Motion of an Electron in the Field of a Magnetic Pole

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The motion of an electron in the field of a magnetic pole is considered. It is shown that in spite of its magnetic moment the electron has no bound states.

RECENTLY Dirac has revived interest in a theory of the electromagnetic field proposed by him some time ago[1] which allows the existence of free magnetic poles. As a consequence of his theory he is able to deduce the fact that the charge of any elementary particle is always an integral multiple of a certain fixed unit. The motion of an electron in the field of a magnetic pole was considered by Dirac[1] and he found that the electron cannot have any bound states. However, Dirac did not take into account the spin of the electron and since the spin gives rise to a magnetic moment, it seemed conceivable that the above conclusion might cease to hold if one used the correct equation of motion. The object of this note is to investigate this point. It turns out that Dirac's result is still valid and so the electron cannot be bound to a magnetic pole.

We put the velocity of light and the Planck's constant equal to 1 and 2π respectively. It is convenient to use tensor notation. Let $(x, y, z) = (x^1, x^2, x^3)$ be the Cartesian coordinates in space and let ϵ^{ijk} be the antisymmetric tensor such that $\epsilon^{123}=1$. Further, let g_{ik} be the metric tensor corresponding to the quadratic form

$$dx^2+dy^2+dz^2=g_{ik}dx^i dx^k.$$

The equations determining the vector potential A_k can then be written as

$$\epsilon^{ijk}(\partial A_j/\partial x^i)=H^k, \tag{1}$$

where H_k are the components of the magnetic field. Now since we wish to consider the field of a magnetic pole it is convenient to introduce polar coordinates (r, θ, φ) given by

$$x=r\sin\theta\cos\varphi, \quad y=r\sin\theta\sin\varphi, \quad z=r\cos\theta,$$

so that

$$dx^2+dy^2+dz^2=dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2.$$

[1] P. A. M. Dirac, Proc. Roy. Soc. **133**, 60 (1931).

Therefore if $g_{\alpha\beta}(\alpha, \beta$ running over $r, \theta, \varphi)$ denotes the metric tensor in the polar coordinates, we have

$$\begin{aligned} &g_{rr}=1, \quad g_{\theta\theta}=r^2, \quad g_{\varphi\varphi}=r^2\sin^2\theta, \\ &g^{rr}=1, \quad g^{\theta\theta}=1/r^2, \quad g^{\varphi\varphi}=1/r^2\sin^2\theta, \\ &g_{\alpha\beta}=g^{\alpha\beta}=0, \quad \alpha\neq\beta. \end{aligned} \tag{2}$$

On transforming (1) to the new coordinate system (r, θ, φ) according to the usual rules of tensor calculus we get

$$\epsilon^{\alpha\beta\gamma}(\partial A_\beta/\partial\xi^\alpha)=H^\gamma, \tag{3}$$

where α, β, γ run over the three indices r, θ, φ and $(\xi^r, \xi^\theta, \xi^\varphi)=(r, \theta, \varphi)$. It follows from the transformation laws that

$$\epsilon^{r\theta\varphi}=1/(g)^{\frac{1}{2}}=1/r^2\sin\theta,$$

where $g=r^4\sin^2\theta$ is the determinant of the matrix formed by the components $g_{\alpha\beta}$. Let $-e$ be the charge of the electron and n an integer and consider a magnetic pole of strength $\frac{1}{2}n/e$ at the origin. Then $H^\theta=H^\varphi=0$ and $H^r=\frac{1}{2}n/er^2$. A possible solution of (3) is now obtained by putting $A_r=A_\theta=0$ and choosing A_φ such that

$$\frac{1}{r^2\sin\theta}\frac{\partial A_\varphi}{\partial\theta}=\frac{n}{2er^2}. \tag{4}$$

This gives $A_\varphi=(n/2e)(1-\cos\theta)$, the constant of integration having been so chosen as to make $A_\varphi=0$ for $\theta=0$ so that the nodal line[1] runs from the origin along the line $\theta=\pi$. This is seen as follows. Consider the integral

$$\int_c A_k dx^k=\int_c A_\alpha d\xi^\alpha=\int_c A_\varphi d\varphi,$$

taken round a closed curve c surrounding the line $\theta=\theta_0$. Clearly if $\theta_0\neq 0$ or π, $\int_c d\varphi=0$ and therefore the integral tends to zero as c shrinks to a point on $\theta=\theta_0$. On the other hand if $\theta_0=0$ or π, $\int_c d\varphi=2\pi$ and the value of the integral tends to $(\pi n/e)(1-\cos\theta_0)$ i.e., 0 or $2\pi n/e$ according as $\theta_0=0$ or π.

Now we come to the equation of motion of the electron. As usual let σ_1, σ_2, σ_3 and ρ_1, ρ_2, ρ_3 be two independent sets of Pauli matrices.[2] Transform σ_1, σ_2, σ_3 as components of a vector to polar coordinates so that

$$\begin{aligned}\sigma_r&=\sigma_k\frac{\partial x^k}{\partial r}=\frac{1}{r}(\boldsymbol{\sigma}\mathbf{x}),\\ \sigma_\theta&=\sigma_k\frac{\partial x^k}{\partial \theta}=\frac{1}{r\sin\theta}(\mathbf{x}(\boldsymbol{\sigma}\mathbf{x})-(\mathbf{x}\mathbf{x})\boldsymbol{\sigma})_3,\\ \sigma_\varphi&=\sigma_k\frac{\partial x^k}{\partial \varphi}=[\mathbf{x}\times\boldsymbol{\sigma}]_3,\end{aligned}\tag{5}$$

where $\mathbf{x}=(x_1,x_2,x_3)$ and $\boldsymbol{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$ and the usual notations for scalar and vector products are used. The Hamiltonian H for the electron can now be written as

$$\begin{aligned}H&=-\rho_1(\boldsymbol{\sigma}\mathbf{p})-\rho_3\mu-\rho_1 e\sigma^\varphi A_\varphi\\ &=-\rho_1(\boldsymbol{\sigma}\mathbf{p})-\rho_3\mu-\rho_1\frac{[\mathbf{x}\times\boldsymbol{\sigma}]_3}{r^2\sin^2\theta}\frac{n}{2}(1-\cos\theta),\end{aligned}\tag{6}$$

where $\mathbf{p}=1/i(\partial/\partial x,\partial/\partial y,\partial/\partial z)$ is the momentum and μ the mass of the electron. Our problem is to find a wave function ψ such that

$$H\psi=E\psi,\tag{7}$$

where E is some eigenvalue of H. Now notice that

$$[\mathbf{x}\times\mathbf{p}]_3=\frac{1}{i}\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)=\frac{1}{i}\frac{\partial}{\partial\varphi},$$

and $[\mathbf{x}\times\mathbf{p}]+\frac{1}{2}\boldsymbol{\sigma}$ commutes with $H_0=-\rho_1(\boldsymbol{\sigma}\mathbf{p})-\rho_3\mu$ while $[\mathbf{x}\times\mathbf{p}]_3+\frac{1}{2}\sigma_3$ commutes with $[\mathbf{x}\times\boldsymbol{\sigma}]_3$. Therefore

$$[\mathbf{x}\times\mathbf{p}]_3+\tfrac{1}{2}\sigma_3=\frac{1}{i}\frac{\partial}{\partial\varphi}+\tfrac{1}{2}\sigma_3,$$

commutes with H. Hence we can choose ψ in such a way that

$$\left(\frac{1}{i}\frac{\partial}{\partial\varphi}+\tfrac{1}{2}\sigma_3\right)\psi=M\psi,$$

where M is half-an-odd integer. Therefore

$$\psi=e^{i(M-\frac{1}{2}\sigma_3)\varphi}\psi',\tag{8}$$

where ψ' is independent of φ. Now

$$\sigma^k\frac{\partial}{\partial x^k}=\sigma^\alpha\frac{\partial}{\partial\xi^\alpha}=\sigma_r\frac{\partial}{\partial r}+\frac{\sigma_\theta}{r^2}\frac{\partial}{\partial\theta}+\frac{\sigma_\varphi}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi}.\tag{9}$$

From (5) we get

$$\begin{aligned}\sigma_r&=\sin\theta(\sigma_1\cos\varphi+\sigma_2\sin\varphi)+\sigma_3\cos\theta\\ &=\sin\theta\sigma_1 e^{i\sigma_3\varphi}+\sigma_3\cos\theta\\ &=e^{-i\sigma_3\varphi/2}(\sigma_3\cos\theta+\sigma_1\sin\theta)e^{i\sigma_3\varphi/2}\\ &=e^{-i\sigma_3\varphi/2}e^{-i\sigma_2\theta/2}\sigma_3e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2}.\end{aligned}\tag{10a}$$

Similarly

$$\begin{aligned}\sigma_\theta&=r\cos\theta(\sigma_1\cos\varphi+\sigma_2\sin\varphi)-\sigma_3r\sin\theta\\ &=r\cos\theta\sigma_1e^{i\sigma_3\varphi}-\sigma_3r\sin\theta\\ &=re^{-i\sigma_3\varphi/2}(\sigma_1\cos\theta-\sigma_3\sin\theta)e^{i\sigma_3\varphi/2}\\ &=re^{-i\sigma_3\varphi/2}e^{-i\sigma_2\theta/2}\sigma_1e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2},\end{aligned}\tag{10b}$$

$$\begin{aligned}\sigma_\varphi&=r\sin\theta(\sigma_2\cos\varphi-\sigma_1\sin\varphi)\\ &=r\sin\theta e^{-i\sigma_3\varphi/2}e^{-i\sigma_2\theta/2}\sigma_2e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2}.\end{aligned}\tag{10c}$$

Also

$$e^{i\sigma_2\theta/2}\frac{\partial}{\partial\theta}=\left(\frac{\partial}{\partial\theta}-i\frac{\sigma_2}{2}\right)e^{i\sigma_2\theta/2},$$

$$\begin{aligned}e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2}\frac{\partial}{\partial\varphi}&=e^{i\sigma_2\theta/2}\left(\frac{\partial}{\partial\varphi}-i\frac{\sigma_3}{2}\right)e^{i\sigma_3\varphi/2}\\ &=\left(\frac{\partial}{\partial\varphi}-\frac{i}{2}e^{i\sigma_2\theta}\sigma_3\right)e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2}\\ &=\left\{\frac{\partial}{\partial\varphi}-\frac{i}{2}(\sigma_3\cos\theta-\sigma_1\sin\theta)\right\}e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2}.\end{aligned}$$

Hence

$$\begin{aligned}\sigma^k\partial/\partial x^k=e^{-i\sigma_2\theta/2}e^{-i\sigma_3\varphi/2}\Bigg[&\sigma_3\frac{\partial}{\partial r}+\frac{\sigma_1}{r}\left(\frac{\partial}{\partial\theta}-\frac{i\sigma_2}{r}\right)\\ &+\frac{\sigma_2}{r\sin\theta}\left\{\frac{\partial}{\partial\varphi}-\frac{i}{2}(\sigma_3\cos\theta-\sigma_1\sin\theta)\right\}\Bigg]e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2}.\end{aligned}$$

Similarly

$$\begin{aligned}ie\sigma^\varphi A_\varphi&=i\frac{\sigma_\varphi}{r^2\sin^2\theta}\frac{n}{2}(1-\cos\theta)\\ &=ie^{-i\sigma_3\varphi/2}e^{-i\sigma_2\theta/2}\frac{\sigma_2}{r\sin\theta}\frac{n}{2}(1-\cos\theta)\\ &\qquad\times e^{i\sigma_2\theta/2}e^{i\sigma_3\varphi/2}\end{aligned}$$

[2] See P. A. M. Dirac, *Principles of Quantum Mechanics* (Clarendon Press, Oxford, 1947), Chap. XI.

Hence on substituting (8) in (7) and putting $e^{i\sigma_2\theta/2}\psi'=\psi_0$, we get

$$\left[\tfrac{1}{2}\rho_1\left\{\sigma_3\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)+\frac{\sigma_1}{r}\left(\frac{\partial}{\partial\theta}-\frac{\sigma_3}{\sin\theta}\left\{M+\frac{n}{2}(1-\cos\theta)-\frac{\sigma_3}{2}\cos\theta\right\}\right)\right\}+\rho_3\mu+E\right]\psi_0=0. \quad (11)$$

It is clear that

$$-K^2=\left\{\sigma_1\left[\frac{\partial}{\partial\theta}-\frac{\sigma_3}{\sin\theta}\left(M+\frac{n}{2}(1-\cos\theta)-\frac{\sigma_3\cos\theta}{2}\right)\right]\right\}^2,$$

commutes with the operator acting on ψ_0 in (11). Hence we can choose ψ_0 to be an eigenvector of K^2. But

$$-K^2=\left\{\frac{\partial}{\partial\theta}+\tfrac{1}{2}\cot\theta+\frac{\sigma_3}{\sin\theta}\left(M+\frac{n}{2}(1-\cos\theta)\right)\right\}\left\{\frac{\partial}{\partial\theta}+\tfrac{1}{2}\cot\theta-\frac{\sigma_3}{\sin\theta}\left(M+\frac{n}{2}(1-\cos\theta)\right)\right\}$$

$$=\frac{1}{\sin\theta}\left\{\frac{\partial}{\partial\theta}-\tfrac{1}{2}\cot\theta+\frac{\sigma_3}{\sin\theta}\left(M+\frac{n}{2}(1-\cos\theta)\right)\right\}\left\{\sin\theta\frac{\partial}{\partial\theta}+\tfrac{1}{2}\cos\theta-\sigma_3\left(M+\frac{n}{2}(1-\cos\theta)\right)\right\}$$

$$=\frac{1}{(1-u^2)}\left\{(1-u^2)\frac{d}{du}-\sigma_3\left(M+\frac{n}{2}\right)+\left(\frac{n}{2}\sigma_3+\tfrac{1}{2}\right)u\right\}\left\{(1-u^2)\frac{d}{du}+\sigma_3\left(M+\frac{n}{2}\right)-\left(\frac{n}{2}\sigma_3+\tfrac{1}{2}\right)u\right\}$$

$$=(1-u^2)\frac{d^2}{du^2}-2u\frac{d}{du}-\frac{\left\{\sigma_3\left(M+\frac{n}{2}\right)-\left(\frac{n}{2}\sigma_3+\frac{1}{2}\right)u\right\}^2}{1-u^2}-\left(\frac{n}{2}\sigma_3+\tfrac{1}{2}\right)^2$$

$$=(1-u^2)\frac{d^2}{du^2}-2u\frac{d}{du}-\frac{\left\{\left(M+\frac{n}{2}\right)-\left(\frac{n}{2}+\frac{\sigma_3}{2}\right)u\right\}^2}{1-u^2}-\left(\frac{n}{2}+\frac{\sigma_3}{2}\right)^2+\frac{n^2-1}{4}, \quad (12)$$

where $u=\cos\theta$. Now it is known that if m and j are both integral or both half-integral the only eigenfunctions of the operator

$$(1-u^2)\frac{d^2}{du^2}-2u\frac{d}{du}-\frac{(m-ju)^2}{1-u^2}-j^2, \quad (13)$$

corresponding to the interval $-1\leqslant u\leqslant 1$ are the Jacobi polynomials $P^k{}_{m,j}(u)$ and the corresponding eigenvalues are $-k(k+1)$, where k is to be so chosen that $k\geqslant|m|$, $|j|$, and $k-j$ is an integer. $P^k{}_{m,j}(\cos\theta)$ is defined by the identity

$$\frac{\left(t_1\cos\frac{\theta}{2}+t_2\sin\frac{\theta}{2}\right)^{k-j}\left(-t_1\sin\frac{\theta}{2}+t_2\cos\frac{\theta}{2}\right)^{k+j}}{(k-j!k+j!)^{\frac{1}{2}}}=\sum_m\frac{t_1^{k-m}t_2^{k+m}}{(k-m!k+m!)^{\frac{1}{2}}}P^k{}_{m,j}(\cos\theta), \quad (14)$$

where m runs through the set of values k, $k-1$, $\cdots$, $-k$. Write

$$w_1=t_1\cos\theta/2+t_2\sin\theta/2,$$
$$w_2=t_2\cos\theta/2-t_1\sin\theta/2.$$

If we observe that $\partial w_1/\partial\theta=w_2$, $\partial w_2/\partial\theta=-w_1$, we get immediately on differentiating (14) with respect to θ

$$((k-j)(k+j+1))^{\frac{1}{2}}P^k{}_{m,j+1}(\cos\theta)-((k+j)(k-j+1))^{\frac{1}{2}}P^k{}_{m,j-1}(\cos\theta)=2\frac{d}{d\theta}P^k{}_{m,j}(\cos\theta). \quad (15)$$

Now keep θ fixed and transform from the variables t_1, t_2 to w_1, w_2. Then

$$t_1\frac{\partial}{\partial t_1}-t_2\frac{\partial}{\partial t_2}=\cos\theta\left(w_1\frac{\partial}{\partial w_1}-w_2\frac{\partial}{\partial w_2}\right)-\sin\theta\left(w_1\frac{\partial}{\partial w_2}+w_2\frac{\partial}{\partial w_1}\right).$$

On applying this operator to (14) we get

$$-2j\cos\theta P^k_{m,j}(\cos\theta)$$
$$-\sin\theta(((k-j)(k+j+1))^{\frac{1}{2}}P^k_{m,j+1}(\cos\theta)$$
$$+((k+j)(k-j+1))^{\frac{1}{2}}P^k_{m,j-1}(\cos\theta))$$
$$=-2mP^k_{m,j}(\cos\theta). \quad (16)$$

From (15) and (16) we find at once that

$$\left\{(1-u^2)\frac{d}{du}+m-ju\right\}P^k_{m,j}(u)$$
$$=((k+j)(k-j+1))^{\frac{1}{2}}(1-u^2)^{\frac{1}{2}}P^k_{m,j-1}(u), \quad (17a)$$

$$\left\{(1-u^2)\frac{d}{du}-m+ju\right\}P^k_{m,j}(u)$$
$$=((k-j)(k+j+1))^{\frac{1}{2}}(1-u^2)^{\frac{1}{2}}P^k_{m,j+1}(u). \quad (17b)$$

Now for (12) $m=M+n/2$ and

$$j=(n/2)+(\sigma_3/2)=\begin{pmatrix}\frac{n+1}{2}\\ \frac{n-1}{2}\end{pmatrix}$$

if we choose a representation in which σ_3 is diagonal. Therefore the eigenfunction ψ_0 can be written as

$$\psi_0=\begin{pmatrix}P^k{}_{M+(n/2),(n+1)/2}(\cos\theta) & \psi^+\\ P^k{}_{M+(n/2),(n-1)/2}(\cos\theta) & \psi^-\end{pmatrix},$$

where ψ^+, ψ^- depend only on r and

$$k\geqslant\left|M+\frac{n}{2}\right|,\ \left|\frac{n-1}{2}\right|,\ \left|\frac{n+1}{2}\right| \quad\text{and}\quad k+\frac{n+1}{2}$$

is an integer. Making use of (17) we find that

$$\sigma_1\left[\frac{\partial}{\partial\theta}-\frac{\sigma_3}{\sin\theta}\left(M+\frac{n}{2}-\frac{n+\sigma_3}{2}\cos\theta\right)\right]$$
$$\times\begin{pmatrix}P^k{}_{M+(n/2),(n+1)/2} & \psi^+\\ P^k{}_{M+(n/2),(n-1)/2} & \psi^-\end{pmatrix}$$
$$=-\left(\left(k+\frac{n}{2}+\frac{1}{2}\right)\left(k-\frac{n}{2}+\frac{1}{2}\right)\right)^{\frac{1}{2}}\sigma_1$$
$$\times\begin{pmatrix}P^k{}_{M+(n/2),(n-1)/2} & \psi^+\\ P^k{}_{M+(n/2),(n+1)/2} & \psi^-\end{pmatrix}$$
$$=-\left(\left(k+\frac{n}{2}+\frac{1}{2}\right)\left(k-\frac{n}{2}+\frac{1}{2}\right)\right)^{\frac{1}{2}}$$
$$\times\begin{pmatrix}P^k{}_{M+(n/2),(n+1)/2} & \psi^-\\ P^k{}_{M+(n/2),(n-1)/2} & \psi^+\end{pmatrix}.$$

Therefore (11) now becomes

$$\left[\frac{1}{i}\rho_1\left\{\sigma_3\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)-\frac{\sigma_1}{r}K\right\}+\rho_3\mu+E\right]\begin{pmatrix}\psi^+\\ \psi^-\end{pmatrix}=0,$$

where

$$K=\left(\left(k+\frac{n+1}{2}\right)\left(k-\frac{n-1}{2}\right)\right)^{\frac{1}{2}}.$$

If we write $\psi=\begin{pmatrix}\psi^+\\ \psi^-\end{pmatrix}$ this equation may be written as

$$\left\{\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)+\frac{i\sigma_2}{r}K\right\}\psi=i\sigma_3(i\rho_2-E\rho_1)\psi. \quad (19)$$

Choose ρ_3 diagonal and split ψ into $\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ with respect to ρ_3 so that

$$\rho_3\psi=\rho_3\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\begin{pmatrix}\psi_1\\ -\psi_2\end{pmatrix}.$$

Then

$$(i\rho_2\mu-E\rho_2)\psi=\begin{pmatrix}0 & \mu & -E\\ -\mu & -E & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}.$$

Put

$$\mu+E=\frac{1}{a_1},\quad \mu-E=\frac{1}{a_2}.$$

Then (19) can be written as

$$\left\{\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)+\frac{i\sigma_2}{r}K\right\}\psi_1=\frac{i\sigma_3}{a_2}\psi_2, \quad (20a)$$

$$\left\{\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)+\frac{i\sigma_2}{r}K\right\}\psi_2=-\frac{i\sigma_3}{a_1}\psi_1. \quad (20b)$$

Now choose a representation in which σ_2 is diagonal so that

$$\sigma_2=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \sigma_3=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_1=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}.$$

Write

$$\psi_1=\begin{pmatrix}\psi_1{}^+\\ \psi_1{}^-\end{pmatrix},\quad \psi_2=\begin{pmatrix}\psi_2{}^+\\ \psi_2{}^-\end{pmatrix}.$$

Then (20) is equivalent to the set of equations

$$\left\{\frac{\partial}{\partial r}+\frac{1}{r}+\frac{iK}{r}\right\}\psi_1{}^+=\frac{i}{a_2}\psi_2{}^-, \qquad (21a)$$

$$\left\{\frac{\partial}{\partial r}+\frac{1}{r}-\frac{iK}{r}\right\}\psi_2{}^-=-\frac{i}{a_2}\psi_1{}^+, \qquad (21b)$$

$$\left\{\frac{\partial}{\partial r}+\frac{1}{r}-\frac{iK}{r}\right\}\psi_1{}^-=\frac{i}{a_2}\psi_2{}^+, \qquad (21c)$$

$$\left\{\frac{\partial}{\partial r}+\frac{1}{r}+\frac{iK}{r}\right\}\psi_2{}^+=-\frac{i}{a_2}\psi_1{}^-. \qquad (21d)$$

Put $a=(a_1a_2)^{\frac{1}{2}}$ the square root being positive in case $a_1a_2>0$. Then we get

$$\left(\frac{\partial}{\partial r}+\frac{1}{r}-\frac{iK}{r}\right)\left(\frac{\partial}{\partial r}+\frac{1}{r}+\frac{iK}{r}\right)\psi_1{}^+ =\left(\frac{i}{a_1}\right)\left(\frac{-i}{a_2}\right)\psi_1{}^+,$$

i.e.,

$$\left\{\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{K^2-iK}{r^2}-\frac{1}{a^2}\right\}\psi_1{}^+=0. \qquad (22a)$$

Similarly

$$\left\{\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{K^2+iK}{r^2}-\frac{1}{a^2}\right\}\psi_1{}^-=0, \qquad (22b)$$

(22) is completely equivalent to (21) since (21a) and (21c) can be regarded as the definitions of $\psi_2{}^-$ and $\psi_2{}^+$ respectively. Since the operator

$$\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{K^2\pm iK}{r^2}$$

is homogeneous in r it is clear that if $\psi(r)$ is one of its eigenfunctions corresponding to the eigenvalue $1/a^2$ and α is any real constant, the function $\psi(\alpha r)$ is another eigenfunction corresponding to the eigenvalue α^2/a^2. Hence the eigenvalues $E=(\mu^2-1/a^2)^{\frac{1}{2}}$ cannot form a discrete spectrum. We shall now show that for all permissible solutions of (22) $a^2<0$. It is sufficient to consider (22a). Put $\varphi_1{}^+=fe^{-r/a}$. Then

$$\left\{\frac{\partial^2}{\partial r^2}+\left(\frac{2}{r}-\frac{2}{a}\right)\frac{\partial}{\partial r}+\frac{K^2-iK}{r^2}-\frac{2}{ra}\right\}f=0.$$

The point $r=0$ is a singular point of this equation. According to the usual procedure for solving second-order linear differential equations with analytic coefficients, we make the substitution

$$f=\sum_{\nu\geq 0} c_\nu r^{\nu+\alpha},$$

where $c_0=1$ and ν runs over all non-negative integers. The indical equation is

$$\alpha(\alpha-1)+2\alpha+K^2-iK=0,$$

i.e.,

$$(\alpha+\tfrac{1}{2})^2+\left(K-\frac{i}{2}\right)^2=0.$$

Hence $\alpha=iK$ or $-iK-1$. However the boundary condition at $r=0$ requires[3] that $r\psi_1{}^+\to 0$ as $r\to 0$. Hence only $\alpha=iK$ is permissible. On substituting

$$f=\sum_{\nu\geq 0} c_\nu r^{\nu+iK},$$

in (23) and equating the coefficients of the various powers of r to zero we get the recurrence relation

$$c_{\nu+1}=\frac{2}{a}\frac{\nu+iK+1}{(\nu+1)(\nu+2iK+2)}.$$

Since K is real it is clear that the series cannot terminate. It converges like $e^{2r/a}$ and therefore for large r, $\psi_1{}^+=fe^{-r/a}$ behaves like $e^{r/a}$. Therefore from the boundary condition at infinity it follows that only those solutions are permissible for which a is pure imaginary, i.e., $a^2<0$, or $E^2>\mu^2$. Thus the electron is never bound to the magnetic pole.

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[3] See reference 2, p. 269.