ties with the fresh methane filling than it did with the methane plus decomposition fragments present.

It was found that after a methane counter had deteriorated it could always be recovered by a thorough cleaning of the electrodes. Usually washing the counter very thoroughly with distilled water, alcohol, benzene, and ether was sufficient. Occasionally a counter did not respond to this treatment, and in these cases, removal of the anode wire showed it was covered with a heavy brown coating which on heating left a carbon black on the wire.

CONCLUSION

The results of these experiments show that methane Geiger-Müller counters deteriorate because of the decomposition of the gas and that the change in the gas composition does not lead to the counter failure. The counter deterioration can be explained by the heavy hydrocarbon decomposition fragments of the methane deposited on both the cathode cylinder and on the anode wire.

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The Factorization Method, Hydrogen Intensities, and Related Problems

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The factorization method enables us to calculate in an elementary way the discrete-discrete and discrete-continuous transition probabilities of hydrogen atoms by means of recurrence formulae. From a key intensity all others are found by a repeated application of an l-changing recurrence formula or an n'-changing operator. The results are given:

for l-changing in the formulae: (4.2), (4.3),
for n'-changing in the formulae: (5.1), (5.2) with (3.2).

Some of the formulae apply to more general matrix components.

I. INTRODUCTION

The value of the Schrödinger hydrogen intensity integral

$$I_{n'\,n}{}^{l-1\,l}=\int_0^\infty rR_{n'}{}^{l-1}R_n{}^l dr, \qquad (1.1)$$

(where $R_n{}^l=r$ times the Schrödinger normalized hydrogen radial function) has been calculated many times. Originally Schrödinger[1] calculated it for special cases using the generating function for Laguerre polynomials. Wheeler[2] has recently applied this method to the general case of discrete-discrete transitions. Epstein[3] used the theory of hypergeometric functions to solve the same problem, while Eckart[4] evaluated the integral directly. Gordon[5] has treated the discrete-continuous and continuous-continuous as well as the discrete-discrete transitions. We want to show that the factorization method leads to a simple treatment of this problem.

The factorization method gives the *solutions* of a second-order differential equation by means of recurrence formulae[6]; the idea here is to develop recurrence formulae for the integrals involving these solutions—the intensity integral in particular. Besides the recurrence formulae a starting point is needed; in what follows this is found by the method of the Laplace transform. Some of the formulae apply to the calculation of other integrals involved in the Kepler problem.

[1] Schrödinger, *Wave Mechanics* (Blackie & Son, London, 1928), p. 99.
[2] Wheeler, Proc. Roy. Irish Acad. **50**, Sec. A, 3 (1944).
[3] Epstein, Proc. Nat. Acad. Sci. **12**, 629 (1926).
[4] Eckart, Phys. Rev. **28**, 927 (1926).
[5] Gordon, Ann. d. Physik **2**, 1031 (1929).
[6] L. Infeld, Phys. Rev. **59**, 737 (1941).

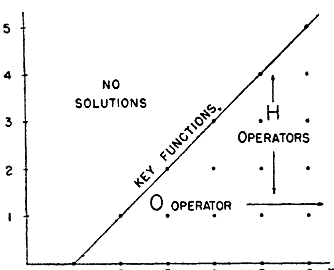

FIG. 1.

II. SOLUTION OF THE RADIAL EQUATION

The factorization method enables us to solve quickly the Schrödinger equation for a Coulomb field. We quote the result for the radial part of that equation.[7]

The equation for the discrete spectrum is:

$$R''+(2/r)R-[l(l+1)/r^2]R-(1/n^2)R=0, \quad (2.1)$$

and the normalized solutions are

$$R_n^{n-1}=(2/n)^n(1/n)[(2n-1)!]^{-\frac{1}{2}}r^n \times\exp(-r/n), \quad (2.2)$$

$$R_n^{l-1}={}^{+}H_n^lR_n^l, \quad (2.3a)$$

$$R_n^l={}^{-}H_n^lR_n^{l-1}, \quad (2.3b)$$

where

$$^{\pm}H_n^l=nl[(n-l)(n+l)]^{-\frac{1}{2}}\left\{\frac{l}{r}-\frac{1}{l}\pm\frac{d}{dr}\right\}$$
$$=(1/A_n^l)\left\{\frac{l}{r}-\frac{1}{l}\pm\frac{d}{dr}\right\}. \quad (2.4)$$

There is thus one solution for each pair of integers (n, l) provided $l\leq n-1$ and n is a positive integer. The solutions are represented by dots in Fig. 1. The key functions (2.2) are those on the line $l=n-1$. Using the recurrence formulae (2.3) one can, so to speak, move up or down the ladders if n equals a constant. Replacing n by in in (2.3) and (2.4) gives recurrence formulae for eigenfunctions of the continuous spectrum but, of course, there is no starting point corresponding to (2.2).

III. A NEW RECURRENCE FORMULA

We shall now develop a new recurrence formula which enables us to move to the right along the horizontal lines $l=$constant.[8] To do this we introduce a new function

$$R_n^l(s).$$

This function is defined by the same recurrence formula (2.3a) as the corresponding function R_n^l. The only difference is that the key functions are now taken to be

$$R_n^{n-1}(s)=(2/n)^n(1/n)[(2n-1)!]^{-\frac{1}{2}}r^n \times[\exp(-sr)]. \quad (3.1)$$

Of course the $R_n^l(s)$ are neither orthogonal nor do they satisfy our differential equation, but they have the following important property:

$$[R_n^l(s)]_{s=1/n}=R_n^l. \quad (3.2)$$

We can now find an operator which enables us to change $R_n^l(s)$ into $R_{n+1}^l(s)$ from which our solutions can be reached by (3.2). In fact

$$R_{n+1}^l(s)=O_{n+1}^lR_n^l(s),$$

where

$$O_{n+1}^l=\frac{n^{l+2}}{(n+1)^{l+2}(2n+1)}\left[\frac{n+l+1}{n-l}\right]^{\frac{1}{2}} \times\left\{2n+1+\left(\frac{1}{n}+s\right)\frac{d}{ds}\right\}. \quad (3.3)$$

We shall prove this theorem by induction: from (2.4) and (3.3) it is easily seen that

$$^{+}H_{n+1}^l=\frac{n+1}{n}\left[\frac{(n-l)(n+l)}{(n+1-l)(n+1+l)}\right]^{\frac{1}{2}}{}^{+}H_n^l,$$

and

$$O_{n+1}^{l-1}=\frac{n+1}{n}\left[\frac{(n-l)(n+l)}{(n+1-l)(n+1+l)}\right]^{\frac{1}{2}}O_{n+1}^l.$$

Using (2.3a), (3.3) and the above equations:

$$R_{n+1}^{l-1}(s)={}^{+}H_{n+1}^lR_{n+1}^l(s)$$
$$=\frac{n+1}{n}\left[\frac{(n-l)(n+l)}{(n+1-l)(n+1+l)}\right]^{\frac{1}{2}} \times{}^{+}H_n^lO_{n+1}^lR_n^l(s)$$
$$=O_{n+1}^{l-1}R_n^{l-1}(s),$$

since the H and O operators commute. Therefore,

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[7] L. Infeld, Phys. Rev. **59**, 743 (1941); the unit of length is $\hbar^2/me^2Z$ and $1/n=\hbar(-2E)^{\frac{1}{2}}/m^{\frac{1}{2}}e^2Z$.

[8] Schrödinger has developed an n-changing operator of a different kind; see Proc. Roy. Irish Acad. **46**, Sec. A, (1940).

if (3.3) is true for the quantum number l, it is true for $l-1$. It is a straightforward matter to check that (3.3) is true for $l=n-1$; the theorem is then established.

Thus with the help of (3.3) we can move to the right along the ladders $l=$constant.

IV. ALGEBRAIC RECURRENCE FORMULAE FOR THE INTENSITIES

Each of the H and O operators leads to a recurrence formula for the intensity integral, the former to an algebraic one as follows: it is not difficult to verify from (2.4) that

$$2lA_{n'}{}^{l+}H_{n'}{}^{l}=(2l+1)A_{n}{}^{l+1}{}^{+}H_{n}{}^{l+1}+A_{n'}{}^{l+1}{}^{-}H_{n'}{}^{l+1}+(\text{constant})/r. \quad (4.1)$$

Multiplying (4.1) on the left by $rR_n{}^l$, on the right by $R_{n'}{}^l$ and integrating gives

$$\begin{aligned}2lA_{n'}{}^{l}\int_0^\infty rR_n{}^{l+}H_{n'}{}^{l}R_{n'}{}^{l}dr &= (2l+1)A_n{}^{l+1}\int_0^\infty rR_n{}^{l+}H_n{}^{l+1}R_{n'}{}^{l}dr \\ &\quad + A_{n'}{}^{l+1}\int_0^\infty rR_n{}^{l-}H_{n'}{}^{l+1}R_{n'}{}^{l}dr \\ &= (2l+1)A_n{}^{l+1}\int_0^\infty r(-H_n{}^{l+1}R_n{}^{l})R_{n'}{}^{l}dr \\ &\quad + A_{n'}{}^{l+1}\int_0^\infty rR_n{}^{l-}H_{n'}{}^{l+1}R_{n'}{}^{l}dr,\end{aligned}$$

because $R_n{}^l$, $R_{n'}{}^l$ are orthogonal and vanish at $r=0$, ∞. We have then:

$$2lA_{n'}{}^{l}I_{n'\,n}{}^{l-1\,l}=(2l+1)A_n{}^{l+1}I_{n'\,n}{}^{l\,l+1}+A_{n'}{}^{l+1}I_{n'\,n}{}^{l+1\,l}.$$

By interchanging n, n' we obtain

$$2lA_n{}^{l}I_{n'\,n}{}^{l\,l-1}=A_n{}^{l+1}I_{n'\,n}{}^{l\,l+1}+(2l+1)A_{n'}{}^{l+1}I_{n'\,n}{}^{l+1\,l}, \quad (4.2)$$

$$A_n{}^l=[(n-l)(n+l)]^{\frac{1}{2}}/nl.$$

Our derivation (and hence this result) is valid for the discrete-continuous transitions once one replaces n by in.

These are algebraic formulae giving a pair of intensities in terms of the next highest pair in the scheme of Fig. 1.

All intensities can now be calculated once a starting point is found.—An obvious choice is the pair at the top of the n' ladder: $I_{n'\,n}{}^{n'\,n'-1}$ and $I_{n'\,n}{}^{n'-1\,n'}$ where

$$I_{n'\,n}{}^{n'\,n'-1}=0. \quad (4.3a)$$

(We will adopt the convention that n' always refers to the discrete spectrum.) The method of calculation of the other expression is indicated in the appendix. The result is:

$$I_{n'\,n}{}^{n'-1\,n'}=2^{2n'+2}(nn')^{n'+2}\times\left[\frac{(n+n')!}{(n-n'-1)!(2n'-1)!}\right]^{\frac{1}{2}}\frac{(n-n')^{n-n'-2}}{(n+n')^{n+n'+2}}, \quad (4.3b)$$

or, for the discrete-continuous transition,

$$I_{n'\,in}{}^{n'-1\,n'}=2^{2n'+2}(nn')^{n'+2}\times\left[\frac{n^3\prod_{p=1}^{n'}(p^2+n^2)}{\{\exp(2n\pi)-1\}(2n'-1)!}\right]^{\frac{1}{2}}\times\frac{\exp[2n\tan^{-1}(n/n')]}{(n^2+n'^2)^{n'+2}}. \quad (4.3c)$$

From (4.3) we can now calculate pairs of intensities by successive application of (4.2)—the important intensities requiring at most but a few steps.

V. OPERATOR RECURRENCE RELATION FOR INTENSITIES

The results of §3 will now be used to find an n'-changing recurrence relation for the intensities. Indeed, it follows immediately from (3.3) that

$$I_{n'+1\,n}{}^{l-1\,l}(s)=O_{n'+1}{}^{l-1}I_{n'\,n}{}^{l-1\,l}(s),$$

or

$$I_{n'+1\,in}{}^{l-1\,l}(s)=O_{n'+1}{}^{l-1}I_{n'\,in}{}^{l-1\,l}(s),$$

$$O_{n'+1}{}^{l-1}=\frac{n'^{l+1}}{(n'+1)^{l+1}(2n'+1)}\times\left[\frac{n'+l}{n'-l+1}\right]^{\frac{1}{2}}\left\{2n'+1+\left(\frac{1}{n'}+s\right)\frac{d}{ds}\right\}, \quad (5.1)$$

where the intensity *function* is defined by

$$I_{n'\,n}{}^{l-1\,l}(s)=\int_0^\infty rR_{n'}{}^{l-1}(s)R_n{}^l dr.$$

The starting point needed here is

$$I_{l\,n}{}^{l-1\,l}(s)=\frac{1}{n}\left(\frac{4}{nl}\right)^{l+1}\left[\frac{(n+l)!}{(n-l-1)!(2l-1)!}\right]^{\frac{1}{2}}\times\frac{\left(s-\frac{1}{n}\right)^{n-l-2}}{\left(s+\frac{1}{n}\right)^{n+l+2}}[(l+1)s-1], \quad (5.2a)$$

or[9]

$$I_{l\,in}{}^{l-1\,l}(s)=\left(\frac{4}{nl}\right)^{l+1}\left[\frac{n\prod_{p=1}^{l}(p^2+n^2)}{\{\exp(2n\pi)-1\}(2l-1)!}\right]^{\frac{1}{2}}\times\frac{\exp(2n\tan^{-1}ns)}{\left(s^2+\frac{1}{n^2}\right)^{l+2}}[(l+1)s-1]. \quad (5.2b)$$

The derivation of (5.2) is indicated in the appendix.

Using (5.1) and (5.2a) we can find $I_{l+1\,n}{}^{l-1\,l}(s)$, $I_{l+2\,n}{}^{l-1\,l}(s)\cdots I_{n'\,n}{}^{l-1\,l}(s)$ which, with (3.2), give the intensities. Similarly from (5.2b) we can find . . . $I_{n'\,in}{}^{l-1\,l}$.

VI. REMARKS

(1) The set of values of the quantum numbers for which the intensities are required will determine which of the above two methods should be used. Being algebraic (4.2) is simpler whereas (5.1) has the special characteristic that it is applicable to the problem of calculating more general matrix components

$$\int_0^\infty \phi(r)R_{n'}{}^{l'}R_n{}^l dr.$$

(2) *Example of a calculation:*
To find

$$I_{2n}{}^{01}=\int_0^\infty rR_2{}^0R_n{}^1dr,$$

we can use (4.3) to get immediately

$$I_{2n}{}^{21}=0,$$
$$I_{2n}{}^{12}=3^{-1/2}2^{19/2}n^{9/2}(n^2-1)^{1/2}(n-2)^{n-7/2}(n+2)^{-n-7/2}.$$

From (4.2)

$$I_{2n}{}^{01}=(2A_2{}^1)^{-1}3A_n{}^2I_{2n}{}^{12}+0$$
$$=2^{17/2}n^{7/2}(n^2-1)^{1/2}(n-2)^{n-3}(n+2)^{-n-3}.$$

Alternatively, we can use (5.2a) to get

$$I_{1n}{}^{01}(s)=2^4n^{-5/2}(n^2-1)^{1/2}(s-1/n)^{n-3}\times(s+1/n)^{-n-3}(2s-1).$$

Operating on this expression with

$$O_2{}^0=2^{-\frac{3}{2}}3^{-1}\left\{3+(1+s)\frac{d}{ds}\right\},$$

and then putting $s=\frac{1}{2}$ we get the value above for $I_{2n}{}^{01}$ which is the same as that given by Condon and Shortley[10] for the transition $2s-np$.

(3) *Explicit forms:* The above methods can lead to explicit forms of the intensity integral except in the case of continuous-continuous transitions (though the *algebraic* formulae are valid for these transitions if n', n are replaced by in', in). For example, by means of (4.2), (4.3) and known relations between contiguous hypergeometric functions the results given by Gordon[11] can be proven by induction.

APPENDIX

The Calculation of (4.3) and (5.2)

Define

$$I_n=\int_0^\infty r^l\exp(-sr)R_n{}^ldr. \quad (A.1)$$

We must distinguish between two cases:

Case I: Discrete-discrete

From (2.1):

$$\int_0^\infty r^{l+1}\exp(-sr)\{R''+(2/r)R-[l(l+1)/r^2]R-(1/n^2)R\}dr=0.$$

After two partial integrations of the first term we obtain

$$(s^2-1/n^2)I_n'+[2(l+1)s-2]I_n=0.$$

$$\therefore\quad I_n=C\frac{(s-1/n)^{n-l-1}}{(s+1/n)^{n+l+1}}.$$

[9] Another starting point which may be needed is $I_{l+1\,in}{}^{l\,l-1}(s)$; it is slightly more complicated but can be found easily from the formulas in the appendix.

[10] Condon and Shortley, *Theory of Atomic Spectra* (Cambridge University Press, Teddington, 1935), p. 133.

[11] See Reference 10, p. 1051.

I_n is the Laplace transform of $r^l R_n^l$ and therefore[12]

$$R_n^l = r^{-l}[\text{residue of } [\exp(sr)I_n] \text{ at } s=-1/n].$$

The coefficient of the lowest power $(l+1)$ of r turns out to be

$$C[\exp(-r/n)]/(2l+1)!. \tag{A.2}$$

But from (2.3a), we have:

$$R_n^l = {}^+H_n^{l+1}\,{}^+H_n^{l+2}\cdots{}^+H_n^{n-1}R_n^{n-1}.$$

Using (2.2) and (2.4) the coefficient of r^{l+1} can be picked out easily; it is

$$\frac{2^n(n-1)!}{n^{l+2}(2n-1)!l!}\left[\frac{(n+l)!}{(n-l-1)!}\right]^{\frac{1}{2}} \times[(2l+3)(2l+5)\dots(2n-1)] \times[\exp(-r/n)]. \tag{A.3}$$

Equating (A.2) and (A.3) we find C so that

$$I_n = \frac{2^{l+1}}{n^{l+2}}\left[\frac{(n+l)!}{(n-l-1)!}\right]^{\frac{1}{2}}\frac{(s-1/n)^{n-l-1}}{(s+1/n)^{n+l+1}}. \tag{A.4}$$

Then

$$I_{l\,n}{}^{l-1\,l}(s) = \left(\frac{2}{l}\right)^l \frac{1}{l[(2l-1)!]^{\frac{1}{2}}} \times\int_0^\infty r^{l+1}\exp(-sr)R_n^l dr$$

$$= \left(\frac{2}{l}\right)^l \frac{1}{l[(2l-1)!]^{\frac{1}{2}}}\left[-\frac{dI_n}{ds}\right].$$

$$I_{l\,n}{}^{l-1\,l}(s) = \frac{1}{n}\left(\frac{4}{nl}\right)^{l+1}\left[\frac{(n+l)!}{(n-l-1)!(2l-1)!}\right]^{\frac{1}{2}} \times\frac{(s-1/n)^{n-l-2}}{(s+1/n)^{n+l+2}}[(l+1)s-1]. \tag{5.2a}$$

[12] See for example: R. V. Churchill, *Modern Operational Mathematics in Engineering* (McGraw-Hill Book Company, New York, 1944), pp. 170–171.

By putting $l=n'$ and $s=1/n'$, we obtain

$$I_{n'\,n}{}^{n'-1\,n'} = 2^{2n'+2}(nn')^{n'+2} \times\left[\frac{(n+n')!}{(n-n'-1)!(2n'-1)!}\right]^{\frac{1}{2}} \times\frac{(n-n')^{n-n'-2}}{(n+n')^{n+n'+2}}. \tag{4.3b}$$

Case II: Discrete-continuous

In this case I_{in} turns out to be

$$I_{in} = C(s-i/n)^{-in-l-1}(s+i/n)^{in-l-1}$$

$$= C\frac{\exp(2n\tan^{-1}ns - n\pi)}{(s^2+1/n^2)^{l+1}};\quad 0<\tan^{-1}ns<\frac{\pi}{2},$$

where the constant $n\pi$ was determined by the condition that I_{in} remain finite as $n\to\infty$.

The inverse transform here is[13]

$$R_{in}^l = \frac{Cr^{-l}}{2\pi i}\int \exp(sr)(s-i/n)^{-in-l-1} \times(s+i/n)^{in-l-1}ds.$$

From an asymptotic expression for R_{in}^l we find[14] the value of C for the usual normalization. The final result in our notation is

$$I_{in} = \left(\frac{2}{n}\right)^{l+\frac{1}{2}}\left[\frac{2\prod_{p=1}^{l}(p^2+n^2)}{\exp(2n\pi)-1}\right]^{\frac{1}{2}}\frac{\exp(2n\tan^{-1}ns)}{(s^2+1/n^2)^{l+1}},$$

and the corresponding starting points are found to be as given in (5.2b) and (4.3c).

[13] Also given by Schrödinger, Ann. der Phys. **79**, 361 (1926).

[14] See for example Bethe, *Handbuch der Physik* (Berlin, Verlag Julius Springer, 1933), 2nd ed., pp. 290–292.