}' e_a e_b\int\int\delta(s_{ab}^2)da_\mu db_\mu \quad (1)$$

is an extremum. Here a_μ represents, for $\mu=1$ to 4, the three space coordinates and the time coordinate of a particle a of mass m_a, charge e_a. We shall later consider them as functions of a parameter α, say. The b_μ are corresponding quantities for a particle b, etc. The symbol $x_\mu y_\mu$ means $x_4y_4-x_1y_1-x_2y_2-x_3y_3$ and $s_{ab}^2=(a_\mu-b_\mu)(a_\mu-b_\mu)$. The δ is Dirac's delta function.

The integrals are taken over the trajectories of the particles. The $\sum'$ means the sum over all pairs a, b with $a\neq b$. We consider varying the path $a_\mu(\alpha)$ of particle a. Defining

$$A_\mu^{(b)}(x)=e_b\int\delta(s_{xb}^2)db_\mu, \quad (2)$$

where x stands for x_μ, a point in space time, we can write (1) as

$$S=\sum_a m_a\int(da_\mu da_\mu)^{\frac{1}{2}}+\sum_a\sum_{b\neq a}e_a\int A_\mu^{(b)}(a)da_\mu.$$

The result of seeking an extremum of this is to lead in the well-known way to the equations of motion,

$$m_a\frac{d}{d\tau_a}\left(\frac{da_\nu}{d\tau_a}\right)=e_a\frac{da_\mu}{d\tau_a}\sum_{b\neq a}F_{\mu\nu}^{(b)}(a), \quad (3)$$

where we can call $F_{\mu\nu}^{(b)}(x)$ the field at x caused by particle b. It is given by

$$F_{\mu\nu}^{(b)}(x)=\partial A_\mu^{(b)}(x)/\partial x_\nu-\partial A_\nu^{(b)}(x)/\partial x_\mu.$$

We have written $d\tau_a=(da_\mu da_\mu)^{\frac{1}{2}}$ for the proper time along the path of a.

Since $\square^2\delta(s_{xb}^2)=4\pi\delta(x_1-b_1)\delta(x_2-b_2)\delta(x_3-b_3)\times\delta(x_4-b_4)$, where $\square^2=(\partial/\partial x_\mu)(\partial/\partial x_\mu)$, Eq. (2) gives

$$\square^2A_\mu^{(b)}(x)=4\pi e_b\int\delta(x_1-b_1)\delta(x_2-b_2)\times\delta(x_3-b_3)\delta(x_4-b_4)db_\mu, \quad (4)$$

which is 4π times the current four-vector of a point charge e_b. Thus $F_{\mu\nu}^{(b)}(x)$ satisfies Maxwell's equations. But the special solution (2) is not the usual retarded solution but is rather half the retarded plus half the advanced solution of Lienard and Wiechert[5] (since $\delta(t^2-r^2)=(1/2r)\times(\delta(t+r)+\delta(t-r).)$ Thus we may write (dots representing derivatives with respect to τ_a, and the fields being calculated at the point $x_\mu=a_\mu$),

$$m_a\ddot{a}_\nu=e_a\dot{a}_\mu\sum_{b\neq a}(\tfrac{1}{2}F^{(b)}{}_{\mu\nu\ \mathrm{ret}}+\tfrac{1}{2}F^{(b)}{}_{\mu\nu\ \mathrm{adv}}). \quad (5)$$

This can be compared to the usual theory which just uses retarded effects by writing it in the form

$$m_a\ddot{a}_\nu=e_a\dot{a}_\mu\left\{\sum_{b\neq a}F^{(b)}{}_{\mu\nu\ \mathrm{ret}}+\tfrac{1}{2}\sum_{\mathrm{all}\ b}[F^{(b)}{}_{\mu\nu\ \mathrm{adv}}-F^{(b)}{}_{\mu\nu\ \mathrm{ret}}]-\tfrac{1}{2}[F^{(a)}{}_{\mu\nu\ \mathrm{adv}}-F^{(a)}{}_{\mu\nu\ \mathrm{ret}}]\right\}, \quad (6)$$

as in the paper[3] by Wheeler and Feynman. As in that paper the first term is the retarded field of other charges, the second term vanishes in a world where all emitted light is eventually ab-

[5] This use of advanced and retarded potentials is really unnecessary for an understanding of the modifications of electrodynamics which is the main point of the paper. It results from the aurhor's desire to start with a principle of least action, for it is in this form that the transition to quantum theory can be made.

sorbed,[6] and the third term, depending only on the motion of a, is the force of radiative damping. Thus (1) is equivalent to (6) and thus satisfactorily describes the known laws of classical electrodynamics. There is no self-energy.

According to the above, a particle does not act upon itself, as the term with $a=b$ in the sum $\sum'_{a,b}e_ae_b\int\cdots$ in the action has been omitted. (Radiation resistance is pictured as in indirect effect of source on absorber and absorber on source.) The field of each particle must be kept separate in order to exclude, when asking for the force on a particle, the field of the particle itself.

There is no *need* to do so, but it is an interesting question to try to reinstate the idea of a universal field. This requires that a particle be allowed to act on itself and the term $a=b$ included in the action sum. This leads immediately to an infinite self force. This difficulty can be eliminated if the $\delta(s_{ab}^2)$ is replaced, as Bopp[4] has suggested, by some other function $f(s_{ab}^2)$ of the invariant s_{ab}^2, which behaves like $\delta(s_{ab}^2)$ for large dimensions but differs for small. (We shall discuss the properties of this function later, but as an example to keep in mind, consider $f(s^2)=(1/2a^2)\exp(-|s|/a)$ for $s^2>0$, and $f(s^2)=0$ for $s^2<0$ with a of order of the electron radius e^2/mc^2.)

We study the consequences of replacing (1) by the law that S is extremum if

$$S=\sum_a m_a\int(da_\mu da_\mu)^{\frac{1}{2}}+\tfrac{1}{2}\sum_a\sum_b e_ae_b\int\int f(s_{ab}^2)da_\mu db_\mu. \quad (7)$$

The term with $a=b$ may be written

$$\tfrac{1}{2}e_a^2\int\int f(s^2_{aa'})da_\mu da_\mu', \quad (8)$$

where a and a' are two points on the world-line of a. The variation problem clearly leads to

$$m_a\ddot{a}_\nu=e_a\dot{a}_\mu\left[\sum_{b\neq a}\bar{F}_{\mu\nu}^{(b)}(a)+\bar{F}_{\mu\nu}^{(a)}(a)\right], \quad (9)$$

where

$$\bar{F}_{\mu\nu}^{(b)}(x)=\partial\bar{A}_\mu^{(b)}(x)/\partial x_\nu-\partial\bar{A}_\nu^{(b)}(x)/\partial x_\mu \quad (10)$$

and the bar over the field quantities indicate that they are calculated from the f function rather than the δ function. That is,

$$\bar{A}_\mu^{(b)}(x)=e_b\int f(s_{xb}^2)db_\mu. \quad (11)$$

This theory differs from the usual in two respects: A. There is an extra force $h_\nu=e_a\dot{a}_\mu\bar{F}_{\mu\nu}^{(a)}(a)$ on particle a depending only on the motion of a. This we shall study in a moment and show that it represents inertia. B. The fields of other particles are given by the curl of a potential but the potential (11) no longer solves the Maxwell equations (4). However, since $f(s^2)$ is close to $\delta(s^2)$ this means that except for particles very close together nothing is changed very much. Thus $f(t^2-r^2)$ is large only when $t=r$ is nearly satisfied, but for large t near $+r$, say, $f(t^2-r^2)\cong f(2t(t-r))$ so that the function which has width a^2 in its argument s^2 has width $a^2/2t$ in $t-r$. Thus for increasing distances from the source the potentials satisfy Maxwell's equations ever more accurately.

The analog of Eq. (6) becomes

$$m_a\ddot{a}_\nu=e_a\dot{a}_\mu\left\{\sum_{b\neq a}\langle F\rangle^{(b)}{}_{\mu\nu\,\mathrm{ret}}+\tfrac{1}{2}\sum_{\mathrm{all}\,b}[F^{(b)}{}_{\mu\nu\,\mathrm{adv}}-F^{(b)}{}_{\mu\nu\,\mathrm{ret}}]-\tfrac{1}{2}[F^{(a)}{}_{\mu\nu\,\mathrm{adv}}-F^{(a)}{}_{\mu\nu\,\mathrm{ret}}]+\bar{F}^{(a)}{}_{\mu\nu}\right\}, \quad (12)$$

where we define $\langle F\rangle_{\mathrm{ret}}=\bar{F}+\frac{1}{2}F_{\mathrm{ret}}-\frac{1}{2}F_{\mathrm{adv}}$. Thus only the δ part, so to speak, of the fields becomes retarded. It would not do to replace F by $\bar{F}$ throughout in (6) for then we could not deduce that the second term is zero at the source because it was zero at infinity for it would not then be a solution of Maxwell's equation in empty space. The damping term is unaltered. It plus the self-force can be written $\langle F\rangle^{(a)}{}_{\mu\nu\,\mathrm{ret}}$ (see

[6] That the second term vanishes in these circumstances may be seen as follows. If a source radiates for a time, at a very long time afterwards the total retarded field vanishes, for all the light is absorbed. But also the total advanced field vanishes at this time (for charges are no longer accelerating and the advanced field exists only at times previous to their motion). Hence, the difference vanishes everywhere at this time and, since it is a solution of Maxwell's homogeneous equations, at all times.

footnote 5), so in practice one can write simply

$$m_a\ddot{a}_\nu = e_a\dot{a}_\mu \sum_{\text{all } b} \langle F\rangle^{(b)}{}_{\mu\nu\text{ ret}}.$$

The effect of the modification in the theory using retarded fields is therefore to change, slightly, the field of one particle on another when they are very close, and to add a self-force h_μ.

We now turn to a study of the self-force h_μ. This can be calculated directly from the formulae (10), (11) but a simpler way is from the action term (8). This term in the action can be re-expressed approximately if we assume that the accelerations are not too great. Only values of a' near a are important. Let us define a parameter along the path and say a corresponds to the value α of this parameter, a' to the value $\alpha'=\alpha+\epsilon$. Assuming a' not to vary too rapidly with ϵ we can approximate $s_{aa'}{}^2=(a_\mu-a_\mu')\times(a_\mu-a_\mu')$ by $\epsilon^2(da_\mu/d\alpha)(da_\mu/d\alpha)=\epsilon^2(d\tau_a/d\alpha)^2$. Likewise $da_\mu da_\mu'$ is to sufficient accuracy $(da_\mu/d\alpha)(da_\mu/d\alpha)d\alpha d\epsilon$. Thus the self-action term is approximately $\frac{1}{2}e_a{}^2\int\int f(\epsilon^2(d\tau_a/d\alpha)^2)\cdot(d\tau_a/d\alpha)^2\times d\epsilon d\alpha$. Then calling $\eta=\epsilon(d\tau_a/d\alpha)$ we can write this as

$$\mu_a\int(d\tau_a/d\alpha)d\alpha=\mu_a\int(da_\mu da_\mu)^{\frac{1}{2}}, \qquad (13)$$

where we have set

$$\mu_a=\tfrac{1}{2}e_a{}^2\int_{-\infty}^{\infty}f(\eta^2)d\eta. \qquad (14)$$

That is, the self-action term to this approximation represents pure electrodynamic mass. The term readily combines with $m_a\int d\tau_a$ for the mass is correctly invariant. We can go further and assume that originally m_a is zero and all mass of electrons is electrodynamic, but for protons this would then not be so.

The function $f(s^2)$ is to be normalized such that

$$\int_{-\infty}^{\infty}f(s^2)d(s^2)=1. \qquad (15)$$

The condition (14) says the range in η of $f(\eta^2)$ is of order e^2/μ, or if μ is the electron mass, of order of the electron radius. The function $f(s^2)$ is chosen so that it is symmetrical near past and future light cones since any asymmetry drops out in the form (7) of the action. Other than these conditions, there are strictly no further conditions on $f(s^2)$. It is convenient to assume $f(s^2)$ to be zero if s^2 is negative (space-like). It is also very desirable to have $f(s^2)$ fall rapidly away from the light cone, rather than oscillate indefinitely, and to have $f(s^2)$ finite everywhere.

By taking the Fourier transform of (11), one can represent the field as a superposition of the effects of harmonic oscillators in the usual way. However, the oscillators corresponding to waves of wave number k_1, k_2, k_3 need not have a frequency k_4 equal to the magnitude of the wave number. Instead we can take the density of the oscillators to be k_4 times, $g(k_\mu k_\mu)dk_1dk_2dk_3dk_4$ where g is defined for positive k_4 only, and is

$$g(k_\mu k_\mu)=(1/4\pi^2)\int f(s^2{}_{xy})\cos(k_\mu(x_\mu-y_\mu))\times dx_1dx_2dx_3dx_4.$$

It is a function of the invariant $k_\mu k_\mu$ only. The ordinary case, $f(s^2)=\delta(s^2)$ corresponds to $g(k_\mu k_\mu)=\delta(k_\mu k_\mu)$. The condition that $f(s^2)$ be finite on the light cone implies that $g(k_\mu k_\mu)$ can be written in the form

$$g(k_\mu k_\mu)=\int_0^\infty[\delta(k_\mu k_\mu)-\delta(k_\mu k_\mu-\lambda^2)]G(\lambda)d\lambda. \qquad (16)$$

Here $G(\lambda)$ is normalized such that $\int_0^\infty G(\lambda)d\lambda=1$, in view of (15). It is otherwise arbitrary, as $f(s^2)$ is. The λ values for which g must exist must be large, going up to order μ/e^2.

If G is chosen as $\delta(\lambda-\lambda_0)$ the resulting $f(s^2)$ is (for $s^2\geqslant 0$) the Bessel function, $\lambda_0J_1(\lambda_0 s)/s$. For large t, if $s=(t^2-r^2)^{\frac{1}{2}}$, this does not die off fast with $t-r$, but oscillates with phase varying as $\lambda_0(t^2-r^2)^{\frac{1}{2}}$. That is, it oscillates with frequency $\lambda_0(1-r^2/t^2)^{-\frac{1}{2}}$ at a time corresponding to arrival of signals with velocity r/t and thus in quantum mechanics would represent arrival of radiated "particles" of mass $\hbar\lambda_0$. The free emission of such "particles" is removed in classical theory by interference among the various values of λ if a smooth distribution, $G(\lambda)$, of λ is used. This is required if f is to represent say a function decaying rather than oscillating (see appendix).

It appears that the quantum mechanical result is simply this: For processes without permanent radiation the oscillator density g is to replace $\delta(k_\mu k_\mu)$. The negative sign in (16) proves embarrassing (see appendix) if quanta of mass λ_0 can be freely radiated so a wide distribution in λ corresponding to a monatonic $f(s^2)$ is preferable. As an example, for $f(s^2)=(1/2a^2)\times\exp(-|s|/a)$ find $G(\lambda)=(3a^2\lambda)(1+a^2\lambda^2)^{-5/2}$.

The electrostatic potential at a distance r from a stationary charge, is according to (11),

$$\bar{A}_4(r)=e\int_{-\infty}^{\infty}f(t^2-r^2)dt. \tag{17}$$

For large r, in view of (15) this is readily seen to be e/r. At the origin $r=0$ however, it is finite being $e\int_{-\infty}^{\infty}f(t^2)dt$ or $2\mu/e$. This has a simple interpretation if all mass is electromagnetic. The energy released in bringing a positron and electron charge together and so canceling out all external fields is just 2μ, the rest mass these particles have in virtue of their fields. Or put otherwise, the rest mass particles have is simply the work done in separating them against their mutual attraction after they are created. No energy is needed to create a pair of particles at the same place. (These ideas do not have direct quantum counterparts since in quantum theory all mass does not appear to be electromagnetic self energy, at least in the same simple way.) There may be a maximum field of attraction between two like charges at some separation since, for some functions f the force arising from (17) vanishes at the origin, and of course again at infinity.

There remains to discuss a curious point about the solutions of the least action principle (7) with the mechanical mass term m_a absent. First let us study the simple problem of an ordinary single particle of mass μ in a potential A_4 (caused by other charges) which depends only on one coordinate x. Call the time t, and use this for the parameter α. The action is

$$S=\mu\int(1-\dot{x}^2)^{\frac{1}{2}}dt+e\int A_4dt. \tag{18}$$

Now suppose the potential A_4 is zero outside a small band in x say $|x|<b/2$ (potential barrier) and that it is large positive, and constant within the region. Consider, in Fig. 1, the paths from a point 1 to a point 2 which make S a local maximum. A typical solution is the solid line which is kinked out of the straight line so as to increase the time integral of A_4. This represents a particle moving from 1 rapidly toward the barrier, entering the region of high potential, losing energy and thus going slower in this region. The high velocity is regained on passing out of the region to 2. Slow particles cannot penetrate the barrier.

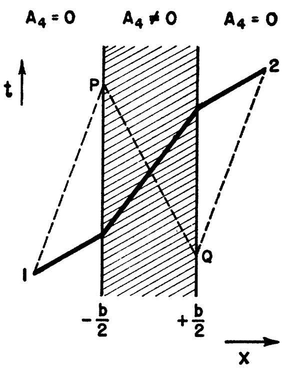

FIG. 1. If two points 1, 2 are separated by a high potential barrier, there are two paths which make action an extremum. One (solid line) represents passage of a fast electron. The other (dotted line) has a section reversed in time and is interpreted as the effective penetration of the barrier by a slow electron by means of a pair production at Q and annihilation at P, section PQ representing the motion of the positron.

But there may be another local maximum. Consider the path $1PQ2$. In the interval PQ the proper time integral must be taken positively as can be verified from a study of the derivation of (13). Now moving the point P upward by Δt might be expected to increase the action by over $2\mu\Delta t$ because the length of $P1$ and PQ are both increased. On the other hand, the integral $\int A_4dt$ is now negative and if A_4 exceeds 2μ such a curve may be a local maximum. Thus for A_4 greater than 2μ there is a new way that slow particles can penetrate the barrier. This is a classical analog of the Klein Paradox.

How would such a path appear to someone whose future gradually becomes past through a moving present? He would first see a single particle at 1, then at Q two new particles would suddenly appear, one moving into the potential to the left, the other out to the right. Later at P the one moving to the left combines with the original particle at 1 and they both disappear, leaving the right moving member of the original pair to arrive at 2. We therefore have a classical description of pair production and annihilation. The particle whose trajectory has its proper time opposed in sign to the true time t (section

PQ) would behave as a particle of opposite sign, for changing the sign of db_μ in (7) is equivalent to changing the sign of e_b. This idea that positrons might be electrons with the proper time reversed was suggested to me by Professor J. A. Wheeler in 1941.

The field at $x=\pm b/2$ is infinite. If it is finite the action (18) does not show such a local maximum, the sharp corner at P becoming a cusp which can go indefinitely into the future. On the other hand, if the correct self-force from (7) is used instead of the approximation (13), a path reversal again becomes a possibility. It is only necessary that the field exceed a critical value, namely, that maximum value of attraction of two unlike particles mentioned above. This field represents a potential of 2μ in a distance of the order of an electron radius and must be as great as this to get the pair of newly created particles apart over the potential barrier of their mutual attraction. (The actual field required to produce pairs in quantum mechanics is 137 times weaker. One might ascribe this to a quantum mechanical penetration of the potential barrier over a Compton wave-length.)

There are many interesting problems presented by these ideas. For example, will pairs be produced *ad infinitum* by the field, or only to that extent that we can guarantee that the positrons will be annihilated by electrons in the future? Again, in a weak field can a large number of pairs be created which separate slightly in the field (which is insufficient to tear the two apart) and thus produce a large polarization of that field? It is hoped that an application of these ideas to a study of positron hole theory will appear in a future paper.

I should like to thank Professor J. A. Wheeler for inoculating me with many ideas without which this work would not have been done.

APPENDIX

The difficulties in a theory such as the one presented here have been discussed by many authors. A very brief discussion of them will be given in this appendix.

The first point is that the action S defined in (1) or (7) is infinite and meaningless because of the infinite extent of the integrals along the path of the particles. The principle of extreme action which we mean to apply can be more rigorously defined as follows. Consider any given variation in paths δa_μ such that $\delta a_\mu \to 0$ as $\alpha \to \pm\infty$. Then define δS as the limit of the quantity δS calculated from (7) with this path variation, the limit being taken as the range of integration passes to infinity. Then the law of motion shall be $\delta S=0$ for all variations which satisfy $\delta a_\mu \to 0$ as $\alpha \to \pm\infty$. The equations of motion (9) are then consequences of the action principle, of course, but not all solutions of these equations satisfy the principle of least action as defined here. There are certain runaway solutions of the equations of motion, such as those discussed by Dirac[2] in the case of Eq. (6), in which the kinetic energy and momentum of a particle increase exponentially with time. These are excluded for they do not satisfy the principle of least action.

Bopp[4] has studied in great detail the consequences of equations of motion. However, he assumes that the function f acts only at retarded times. He finds that the radiation resistance of an oscillator decreases below the normal value with increasing frequency of the oscillator. However, if an oscillator is enclosed in a large, light tight box the fields at the walls of the box are effectively unchanged by the use of f rather than δ. (We are assuming that f decays and does not oscillate. Below, we discuss the situation if f oscillates.) Hence the energy absorbed by the walls will not, apparently, decrease with the increase in oscillator frequency and the radiation resistance will not keep up with it. In the modification described in this paper, in such a box, only the δ-part of f is to be retarded. The radiation resistance has its normal value at all frequencies, and the energy lost will all be found eventually in the walls of the box.

The conservation of quantities like energy (and momentum) can be demonstrated directly if a theory starts from a principle of least action, the form of which action is invariant under a change of origin of the time (and space). For the action (7) consider the quantity

$$g_\nu=\sum_a \left\{ m_a\dot{a}_\nu+e_a\sum_b \bar{A}_\nu^{(b)}(a)\right\}_{\text{at } a_\mu(\alpha_0)} -\sum_a\sum_b e_ae_b\int_{-\infty}^{\alpha_0}\int_{\beta_0}^{\infty} 2(a_\nu-b_\nu)f'(s_{ab}^2)da_\mu db_\mu \quad (19)$$

where $f'(x)=df(x)/dx$. The points $\alpha_0, \beta_0, \ldots$ are an arbitrary set of points, one chosen on the world-line of each particle. In virtue of the equations of motion (9) the derivative of g_ν with respect to any of the α_0's is zero. This is a generalization of the usual conservation of energy. Ordinarily, we would choose all the α_0 such that all $a_4(\alpha_0)$ are equal, and find g_ν is independent of this common value of a_4. The energy is seen to consist of a self-energy, an energy caused by the presence of the potential on each particle, and (since this last would count each interaction twice) a further correction to interaction energy. This is described as a line integral over the paths of the particles, but since one point is in the future and the other in the past, the actual range of integration does not extend beyond the time during which b could interact with a at α_0 and that a could interact with b at β_0. This is the way in which energy which is usually spoken of as being in the field is represented in a theory of action at a distance between particles. Since it is an integral only over a limited range, the energy of motion of the particles is conserved in the long run. (It is easy to generalize (19) to the case that paths may reverse themselves in time.)

We now consider the situation in which the function f oscillates rather than decays. If, as was discussed by Bopp[4] and others (e.g., B. Podolsky and P. Schwed[4]). f is replaced by a Bessel function $\lambda_0 J_1(\lambda_0 s)/s$, the theory corresponds as we have seen to that of interaction through ordinary "quanta" *minus* those corresponding to a mass $\hbar\lambda_0$. The f function does not appear as a pure δ-function at large distances, but another component appears if the frequency of the source exceeds λ_0. Thus, a source at high frequency ω emits waves of two wave-lengths, light of wave number $K=\omega$ and "λ_0-quanta" of wave number $K=(\omega^2-\lambda_0^2)^{\frac{1}{2}}$. Again Bopp's equations (using retarded potentials only) show that the radiative resistance force on a dipole oscillator of amplitude x, frequency ω, is constant at $2e^2\omega^3x^2/3c^3$ for $\omega<\lambda_0$ and falls off as ω exceeds λ_0, as $(2e^2x^2/3c^3)[\omega^3-(\omega^2+\frac{1}{2}\lambda_0^2)(\omega^2-\lambda_0^2)^{\frac{1}{2}}]$, remaining positive, however, for all frequencies. The decrease at higher frequencies must correspond to a negative contribution to radiation resistance accompanying the emission of "λ_0-quanta." That is, the λ_0-quanta behave as though they had negative energy. That this is so and that it results in fundamental difficulties may be seen from a few examples. If through interference the rate of emission of "λ_0-quanta" can be enhanced relative to the rate for the ordinary light quanta, a net negative radiation resistance will result. For example, a set of two vertical dipoles oscillating in phase (at frequency $\omega=2\lambda_0/3^{\frac{1}{2}}$), separated horizontally by one-half the wave-length of light, and one-fourth the wave-length of the λ_0-quanta of the same frequency, shows a negative net radiation resistance. It would oscillate with ever-increasing amplitude, the large emission of negative energy λ_0-quanta supplying the increase in energy of the oscillators and the energy of the light quanta emitted. Again a beam of λ_0-quanta passing through a medium containing damped (energy-absorbing) oscillators increases in amplitude. The wave of λ_0-quanta scattered by the oscillators in the forward direction which ordinarily interferes destructively with the incident wave, in this case has a reversed sign and enhances the incident wave. (The light scattered forward has the incorrect wave-length to make an appreciable effect by interference.) A beam of λ_0-quanta can be separated from light of the same frequency by having the radiation from a source of a given frequency impinge on a diffraction grating of scattering centers. The λ_0-quanta and light quanta will then be scattered in different directions as they have different wave-lengths.

What results if instead of using only retarded waves for λ_0-quanta, we start from the least action principle and analyze the situation of a source enclosed in a box? Then the derivation of Eq. (12) is still incomplete as $\langle F\rangle_{\text{ret}}$ still contains both advanced and retarded components (of λ_0-quanta) at large distances. We could now split off the advanced parts for λ_0-quanta just as we did for light. The resulting equation is just that used by Bopp, namely all retarded interactions but negative contribution of λ_0-quanta to radiation resistance, and therefore leads this time to conservation of energy but to diverging solutions. Such diverging solutions are, as indicated above, excluded by the least action principle so this form of the equation is not convenient. Non-divergent motions do exist. To see this it is better to split off the retarded part of the λ_0-quanta. What results is that light goes by retarded waves, λ_0-quanta by advanced waves,[7] and the radiation resistance of both contribute positively. Thus an accelerating charge will emit light, but it is predestined that negative energy λ_0-quanta were coming toward it to be absorbed, still further increasing the radiation resistance. This avoids the divergent solutions only to predict observable advanced effects.

For these reasons it is better to restrict oneself to the case of a decaying f-function (distribution of λ) for which a consistent theory can be made. Then the modifications of classical electrodynamics will only appear at very small distances from a charge. On the other hand, these distances are well within the Compton wave-length so that modifications caused by quantum mechanics would in any case appear before the ones here discussed. There is, therefore, little reason to believe that the ideas used here to solve the divergences of classical electrodynamics will prove fruitful for quantum electrodynamics. Nevertheless, the corresponding modifications were attempted with quantum electrodynamics and appear to solve some of the divergence difficulties of that theory. This will be discussed in a future paper.

[7] This may be understood in that, as indicated above, the energy-absorbing walls of the box absorb retarded light waves, but cannot be presumed to absorb retarded λ_0-quanta. Instead, in fact, they spontaneously emit such waves (warming up in the process) and non-divergent solutions result only if they emit just exactly the λ_0-quanta which can later be absorbed by the accelerating charge at the center.

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Concentration of He^3 by Thermal Diffusion

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A thermal diffusion plant for the enrichment of He^3 has been constructed consisting of cylindrical columns followed by a hot wire column. In the case of the cylindrical columns excellent agreement with the theory of Jones and Furry has been obtained. In the case of the hot wire column, discrepancies exist between the observed performance and that predicted by theory. Under continuous operating conditions it was possible to produce with the expenditure of 16.6 kw, 14 std. cc of helium per day having a He^3/He^4 ratio of 0.0021 when well helium was used as a source of gas.

IN view of the great interest in He^3, both as a tool in superflow studies of liquid helium and as one of the simplest nuclei whose properties need to be determined, an investigation has been started to determine the effectiveness of thermal diffusion as a means of concentrating this isotope. A thermal diffusion plant was constructed consisting of two cylindrical columns followed by a hot wire column. While the general design was similar to that proposed by Jones and Furry,[1] the dimensions were somewhat different.

Figure 1 shows a schematic drawing of the arrangement of the columns. The concentric cylindrical tube sections, 1 and 2, consisted of electrically heated steel tubes surrounded by brass water jackets. The hot wire section (3) consisted of a fine platinum wire surrounded by a brass water jacket. Table I gives dimensions and miscellaneous operating details for the plant.

Well helium, $He^3/He^4 = 1.5\times10^{-7}$, is caused to circulate past the bottom of section 1. Since sections 1 and 2 are joined by a tube approximately 11 cm long and 6 cm in diameter, no special circulating system is needed at this point. The circulating system between sections 2 and 3 consists of 900 cm of 0.4-cm i. d. copper tubing in series with a small centrifugal blower. A continuously heated palladium thimble 10 cm long and

[1] R. C. Jones and W. H. Furry, Rev. Mod. Phys. 18, 151 (1946).