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Electrical Breakdown of a Gas between Coaxial Cylinders at Microwave Frequencies*

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The criterion for breakdown of a low pressure gas at microwave frequencies is that the ionization rate equal the diffusion rate for electrons. This principle is applied to compute the breakdown of air in coaxial cylinders using previously reported high frequency ionization coefficient data obtained from breakdown experiments between parallel plates. Agreement between computed and measured results at 9.6-cm wave-length supports the validity of the above criterion for breakdown. The method of computation is illustrative of the general problem of computing breakdown fields for tubes of arbitrary shape and field configuration using ionization coefficient data obtained from uniform field breakdown.

ELECTRICAL breakdown of a gas at microwave frequencies has been discussed by the authors in a recent paper.[1] The condition for breakdown has been postulated as a balance between generation of electrons by ionization and loss of electrons by diffusion, and a mathematical procedure for computing breakdown electric field strengths under specified experimental conditions has been given. An ionization coefficient appropriate to the high frequency case is necessary. This coefficient has been measured by experiments on breakdown between parallel plates.[1] The present paper describes a procedure for computing breakdown between coaxial cylinders using the experimental values of the ionization coefficient, and compares the results of the computation with experiment.

Three purposes are fulfilled by the present work on coaxial cylinders. First, this geometry for a discharge tube has long been useful in radioactivity instrumentation. The microwave discharge has been studied for this application[2] and found to have certain advantages over the conventional d.c. discharge. Second, although the procedure for computing breakdown in an arbitrary cavity is straightforward in principle, the detailed computation requires further mathematical study. This paper illustrates a computing technique which is useful generally. Finally, the basic diffusion theory for breakdown is checked experimentally by using the experimental results obtained from parallel plate breakdown to predict the results to be expected for coaxial breakdown. The agreement with experiment indicates that no mechanisms other than ionization, attachment, and diffusion need to be introduced into the breakdown criterion.

The breakdown condition is obtained from the continuity equation for electrons,[1]

$$\partial n/\partial t = \nu n - \nabla \cdot \mathbf{\Gamma},$$

where n is the electron density, ν is the net production rate of electrons per electron, and $\mathbf{\Gamma}$ is the d.c. electron current density in electrons per unit area per second. The production rate per electron is the difference between the ionization rate and the attachment rate,

$$\nu = \nu_i - \nu_a.$$

If there is no electric field present, ν is zero or negative and $\partial n/\partial t$ is negative, indicating a decaying discharge. As the field is raised, ν increases until the time derivative rises and becomes positive. The threshold of breakdown occurs as it goes through zero. Thus, breakdown is computed as the characteristic value of the electric field obtained from the solution of the equation

$$\nu n - \nabla \cdot \mathbf{\Gamma} = 0, \tag{1}$$

and the boundary condition that the electron density vanish at the cavity surfaces.[1]

The high frequency electric field contributes nothing to the value of $\mathbf{\Gamma}$, so that it is given

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[1] M. A. Herlin and S. C. Brown, Phys. Rev. **74**, 291 (1948).

[2] S. C. Brown, Phys. Rev. **73**, 1240 (1948).

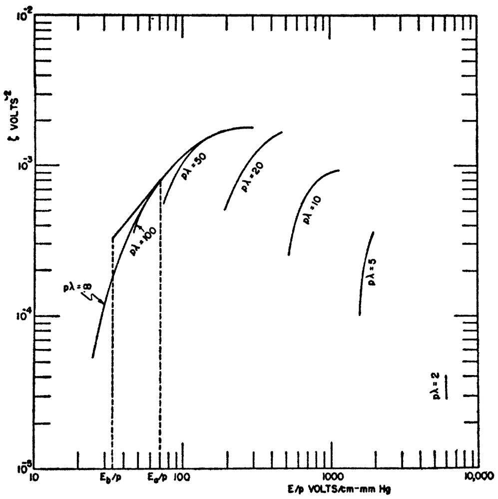

FIG. 1. Ionization coefficient (ionizations/volt²), as a function of E/p (volts/cm-mm Hg) and $p\lambda$ (mm Hg-cm). The straight-line approximation used to compute coaxial breakdown agrees with numerical integration using the correct value to within one percent.

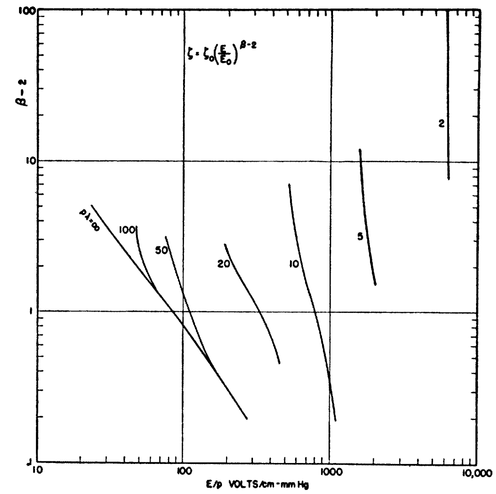

FIG. 2. Slope of ζ *versus* E/p curve on log plot taken from Fig. 1.

entirely by the diffusion equation

$$\mathbf{\Gamma} = -\nabla(Dn) = -\nabla\psi, \tag{2}$$

where D is the electron diffusion coefficient. The diffusion coefficient is left under the gradient sign to account for "thermal" diffusion, important here because the electron kinetic energy varies with position in the cavity as a result of the electric field variation. The product of the diffusion coefficient and the electron density, denoted by ψ, is the electron diffusion current density "potential," similar to the hydrodynamic velocity potential.

Equation (1), with Eq. (2) substituted, is written in terms of ψ and the high frequency ionization coefficient, ζ (ionizations/volt²),

$$\zeta = \nu/DE^2, \tag{3}$$

to yield

$$\nabla^2\psi + \zeta E^2 \psi = 0, \tag{4}$$

where E is the r.m.s. value of the high frequency electric field strength. The value of ν/D is divided by E^2 in the definition of the ionization coefficient for dimensional reasons[1] and for mathematical convenience.

The ionization coefficient has been measured as a function of E/p and $p\lambda$, where p is the gas pressure and λ is the free-space wave-length of the electric field. The variable E/p enters as determining the energy gained by an electron between collisions at zero frequency, and therefore as determining the energy distribution of the electrons at zero frequency. The energy transfer is modified at high frequencies by the oscillatory nature of the field-induced electron motion. This modification is accounted for by the variable $p\lambda$, which is a measure of the ratio of the collision frequency to the field radian frequency.[1] Figure 1 shows the measured values of the ionization coefficient, reproduced from reference 1.

The value of ζE^2 as a function of position in the discharge cavity is obtained from Fig. 1 and from the electric field as a function of position,

$$E = (V/r \ln b/a)\sin \pi z/L,$$

where V is the "voltage" at the center of the coaxial cavity, a and b are the inner and outer

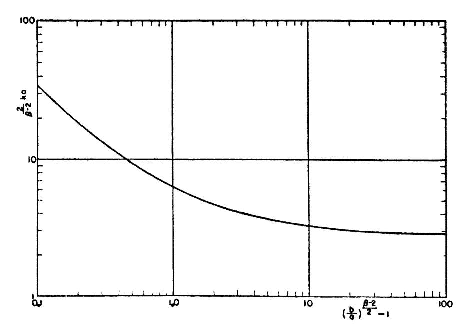

FIG. 3. Solution of the transcendental equation for coaxial breakdown.

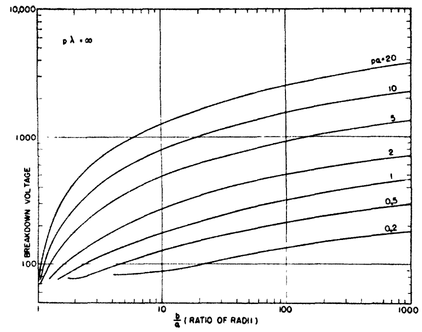

FIG. 4. Breakdown voltage between coaxial cylinders as a function of b/a, ratio of outer conductor radius b to inner conductor radius a, and pa (mm Hg-cm). This family of curves applies for $p\lambda$ = infinity. A separate family is required for each value of $p\lambda$, for a complete set of curves.

conductor radii, L is the length of the cavity, and r and z are the radial and axial space coordinates. This expression applies to a half-wave-length-long cavity, which was used experimentally. The value of E at each point in the cavity leads to a value of ζ, through the experimental ζ-curve of Fig. 1. The boundary condition that ψ be zero at the cavity walls leads then to a characteristic value for V which is the breakdown voltage.

The above expression for the electric field is a function of two space coordinates and leads to a non-separable differential equation. This difficulty can be avoided by choosing an inner conductor radius which is small compared to the length L, which is a half-wave-length. The electric field in the region near the center conductor does not vary much with distance along the conductor, so that the conditions of infinite length are approached. We may consider the field to be given by the formula

$$E = V/r\ln(b/a), \tag{5}$$

from which ζE^2 is computed as a function of position. Then to this approximation Eq. (4) is a second-order linear differential equation in r only.

Since ζ is given from numerical data, the differential equation must be integrated numerically in its present form. The work may be simplified by approximating the value of ζE^2 by an algebraic factor. The approximation used has been compared with the results of numerical integration, with agreement to within one percent over the full range of the variables. The plot of ζ against E/p on a logarithmic scale is replaced by a straight line of the slope obtained at the inner conductor of the cavity. Thus,

$$\zeta E^2 \simeq k^2(E/E_a)^\beta, \tag{6}$$

where

$$k^2 = \zeta_a E_a^{\,2}.$$

The subscript "a" refers to the position $r=a$, and the slope on a logarithmic plot of the curve of Fig. 1 at the value of E/p at $r=a$ is $(\beta-2)$. The appropriate $p\lambda$ = constant curve is chosen. This approximation expresses the ionization coefficient analytically. It is accurate near the center conductor where the ionization is most pronounced. If the ratio b/a is small it is accurate throughout the cavity, and the resulting breakdown computation is expected to be accurate. If b/a is large the approximation is not accurate at the outer part of the cavity, but since the ionization is low there the inaccuracy has little effect on the solution. The slope $(\beta-2)$ as a function of E/p and $p\lambda$ has been taken from Fig. 1 and plotted in Fig. 2.

Equations (4), (5), and (6) now combine into

$$\frac{d^2\psi}{dr^2}+\frac{1}{r}\frac{d\psi}{dr}+k^2\left(\frac{a}{r}\right)^\beta\psi=0, \tag{7}$$

whose solution is

$$\psi = Z_0\left[\frac{2}{\beta-2}ka\left(\frac{a}{r}\right)^{\frac{1}{2}(\beta-2)}\right],$$

where Z_0 is a Bessel function of zero order. The boundary condition that $\psi=0$ at $r=a$ and $r=b$ leads to the following transcendental equation

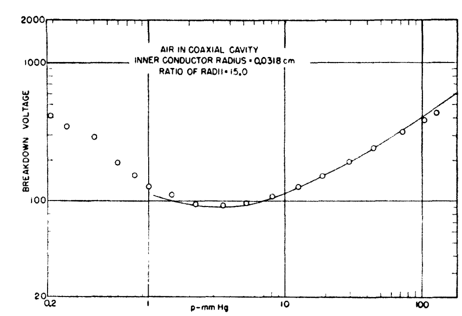

FIG. 5. Comparison of experimental points and theoretical curve for coaxial breakdown.

expressing the breakdown condition

$$J_0(x)N_0[(b/a)^{\frac{1}{2}(\beta-2)}x] - J_0[(b/a)^{\frac{1}{2}(\beta-2)}x]N_0(x)=0, \quad (8)$$

where

$$x=\frac{2}{\beta-2}ka\left(\frac{a}{b}\right)^{\frac{1}{2}(\beta-2)}. \quad (9)$$

Roots of this equation are tabulated,[3] giving x as a function of $(b/a)^{\frac{1}{2}(\beta-2)}$. Multiplying x by $(b/a)^{\frac{1}{2}(\beta-2)}$, the curve of Fig. 3 is obtained, which gives $[2/(\beta-2)]ka$ as a function of $(b/a)^{\frac{1}{2}(\beta-2)}$.

The details of computation are as follows. Values of $p\lambda$ and pa are assumed. Figure 1 with Eq. (5) and Fig. 2 gives k/E_a^2 and $(\beta-2)$, respectively, as functions of E_a/p. Then $2ka/(\beta-2)$ is computed from these, which yields $(b/a)^{\frac{1}{2}(\beta-2)}$ from Fig. 3. From this latter, b/a and V are computed, using Eq. (5).

The results of this calculation for $p\lambda=$ infinity are given in Fig. 4. A separate family would be obtained for each value of $p\lambda$, but these are not given in detail here. A few points at other values of $p\lambda$ have been computed for the special case which was measured experimentally.

Figure 5 shows a particular case which has been computed from the foregoing theory and also measured experimentally. A ratio of outer to inner conductor radii of fifteen was chosen as the greatest possible departure from uniform field conditions consistent with mechanical and electrical limitations in the experiment. The experimental technique is similar to that described in reference 1, using a tunable continuous-wave magnetron as a power source, standing-wave technique for impedance measurements, and a bolometer and bridge for power measurements. The coaxial cavity was made of brass and was resonant at 9.6-cm wave-length. The data are shown as points and the curve is derived from theory. The curve is not extended beyond the limit of validity of diffusion theory on the low pressure side.

The electron production and removal processes which have been included in the breakdown condition are linear processes; that is, the electron production and removal rates are proportional to the electron density. Recombination with positive ions and modification of diffusion by space-charge effects are non-linear and would therefore result in a different relationship between uniform-field breakdown between parallel plates and the present coaxial breakdown. Furthermore, the validity of the boundary condition is checked by the present experiment. The condition used is based on the assumption of negligible effect from surface emission of electrons, which in turn results from the absence of a drift motion of the emitted electrons resulting from the field. If the field-induced drift motion of electrons were operating strongly as an electron removal process, the effect of electrons emitted from the cavity walls would be exaggerated by electron multiplication along their path, as happens in d.c. discharges. Also, the differential equation for the electron density would have to be modified. The agreement between theory and experiment indicates that the diffusion theory for breakdown, as developed here and in reference 1, accounts for all of the important processes that affect high frequency breakdown.

[3] E. Jahnke and F. Emde, *Functionentafeln* (B. G. Teubner, Leipzig, 1933), p. 205.