Enumeration of Potentials for Which One-Particle Schroedinger Equations Are Separable

L. P. EISENHART
Fine Hall, Princeton, New Jersey

(Received March 2, 1948)

IN 1891 Stäckel[1] solved the problem of determining the quantities H_i in the Hamiltonian equation

$$\textstyle\sum_i(1/H_i^2)(\partial\theta/\partial x_i)^2+V-\alpha_1=0,$$

where α_1 is a constant, so that the variables are separable, that is, so that the solution is of the form $\theta=\sum_i X_i$, where X_i is a function of x_i alone. The complete solution is: given any n^2 functions φ_{ij}, where φ_{ij} is a function of x_i alone, φ^{ij} is the cofactor of φ_{ij} in the determinant $\varphi(\neq 0)$ of the functions φ_{ij}; then $H_i^2=\varphi/\varphi^{ij}$. The potential function V in any such coordinate system is

$$V=\textstyle\sum_i(f_i(x_i)/H_i^2), \tag{1}$$

when f_i is an arbitrary function of x_i.

In 1927 Robertson[2] showed that for the Schroedinger equation

$$\textstyle\sum_i(1/H)(\partial/\partial xi)[(H/H_i^2)(\partial\psi/\partial xi)]+k^2(E-V)\psi=0,$$
$$H=\textstyle\prod_i H_i, \quad (i=1,\cdots,n),$$

to admit by separation of the variables a solution of the form $\prod_i X_i$, where X_i is a function of x_i alone, the functions H_i must be of the Stäckel form, the potential function be of the form (1), and the Stäckel determinant φ be such that

$$\varphi=\textstyle\prod_i(H_i/\theta_i), \tag{2}$$

where θ_i is a function of x_i at most.

In 1934 I showed[3] that a necessary and sufficient condition that functions H_i be of the Stäckel form is that the following equations be satisfied:

$$(\partial^2 \log H_i^2/\partial x_i\partial x_j)+(\partial \log H_i^2/\partial x_j)(\partial \log H_j^2/\partial x_i)=0, \quad (i\neq j)$$

$$\frac{\partial^2 \log H_i^2}{\partial x_j\partial x_k}-\frac{\partial \log H_i^2}{\partial x_j}\frac{\partial \log H_i^2}{\partial x_k}+\frac{\partial \log H_i^2}{\partial x_j}\frac{\partial \log H_j^2}{\partial x_k}+\frac{\partial \log H_i^2}{\partial x_k}\frac{\partial \log H_k^2}{\partial x_j}=0, \quad (i,j,k\neq).$$

Proceeding from this result I derived all the real forms of H_i for Euclidean 3-space, ten in number in addition to the Cartesian case $H_i=1$; and showed that the condition (2) is automatically satisfied in Euclidean 3-space. In a note in this journal[4] I listed the ten possible forms of H_i.

For each of the ten possible forms Eq. (1) gives the expression for the potential function V in the particular coordinates x_i of each canonical form. Recently Professor Wigner raised the question as to whether it is possible to obtain the expression for V in Cartesian coordinates in each of the ten cases, as a possible aid to those making an investigation of particular problems. This paper gives an answer to this question, deriving the possible forms of V in Cartesian coordinates x, y, z.

What follows after the expressions for the H's in each case are the expressions of Cartesian coordinates x, y, z in terms of the corresponding x_i. Then follow xi as functions of x, y, z, except in the last two cases where a general solution is impossible. For the possible cases the form of V is given in Cartesian coordinates.

I.
$$H_1^2=1,\quad H_2^2=x_1^2,\quad H_3^2=1,$$
$$x=x_1\cos x_2,\quad y=x_1\sin x_2,\quad z=x_3,$$
$$x_1=(x^2+y^2)^{\frac{1}{2}},\quad \tan x_2=y/x,\quad x_3=z,$$
$$V=\varphi[(x^2+y^2)^{\frac{1}{2}}]+[\psi(y/x)/(x^2+y^2)]+f(z),$$

where here, and in what follows, φ, ψ, and f are arbitrary functions of their arguments.

II. $H_1^2=H_2^2=\frac{1}{2}a^2(\cosh 2x_1-\cos 2x_2),\quad H_3^2=1,$

where here, and in later cases, a is an arbitrary constant.

[1] Paul G. Stäckel, Ueber die Integration des Hamilton—Jacobischen Differential gleichunger Mittelst separation der variabeln, Habilitationschrift, Halle.

[2] H. P. Robertson, "Bemerkung über separierbare Systeme in der Wellenmechanik," Math. Ann. **98**, 749–752 (1927).

[3] L. P. Eisenhart, "Separable systems of Stäckel," Ann. Math. **35**, 284–305 (1934).

[4] L. P. Eisenhart, "Separable systems in Euclidean 3-space," Phys. Rev. **45**, 427–428 (1934).

$$x = a\cosh x_1 \cos x_2, \quad y = a\sinh x_1 \sin x_2, \quad z = x_3,$$
$$\sinh^2 x_1 = A+B, \quad \sin^2 x_2 = -A+B, \quad x_3 = z,$$

where

$$A = (1/2a^2)(x^2+y^2-a^2), \quad B = (A^2+(y^2/a^2))^{\frac{1}{2}},$$
$$V = [a^2/(A^2+(y^2/a^2))^{\frac{1}{2}}] \times [\varphi(A+B)+\psi(-A+B)]+f(z).$$

III. $H_1^2 = H_2^2 = x_1^2+x_2^2, \quad H_3^2 = 1,$

$$x = \tfrac{1}{2}(x_1^2-x_2^2), \quad y = x_1x_2, \quad z = x_3,$$
$$x_1^2 = x+(y^2-x^2)^{\frac{1}{2}}, \quad x_2^2 = -x+(y^2-x^2)^{\frac{1}{2}}, \quad x_3 = z,$$
$$V = [1/2(y^2-x^2)^{\frac{1}{2}}][\varphi(x+(y^2-x^2)^{\frac{1}{2}})^{\frac{1}{2}} +\psi(-x+(y^2-x^2)^{\frac{1}{2}})^{\frac{1}{2}}+f(z).$$

Before proceeding with the other cases we remark that it is evident from the expressions in Cases I, II, and III that, if one puts $z = x_3 = 0$, one has cases of separable systems in Euclidean 2-space. It can be shown that these are the only possibilities.

IV. $H_1^2 = 1, \quad H_2^2 = x_1^2, \quad H_3^2 = x_1^2 \sin^2 x_2,$

$$x = x_1 \sin x_2 \cos x_3, \quad y = x_1 \sin x_2 \sin x_3, \quad z = x_1 \cos x_2,$$
$$x_1 = r = (x^2+y^2+z^2)^{\frac{1}{2}}, \quad \cos x_2 = z/r, \quad \tan x_3 = y/x,$$
$$V = \varphi(r)+(1/r^2)\psi(z/r)+[f(y/x)/(x^2+y^2)].$$

V. $H_1^2 = H_2^2 = x_3^2[k^2cn^2(x_1, k)+k'^2cn^2(x_2, k')], \quad H_3^2 = 1,$

where the constants k and k' of the elliptic functions are in the relation $k^2+k'^2 = 1$.[5]

$$x = x_3dn(x_1, k)sn(x_2, k'),$$
$$y = x_3sn(x_1, k)dn(x_2, k'),$$
$$z = x_3cn(x_1, k)cn(x_2, k').$$

Making use of the relations

$$sn^2+cn^2 = 1, \quad dn^2+k^2sn^2 = 1, \quad dn^2-k^2cn^2 = k'^2,$$

we obtain from the above

$$k^2cn^2(x_1, k) = A+B, \quad k'^2cn^2(x_2, k') = -A+B, \quad x_3^2 = r^2,$$

where

$$A = (1/2r^2)(k^2x^2-k'^2y^2+(k^2-k'^2)z^2),$$
$$B = (A^2+(k^2k'^2z^2/r^2))^{\frac{1}{2}},$$
$$V = (1/2r^2B)[\varphi((A+B)^{\frac{1}{2}}/k) +\psi(-A+B)^{\frac{1}{2}}/k')]+f(r).$$

VI. $H_1^2 = H_2^2 = x_1^2+x_2^2, \quad H_3^2 = x_1^2x_2^2,$

$$x = x_1x_2 \cos x_3, \quad y = x_1x_2 \sin x_3, \quad z = \tfrac{1}{2}(x_1^2-x_2^2),$$
$$x_1^2 = r+z, \quad x_2^2 = r-z, \quad \tan x_3 = y/x,$$
$$V = (1/2r)[\varphi((r+z)^{\frac{1}{2}})+\psi((r-z)^{\frac{1}{2}})] +[1/(x^2+y^2)]f(y/x).$$

VII. $H_1^2 = H_2^2 = a^2(\sinh^2 x_1+\sin^2 x_2),$
$H_3^2 = a^2 \sinh^2 x_1 \sin^2 x_2,$

$$x = a \sinh x_1 \sin x_2 \cos x_3, \quad y = a \sinh x_1 \sin x_2 \sin x_3,$$
$$z = a \cosh x_1 \cos x_2,$$
$$\sinh^2 x_1 = A+B, \quad \sin^2 x_2 = -A+B, \quad \tan x_3 = y/x,$$

where

$$A = (1/2a^2)(r^2-a^2), \quad B = (A^2+(x^2+y^2)/a^2)^{\frac{1}{2}},$$
$$V = (1/2a^2B)[\varphi((A+B)^{\frac{1}{2}})+\psi((-A+B)^{\frac{1}{2}})] +f(y/x)/(x^2+y^2).$$

VIII. $H_1^2 = H_2^2 = a^2(\sinh^2 x_1+\cos^2 x_2),$
$H_3^2 = a^2 \cosh^2 x_1 \sin^2 x_2,$

$$x = a \cosh x_1 \sin x_2 \cos x_3, \quad y = a \cosh x_1 \sin x_2 \sin x_3,$$
$$z = a \sinh x_1 \cos x_2,$$
$$\cosh^2 x_1 = A+B, \quad \cos^2 x_2 = -A+B, \quad \tan x_3 = y/x,$$

where

$$A = (1/2a^2)(r^2-a^2), \quad B = (A^2+(z^2/a^2))^{\frac{1}{2}},$$
$$V = (1/2a^2B)[\varphi((A+B)^{\frac{1}{2}}) +\psi((-A+B)^{\frac{1}{2}})]+f(y/x).$$

IX. $H_i^2 = [(x_i-x_j)(x_i-x_k)/f(x_i)],$

$$f(x_i) = 4(\alpha-x_i)(\beta-x_i)(\gamma-x_i), \quad (i, j, k \neq),$$
$$x^2 = [\textstyle\prod_i(\alpha-x_i)]/(\alpha-\beta)(\alpha-\gamma),$$
$$y^2 = [\textstyle\prod_i(\beta-x_i)]/(\beta-\alpha)(\beta-\gamma),$$
$$z^2 = [\textstyle\prod_i(\gamma-x_i)]/(\gamma-\alpha)(\gamma-\beta),$$

where $\alpha > x_1 > \beta > x_2 > \gamma > x_3$. From these expressions we have

$$x^2/(\alpha-x_i)+y^2/(\beta-x_i)+z^2/(\gamma-x_i) = 1 \quad (i = 1, 2, 3);$$

consequently, the x_i are roots of the cubic

$$t^3-at^2+bt-c = 0, \tag{i}$$

where

$$a = \alpha+\beta+\gamma-r^2,$$
$$b = \alpha\beta+\beta\gamma+\gamma\alpha-(\beta+\gamma)x^2 -(\gamma+\alpha)y^2-(\alpha+\beta)z^2, \tag{ii}$$
$$c = \alpha\beta\gamma-\beta\gamma x^2-\gamma\alpha y^2-\alpha\beta z^2.$$

If in Eq. (i) we put

$$t = u+(a/3) \tag{iii}$$

[5] Cf. Benjamin O. Peirce, *A Short Table of Integrals* (Ginn and Company, Boston, 1929), pp. 84–87.

the resulting equation in (ii) is

$$u^3 - pu^2 + q = 0, \quad p = \tfrac{1}{3}a^2 - b, \quad q = -(2/27)a^3 + (ab/3) - c. \tag{iv}$$

For $q=0$, which is an equation of the third degree in x^2, y^2, z^2, the expressions for x_i are

$$x_1 = (a/3) + (p)^{\frac{1}{2}}, \quad x_2 = a/3, \quad x_3 = (a/3) - (p)^{\frac{1}{2}};$$

p must be positive. Also there are the conditions $\alpha > x_1 > \beta > x_2 > \gamma > x_3$ to be satisfied. In this case

$$H_1^2 = 2p/f(x_1), \quad H_2^2 = -p/f(x_2), \quad H_3^2 = 2p/f(x_3),$$

where $f(x_i) = 4(\alpha - x_i)(\beta - x_i)(\gamma - x_i)$, and thus $f(x_1)$ and $f(x_3)$ are positive, and $f(x_2)$ negative,

$$V = (1/p)[(\varphi_1 f(x_1)/2) - \varphi_2 f(x_2) + (\varphi_3 f(x_3)/2)],$$

where φ_i is an arbitrary function of x_i.

When p and q are different from zero the cubic equation (i) above, having three real solutions, $x_1 > x_2 > x_3$, is irreducible, that is, it is impossible to find x_i as functions of x, y, and z. However for particular numerical values of p and q, such that p and $(p^3/27) - (q^2/4)$ are positive, real solutions of Eq. (iv) are[6]

$$2(p/3)^{\frac{1}{2}} \cos(\theta/3), \quad 2(p/3)^{\frac{1}{2}} \cos[(\theta + 2\pi)/3], \quad 2(p/3)^{\frac{1}{2}} \cos[(\theta + 4\pi)/3],$$

where θ is given by

$$\cos\theta = -(3q/2p)(3/p)^{\frac{1}{2}}.$$

By means of a table of cosines θ can be found, and then the numerical values of the three roots of Eq. (iv). This process yields a numerical expression for V involving three arbitrary functions of constants x_i.

X. $$H_i^2 = (x_i - x_j)(x_i - x_k)/f(x_i), \quad f(x_i) = 4(\alpha - x_i)(\beta - x_i),$$

where $x_1 > \alpha > x_2 > \beta > x_3$.

$$x = (x_1 + x_2 + x_3 - \alpha - \beta)/2,$$
$$y^2 = [\textstyle\prod_i(\alpha - x_i)]/(\beta - \alpha),$$
$$z^2 = [\textstyle\prod_i(\beta - x_i)]/(\alpha - \beta).$$

From these expressions we have

$$y^2/(\alpha - x_i) + z^2/(\beta - x_i) = 2x - x_i, \quad (i = 1, 2, 3).$$

Consequently, the roots x_i are solutions of an equation of the form (iv), where in this case

$$a = -(2x + \alpha + \beta), \quad b = \alpha\beta + 2x(\alpha + \beta) - y^2 - z^2, \quad c = \beta y^2 + \alpha z^2 - 2x\alpha\beta.$$

Unless $q=0$, the equation is irreducible. The procedure in this case is the same as with type IX where now a, b, c are the new functions of x, y, and z.

The foregoing results are the solution of the problem in one system of Cartesian coordinates. They are such that

$$H_i^2 = (\partial x/\partial x_i)^2 + (\partial y/\partial x_i)^2 + (\partial z/\partial x_i)^2 \equiv \textstyle\sum (\partial x/\partial x_i)^2,$$
$$\textstyle\sum (\partial x/\partial x_i)(\partial x/\partial x_j) = 0 \quad (i \neq j),$$

where $\sum$ denotes summation over x, y, and z.

There is, however, no preferred system of Cartesian coordinates. If x, y, z and $\bar{x}, \bar{y}, \bar{z}$ are two such systems, they are related as follows:

$$x = a_1^1\bar{x} + a_2^1\bar{y} + a_3^1\bar{z} + b_1,$$
$$y = a_1^2\bar{x} + a_2^2\bar{y} + a_3^2\bar{z} + b_2,$$
$$z = a_1^3\bar{x} + a_2^3\bar{y} + a_3^3\bar{z} + b_3,$$

where the a's and b's are constants, the former subject to the conditions

$$\textstyle\sum_i a_j^i a_k^i = \delta_{jk}, \quad \sum_i a_i^j a_i^k = \delta^{jk},$$

where

$$\delta_{jk}, \delta^{jk} \text{ are } 1 \text{ for } j = k;\ 0 \text{ for } j \neq k.$$

In view of these conditions we have

$$H_i^2 = \textstyle\sum (\partial x/\partial x_i)^2 = \sum (\partial \bar{x}/\partial x_i)^2,$$

$$0 = \textstyle\sum (\partial x/\partial x_i)(\partial x/\partial x_j) = \sum (\partial \bar{x}/\partial x_i)(\partial \bar{x}/\partial x_j) \quad (i \neq j).$$

Accordingly, the general form of V is obtained by substituting the above expressions for x, y, z, interims of $\bar{x}$, $\bar{y}$, and $\bar{z}$, in the previous results.

[6] Cf. H. B. Fine, *College Algebra* (Ginn and Company, Boston, 1905), pp. 483–491.