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Theory of Oscillating Absorber in a Chain Reactor*

A. M. WEINBERG AND H. C. SCHWEINLER
*Oak Ridge National Laboratory, Oak Ridge, Tennessee***

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The response of a critical chain-reacting pile to a thermal neutron absorber which is oscillated back and forth inside the pile is calculated. It is shown that at frequencies which are low compared to the periods of delayed neutrons, the neutron intensity in the pile rises and falls as a whole, the shape of the stationary distribution always being maintained. As the frequency of the oscillation increases, the nature of the neutron response changes from the over-all fluctuation characteristic at low frequency to a propagated and attenuated neutron wave which emanates from the neighborhood of the oscillator. These neutron waves set up at high frequencies are entirely analogous to the familiar thermal waves which are established in Angstrom's method of measuring thermal conductivities. It is pointed out that since the pile equations are linear, the amplitude of the oscillating response is proportional to the total neutron absorption of the oscillated absorber, and therefore a known and an unknown absorber can be compared by observing the magnitude of the neutron oscillations caused by each.

1. INTRODUCTION

If a chain-reacting pile is modulated, i.e., if its reactivity is subjected to periodic fluctuations about critical, the neutron flux in the reactor will undergo oscillations of the same period. Qualitatively this is evident since the flux waxes while the reactor is super-critical and wanes while the reactor is sub-critical.

If the reactivity modulation is effected by periodically moving a neutron absorber between regions of high and low flux, then, since the pile equations are linear, the amplitude of the resulting neutron fluctuation must be proportional to the magnitude of the absorber. Thus a known absorber can be compared with an unknown by measuring the amplitudes of the neutron responses when each of the absorbers is oscillated in the same way, inside the pile.

In addition to reducing the reactivity of the pile, a localized absorber introduced into a chain reactor also lowers the neutron flux in its immediate vicinity. If the absorber is moved back and forth, this local neutron depression will move with the absorber, and the response in a nearby neutron-sensitive chamber will fluctuate with the same frequency as that of the oscillating absorber. This local response, like the over-all response which arose from the reactivity modulation, is also proportional to the size of the absorber, and hence can also be used to compare two neutron absorbers. Both methods have been used for neutron absorption cross-section measurements, the reactivity response by A. Langs-

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dorf[1] and the local response by Pomerance, Hoover *et al.*[2]

The character of the response to an oscillating absorber depends on the frequency of the oscillation. At low frequencies the reactivity change predominates and the pile response is adiabatic: the neutron intensity rises and falls, always keeping essentially the shape it had before the absorber was oscillated. At high frequencies the positive and negative reactivity changes tend to cancel and what remains is the local depression. This depression in neutron intensity is propagated away from the absorber as a damped wave which is reflected at the pile boundary. The wave-length, velocity, and attenuation length of the waves depend on the properties of the chain-reacting medium. These "neutron waves" are analogous to the thermal waves set up in a conducting medium in which is placed a localized heat source whose strength varies periodically.

In the present paper we calculate the response of a pile to a periodically varying absorber. In particular we trace how the neutron response changes from an over-all fluctuation at low frequencies to a propagated wave which appears at high frequencies. The considerations reported here are a generalization of results first obtained by Cahn, Monk, and Weinberg[3] (for a reactor in which slowing down was ignored) and then extended to age theory slowing down by Wigner.[4] The method used bears considerable resemblance to that employed by Placzek and Volkoff[5] in discussing the response of a sub-critical pile to a stationary neutron source.

2. THE PILE EQUATION

We consider a non-reflected, homogeneous, uniform chain-reacting pile which is just critical. We suppose that the extrapolation distance is independent of the energy so that the density of neutrons of all energies extrapolates to zero at the same distance beyond the pile boundary; this assumption simplifies the analysis enormously without introducing any important error so long as the pile is large compared to the extrapolation distance. The pile equation, which describes the slowing down, diffusion, and multiplication of neutrons, is

$$D\Delta n_0-\sigma_p n_0+\frac{k}{p}(1-\beta)\sigma_p\int_\infty n_0(\mathbf{r}')P(|\mathbf{r}-\mathbf{r}'|)d\mathbf{r}'+\sum_j\int_\infty\frac{c_{j0}(\mathbf{r}')}{\tau_j}P_j(|\mathbf{r}-\mathbf{r}'|)d\mathbf{r}'=0, \quad (1a)$$

$$(k/p)\beta_j\sigma_p n_0-(c_{j0}/\tau_j)=0, \quad (1b)$$

where

$n_0(\mathbf{r})$ = thermal neutron flux at $\mathbf{r}$, i.e., number of neutrons/cc × average velocity, v.

D = thermal neutron diffusion coefficient ÷ v.

σ_p = thermal neutron absorption cross section of pile per cc.

k = multiplication constant; i.e., average number of neutrons produced per neutron absorbed.

p = resonance escape probability; k/p is the average number of neutrons produced per *slow* neutron absorbed.

β_j = fraction of neutrons produced per fission which are delayed with a mean life τ_j. Each delayed neutron comes from a delayed neutron emitting fission fragment.

$\beta=\sum\beta_j$.

$P(|\mathbf{r}-\mathbf{r}'|)$ = the prompt neutron slowing down function, i.e., number of neutrons becoming slow at $\mathbf{r}'$ due to a prompt fission neutron produced at $\mathbf{r}$ in an infinite chain-reacting medium having the same slowing down and resonance absorption characteristics as the actual pile. Since the resonance escape probability is p,

$$\int_\infty P(|\mathbf{r}-\mathbf{r}'|)d\mathbf{r}'=p.$$

$c_{j0}(\mathbf{r})$ = density of fission fragment nuclei at $\mathbf{r}$ which will emit a delayed neutron with mean life τ_j.

$P_j(|\mathbf{r}-\mathbf{r}'|)$ = jth delayed neutron slowing-down function, i.e., number of neutrons becoming slow at $\mathbf{r}'$ due to a jth-type delayed fission neutron produced at $\mathbf{r}$ in an infinite chain-reacting medium. $P_j(|\mathbf{r}-\mathbf{r}'|)$ may differ from $P(|\mathbf{r}-\mathbf{r}'|)$ because the energy of the delayed and prompt neutrons may not be the same.

In Eq. (1a), $D\Delta n_0$ is the net diffusion flow of neutrons into a cubic centimeter, $\sigma_p n_0$ is the number of thermal neutrons absorbed per cc per second, while the two integrals represent the number of neutrons which become thermal per cc per second at $\mathbf{r}$ after birth as prompt or delayed fission neutrons, respectively.

[1] Alexander Langsdorf, Jr., Bull. Am. Phys. Soc. **23**, No. 3, 20 (1948).

[2] H. Pomerance and J. I. Hoover, Phys. Rev. **73**, 1265 (1948); also J. I. Hoover, W. H. Jordan, C. D. Moak, L. A. Pardue, H. Pomerance, J. Strong, and E. O. Wollan, Phys. Rev. **73**, 1259 (1948).

[3] A. Cahn, Jr., A. T. Monk, and A. M. Weinberg, Metallurgical Project Report, CP-2907.

[4] E. P. Wigner, Metallurgical Project Report, CP-3066.

[5] G. Placzek and G. Volkoff, Can. J. Research **A25**, 276 (1947).

The integrals in (1a), as indicated by subscripts ∞, are to be extended over all space. At first sight this would appear incorrect since the reactor is finite. However, as is well known in pile theory, $n_0(\mathbf{r})$, and therefore c_{j0}, satisfy

$$\Delta n_0(\mathbf{r})+\kappa_0^2 n_0(\mathbf{r})=0, \qquad (2)$$

κ_0^2 being a constant; hence, as will be shown below, the integrals

$$q(\mathbf{r})=(k/p)(1-\beta)\sigma_p\int_\infty n_0(\mathbf{r}')P(|\mathbf{r}-\mathbf{r}'|)d\mathbf{r}', \qquad (3)$$

and

$$q_j(\mathbf{r})=\int_\infty \frac{c_{j0}(\mathbf{r}')}{\tau_j}P_j(|\mathbf{r}-\mathbf{r}'|)d\mathbf{r}', \qquad (4)$$

which represent the number of prompt and delayed neutrons becoming slow at $\mathbf{r}$, satisfy the same boundary conditions as $n_0(\mathbf{r})$. Thus the slow neutron production density $q(\mathbf{r})$ [and $q_j(\mathbf{r})$] is a linear superposition of infinite system slowing-down functions which satisfies the same boundary conditions as the thermal neutron density. Since we assume the density of neutrons of all energies extrapolates to zero at the same distance beyond the pile boundary, $q(\mathbf{r})$ and $q_j(\mathbf{r})$ are the proper production densities for a finite pile, i.e., it is correct to extend the integrals in (1) over all space.

The physical meaning of the preceding remarks, which are fundamental in pile theory, is the following:

The neutron distribution in a finite pile (in which the extrapolation distance is energy independent) can be calculated by extending the pile to infinity and finding the asymptotic neutron distribution in this infinite system. This solution oscillates, positive neutron densities alternating with negative ones *ad infinitum.* The positive and negative densities are so distributed that on the extrapolated pile boundary their superposed effect vanishes. In particular, the production density $q(\mathbf{r})$ in a finite pile is the same as that in an infinite system in which the neutron density $n(\mathbf{r})$ is the analytic continuation of the density appropriate to the finite system.

The regions in which the neutron density is negative may be called "negative" piles. The positive and negative piles constitute a system of images of the sort commonly used to solve ordinary boundary-value problems. In problems ordinarily encountered, which lead to non-homogeneous differential equations, e.g., the potential of a pre-assigned source distribution, the system of images can be constructed easily only if the bounding surface has sufficient symmetry. In the pile case, however, even if the boundary is arbitrarily shaped, the proper intensity and distribution of the image piles are automatically given by the analytic continuation of the asymptotic neutron distribution defined by (2) outside the pile.

It remains to show that q, q_j, and n_0 all satisfy the same boundary conditions (actually it will be shown that these functions are everywhere proportional). We represent $n_0(\mathbf{r})$ as a Fourier integral:

$$n_0(\mathbf{r})=\int_\infty A(\boldsymbol{\alpha})e^{i\boldsymbol{\alpha}\cdot\mathbf{r}}d\boldsymbol{\alpha},$$

where $\boldsymbol{\alpha}=\alpha_x\mathbf{i}+\alpha_y\mathbf{j}+\alpha_z\mathbf{k}$ is a vector integration variable. Since $n_0(\mathbf{r})$ satisfies (2), the function $A(\boldsymbol{\alpha})$ must vanish unless $\alpha^2=\kappa_0^2$; i.e., it is proportional to a δ-function on the surface of a sphere of radius κ_0. Now

$$\begin{aligned} q(\mathbf{r}) &= \frac{k}{p}(1-\beta)\sigma_p\int_\infty n_0(\mathbf{r}')P(|\mathbf{r}-\mathbf{r}'|)d\mathbf{r}' \\ &= \frac{k}{p}(1-\beta)\sigma_p\int_\infty\int_\infty A(\boldsymbol{\alpha})e^{i\boldsymbol{\alpha}\cdot\mathbf{r}'}P(|\mathbf{r}-\mathbf{r}'|)d\mathbf{r}'d\boldsymbol{\alpha} \\ &= \frac{k}{p}(1-\beta)\sigma_p\int_\infty A(\boldsymbol{\alpha})e^{i\boldsymbol{\alpha}\cdot\mathbf{r}}d\boldsymbol{\alpha} \\ &\qquad\times\int_\infty e^{i\boldsymbol{\alpha}\cdot(\mathbf{r}'-\mathbf{r})}P(|\mathbf{r}-\mathbf{r}'|)d(\mathbf{r}'-\mathbf{r}), \end{aligned}$$

or, since $A(\boldsymbol{\alpha})$ is a delta-function, i.e., since $A(\boldsymbol{\alpha})=0$ for $\alpha^2\neq\kappa_0^2$, $A(\boldsymbol{\alpha})=\infty$ for $\alpha^2=\kappa_0^2$,

$$q(\mathbf{r})=\frac{k}{p}(1-\beta)\sigma_p\bar{P}(\kappa_0^2)n_0(\mathbf{r}), \qquad (5)$$

where $\bar{P}(\kappa_0^2)$ is the three-dimensional Fourier transform of $P(|\mathbf{r}-\mathbf{r}'|)$:

$$\bar{P}(\kappa_0^2)=4\pi\int_0^\infty P(r)\frac{\sin\kappa_0 r}{\kappa_0 r}r^2dr. \qquad (6)$$

An exactly similar argument shows that q_j is also proportional to $n_0(\mathbf{r})$:

$$q_j(\mathbf{r})=\frac{c_{j0}(\mathbf{r})}{\tau_j}\bar{P}_j(\kappa_0{}^2)=\frac{k}{p}\sigma_p\beta_j n_0(\mathbf{r})\bar{P}_j(\kappa_0{}^2), \quad (7)$$

where $\bar{P}_j(\kappa_0{}^2)$ is the Fourier transform of the jth delayed neutron slowing-down function.

The characteristic equation which κ_0 satisfies is found by substituting the expressions for $q(\mathbf{r})$ and $q_j(\mathbf{r})$ from (5) and (7) into (1). The result is

$$\frac{k}{p}\bar{P}_{\mathrm{Av}}(\kappa_0{}^2)-L^2\kappa_0{}^2-1=0, \quad (8)$$

where

$$\bar{P}_{\mathrm{Av}}(\kappa_0{}^2)=(1-\beta)\bar{P}(\kappa_0{}^2)+\sum_j \beta_j\bar{P}_j(\kappa_0{}^2).$$

Since $\kappa_0{}^2$ depends on the size of the pile—for example, $\kappa_0{}^2=\pi^2/R^2$ for a sphere of radius R—Eq. (8) serves to determine the critical size of a pile in terms of its microscopic properties.

We now suppose that an absorber, whose absorption cross section per cc at position $\mathbf{r}$ and time t is $\sigma_a(\mathbf{r}, t)$, is placed in the pile. We assume that introduction of the absorber does not alter the pile scattering cross section. Since the pile is critical before the absorber was introduced, it will not be so, on the average, after the absorber is in place unless the multiplication constant is increased (by moving a control rod out) to a new value $k'>k$. After this adjustment has been made the pile equation is

$$D\Delta n-[\sigma_p+\sigma_a(\mathbf{r},t)]n+\frac{k'}{p}(1-\beta)\sigma_p\int_\infty n(\mathbf{r}',t)P(|\mathbf{r}-\mathbf{r}'|)d\mathbf{r}'+\sum_j\int_\infty\frac{c_j(\mathbf{r}',t)}{\tau_j}P_j(|\mathbf{r}-\mathbf{r}'|)d\mathbf{r}'=\frac{1}{v}\frac{\partial n}{\partial t}, \quad (9a)$$

$$(k'/p)\beta_j\sigma_p n-(c_j/\tau_j)=\partial c_j/\partial t, \quad (9b)$$

where the subscripts of Eq. (1) have been dropped. To determine how much k' must differ from k in order that the pile remain critical, on the average, we multiply (1a) and (1b) by n, (9a) and (9b) by n_0, subtract, integrate over the pile, and integrate with respect to time from time 0 to ∞. The integrals involving $n_0\Delta n$ and $n\Delta n_0$ vanish by Green's theorem and the homogeneity of the boundary conditions. Now the integral in (9) of the form

$$\int_\infty n(\mathbf{r}',t)P(|\mathbf{r}-\mathbf{r}'|)d\mathbf{r}', \quad (10a)$$

where the integration is extended over all space and $P(|\mathbf{r}-\mathbf{r}'|)$ is the slowing-down function appropriate to an infinite system, is equivalent to

$$\int_\mathfrak{p} n(\mathbf{r}',t)P_\mathfrak{p}(\mathbf{r},\mathbf{r}')d\mathbf{r}', \quad (10b)$$

where the integral extends over the pile ($\mathfrak{p}$), and $P_\mathfrak{p}(\mathbf{r},\mathbf{r}')$, the slowing-down kernel in a finite pile, can be constructed from $P(|\mathbf{r}-\mathbf{r}'|)$ by a suitable superposition of images. The equivalence of the pile integral (10b) and the all-space integral (10a) follows from the fact that the all-space integral vanishes on the extrapolated pile boundary. This, therefore, insures that the linear superposition of infinite system slowing-down kernels represented by the all-space integral is exactly that superposition which satisfies the boundary conditions on the finite pile. To show that (10a) vanishes on the extrapolated boundary of the pile we express $n(\mathbf{r}, t)$ as a superposition of functions $Z_\nu(\mathbf{r})$ which vanish on the extrapolated pile boundary and satisfy the wave equation

$$\Delta Z_\nu(\mathbf{r})+\kappa_\nu{}^2Z_\nu(\mathbf{r})=0;$$

that is,

$$n(\mathbf{r},t)=\sum_\nu n_\nu(t)Z_\nu(\mathbf{r}),$$

$n_\nu(t)$ being expansion coefficients and ν being an index which orders the eigenfunctions. Upon substituting this expansion into (10a) we obtain

$$\int_\infty n(\mathbf{r}',t)P(|\mathbf{r}-\mathbf{r}'|)d\mathbf{r}'=\sum_\nu n_\nu(t)Z_\nu(\mathbf{r})\bar{P}(\kappa_\nu{}^2),$$

which vanishes on the boundary, even for arbitrary $n_\nu(t)$. The double integrals can therefore be transformed thus:

$$\int_\mathfrak{p}\int_\infty n_0(\mathbf{r})P(|\mathbf{r}-\mathbf{r}'|)n(\mathbf{r}',t)d\mathbf{r}'d\mathbf{r}$$
$$=\int_\mathfrak{p}\int_\mathfrak{p} n_0(\mathbf{r})P_\mathfrak{p}(\mathbf{r},\mathbf{r}')n(\mathbf{r}',t)d\mathbf{r}'d\mathbf{r}$$
$$=\int_\infty\int_\mathfrak{p} n_0(\mathbf{r})P(|\mathbf{r}-\mathbf{r}'|)n(\mathbf{r}',t)d\mathbf{r}'d\mathbf{r}$$
$$=\bar{P}(\kappa_0{}^2)\int_\mathfrak{p} n(\mathbf{r}',t)n_0(\mathbf{r}')d\mathbf{r}'.$$

The result of the manipulation with Green's theorem is therefore

$$\frac{(k'-k)}{p}\bar{P}_{\mathrm{Av}}(\kappa_0^2)=\lim_{T\to\infty}\frac{\int_0^T\int_{\mathfrak{p}}\sigma_a(\mathbf{r},t)n(\mathbf{r},t)n_0(\mathbf{r})d\mathbf{r}dt}{\sigma_p\int_0^T\int_{\mathfrak{p}}n(\mathbf{r},t)n_0(\mathbf{r})d\mathbf{r}dt}+\frac{\int_{\mathfrak{p}}n_0(\mathbf{r})\left[\sum_j\bar{P}_j(\kappa_0^2)\{c_j(\mathbf{r},T)-c_j(\mathbf{r},0)\}+\frac{n(\mathbf{r},T)-n(\mathbf{r},0)}{v}\right]d\mathbf{r}}{\sigma_p\int_0^T\int_{\mathfrak{p}}n(\mathbf{r},t)n_0(\mathbf{r})d\mathbf{r}dt}.$$

Since k' is so adjusted that the pile remains critical on the average after the absorber has been introduced, $n(\mathbf{r}, t)$ and $c_j(\mathbf{r}, t)$ are bounded as $t\to\infty$; the second term on the right therefore goes to zero at long times T. Thus the required change in k is

$$k'-k=\frac{p}{\bar{P}_{\mathrm{Av}}(\kappa_0^2)}\frac{\int_{\mathfrak{p}}\langle\sigma_a nn_0\rangle_{\mathrm{Av}}d\mathbf{r}}{\sigma_p\int_{\mathfrak{p}}\langle nn_0\rangle_{\mathrm{Av}}d\mathbf{r}},\tag{11}$$

where

$$\langle\sigma_a nn_0\rangle_{\mathrm{Av}}=\lim_{T\to\infty}\frac{1}{T}\int_0^T\sigma_a nn_0 dt.\tag{12}$$

If we introduce the notation

$$S_a=\lim_{T\to\infty}\left[\frac{1}{T}\int_0^T\int_{\mathfrak{p}}\sigma_a(\mathbf{r},t)d\mathbf{r}dt\right],\tag{13}$$

$$f(\mathbf{r},t)=\frac{\sigma_a(\mathbf{r},t)}{S_a},\tag{14}$$

$$\eta=\lim_{T\to\infty}\frac{\int_0^T\int_{\mathfrak{p}}f(\mathbf{r},t)n(\mathbf{r},t)n_0(\mathbf{r})d\mathbf{r}dt}{\int_0^T\int_{\mathfrak{p}}n(\mathbf{r},t)n(\mathbf{r})d\mathbf{r}dt},\tag{15}$$

the required k change is

$$\delta k=k'-k=\frac{p}{\bar{P}_{\mathrm{Av}}(\kappa_0^2)}\frac{S_a}{\sigma_p}\eta.\tag{16}$$

The quantity S_a is equal to the time average of the absorption cross section per cc of the absorber multiplied by the volume of the absorber; η is the average "statistical weight" of the pile region occupied by the absorber. Thus Eq. (16) is the generalization for a time-varying absorber of the usual statistical weight formula: *the average change in multiplication constant caused by a time-varying absorber is the average cross section multiplied by the average statistical weight of the absorber.* Since the time average is over nn_0, or, approximately, over the square of the neutron density, the average k change is not the same as the k change caused by the absorber if it were stationary in its average position.

The following example illustrates this point. Consider a point absorber which oscillates along the z axis perpendicular to the faces of an infinite slab pile with amplitude ξ and frequency $\omega/2\pi$ around $z=z_0$. Then

$$f(\mathbf{r},t)=\delta(z_0-z+\xi e^{i\omega t})\delta(x)\delta(y),$$

and

$$n(\mathbf{r})\approx\sin\frac{\pi z}{H}\approx n_0(\mathbf{r}),$$

H being the width of the pile, $H\gg\xi$. It is understood throughout that only the real part of the exponential is to be used. If the sink were stationary at its equilibrium point, z_0, the statistical weight, η_0, would be

$$\eta_0=\frac{\sin^2\frac{\pi z_0}{H}}{\int_0^H\sin^2\frac{\pi z}{H}dz}=\frac{2}{H}\sin^2\frac{\pi z_0}{H}.$$

Since the absorber oscillates, its actual average statistical weight is

$$\eta=\frac{\int_0^T\int_0^H\delta(z_0-z+\xi e^{i\omega t})\sin^2\frac{\pi z}{H}dzdt}{\int_0^T\int_0^H\sin^2\frac{\pi z}{H}dzdt}\cong\frac{2}{H}\sin^2\frac{\pi z_0}{H}+\frac{\pi^2\xi^2}{H^3}\cos\frac{2\pi z_0}{H}.$$

If the absorber oscillates around $z_0=H/4$, the average k

change is the same as the k change at the average position, since the two statistical weights are the same. If $3H/4>z_0>H/4$ (toward center), the k change is less than the stationary value, while if $z_0<H/4$, the k change is greater.

3. SOLUTION OF THE PILE EQUATION FOR OSCILLATING ABSORBER

We now calculate the pile response to an absorber of small volume V_a and cross section σ_a which is oscillated back and forth about the point $\mathbf{r}_0$ with frequency $\omega/2\pi$ and amplitude $|\boldsymbol{\varrho}|$ in the direction $\boldsymbol{\varrho}$. We assume that $|\boldsymbol{\varrho}|$ is small compared to any pile dimension. Then by definitions (13) and (14),

$$S_a=\sigma_a V_a, \qquad (17)$$

$$f(\mathbf{r},t)=\delta(\mathbf{r}_0-\mathbf{r}+\boldsymbol{\varrho}e^{i\omega t}), \qquad (18)$$

where $\delta(\mathbf{r})$ is normalized to

$$\int\delta(\mathbf{r})d\mathbf{r}=\int\int\int\delta(x)\delta(y)\delta(z)dxdydz=1.$$

If $|\mathbf{r}_0-\mathbf{r}|$ is large compared to $|\boldsymbol{\varrho}|$, it is appropriate to expand the δ-function formally in Taylor's series and keep only the first two terms:

$$f(\mathbf{r},t)\approx\delta(\mathbf{r}_0-\mathbf{r})+e^{i\omega t}\boldsymbol{\varrho}\cdot\nabla\delta(\mathbf{r}_0-\mathbf{r}), \qquad (19)$$

where ∇ is the gradient with respect to $\mathbf{r}_0$. The higher terms in this expansion have frequencies which are multiples of ω; if the detecting equipment is tuned only to ω, these overtones will not be recorded.

The neutron and delayed emitter densities will consist of a stationary part, $(\bar{n},\bar{c}_j)$, and an oscillatory part, $(y(\mathbf{r})e^{i\omega t},\, w_j(\mathbf{r})e^{i\omega t})$:

$$n(\mathbf{r},t)=\bar{n}(\mathbf{r})+y(\mathbf{r})e^{i\omega t}, \qquad (20a)$$

$$c_j(\mathbf{r},t)=\bar{c}_j(\mathbf{r})+w_j(\mathbf{r})e^{i\omega t}. \qquad (20b)$$

In (20), and in the remaining calculations, only the real part of any complex quantity is significant.

Upon substituting (17), (19), and (20), into the pile equation, (9), and equating time-dependent and time-independent quantities, we obtain for the oscillating part of the flux,

$$D\Delta y-\sigma_p y+\frac{k}{p}(1-\beta)\sigma_p\int_\infty y(\mathbf{r}')P(|\mathbf{r}-\mathbf{r}'|)d\mathbf{r}'+\sum_i\int_\infty\frac{w_j(\mathbf{r}')}{\tau_j}P_j(|\mathbf{r}-\mathbf{r}'|)d\mathbf{r}'-\frac{i\omega}{v}y$$
$$=S_a\bar{n}\boldsymbol{\varrho}\cdot\nabla\delta(\mathbf{r}_0-\mathbf{r})+S_a y\delta(\mathbf{r}_0-\mathbf{r})-\frac{\delta k}{p}(1-\beta)\sigma_p\int_\infty y(\mathbf{r}')P(|\mathbf{r}-\mathbf{r}'|)d\mathbf{r}', \qquad (21a)$$

$$\frac{k}{p}\beta_j\sigma_p y+\frac{\delta k}{p}\beta_j\sigma_p y-\frac{w_j}{\tau_j}-i\omega w_j=0. \qquad (21b)$$

Since y vanishes with S_a, the terms $S_a y$, $y\delta k$, and $\delta k\int y(\mathbf{r}')P(|\mathbf{r}-\mathbf{r}'|)d\mathbf{r}'$ are of second order in S_a, and, on the assumption of weak absorber, can be neglected. Similarly, $\bar{n}(\mathbf{r})$ in the term $S_a\bar{n}\boldsymbol{\varrho}\cdot\nabla\delta(\mathbf{r}_0-\mathbf{r})$ can be replaced by $n_0(\mathbf{r})$, the density before the absorber was introduced (this corresponds to using Born's approximation).

Since the amplitude of motion, $|\boldsymbol{\varrho}|$, is small compared to the pile dimension, $n(\mathbf{r})$ [or $n_0(\mathbf{r})$] in (21) can be expanded around $\mathbf{r}_0$. Thus, for points not too near to the absorber,

$$S_a\bar{n}\boldsymbol{\varrho}\cdot\nabla\delta(\mathbf{r}_0-\mathbf{r})\approx S_a[n_0(\mathbf{r}_0)+(\mathbf{r}-\mathbf{r}_0)\cdot\nabla n_0(\mathbf{r}_0)][\boldsymbol{\varrho}\cdot\nabla\delta(\mathbf{r}-\mathbf{r}_0)]\approx S_a(\boldsymbol{\varrho}\cdot\nabla)[n_0(\mathbf{r}_0)\delta(\mathbf{r}-\mathbf{r}_0)]. \qquad (22)$$

In this approximation, therefore, the solution to (21) can be obtained by solving

$$D\Delta\bar{y}-\sigma_p\bar{y}+\frac{k}{p}(1-\beta)\sigma_p\int_\infty\bar{y}(\mathbf{r}')P(|\mathbf{r}-\mathbf{r}'|)d\mathbf{r}'+\sum_i\int_\infty\frac{\bar{w}_j(\mathbf{r}')}{\tau_j}P_j(|\mathbf{r}-\mathbf{r}'|)d\mathbf{r}'-\frac{i\omega}{v}\bar{y}=S_a n_0(\mathbf{r}_0)\delta(\mathbf{r}-\mathbf{r}_0), \qquad (23a)$$

$$\frac{k}{p}\beta_j\sigma_p\bar{y}-\frac{\bar{w}_j}{\tau_j}-i\omega\bar{w}_j=0, \qquad (23b)$$

and taking the directional derivative $\boldsymbol{\varrho}\cdot\nabla$ of the results:

$$y(\mathbf{r})=(\boldsymbol{\varrho}\cdot\nabla)\bar{y}(\mathbf{r},\mathbf{r}_0), \qquad (24a)$$

$$w(\mathbf{r})=(\boldsymbol{\varrho}\cdot\nabla)\bar{w}(\mathbf{r},\mathbf{r}_0). \qquad (24b)$$

The solution $\bar{y}$ of (23a) when multiplied by $e^{i\omega t}$ represents the oscillatory part of the response to a point absorber at $\mathbf{r}_0$ whose absorption cross section oscillates with amplitude S_a.

To solve (23) we first replace $\bar{w}_j$ in (23a) by its value from (23b):

$$\bar{w}_j = \frac{k}{p} \frac{\beta_j \sigma_p \bar{y}}{i\omega + 1/\tau_j}. \tag{25}$$

We then expand $\bar{y}$ and $\delta(\mathbf{r}-\mathbf{r}_0)$ in the complete orthonormal set

$$Z_\nu(\mathbf{r}),$$

which consists of those solutions of

$$\Delta Z_\nu(\mathbf{r}) + \kappa_\nu^2 Z_\nu(\mathbf{r}) = 0, \tag{26}$$

which vanish on the extrapolated boundary. Thus

$$\bar{y}(\mathbf{r}) = \sum_\nu \bar{y}_\nu Z_\nu(\mathbf{r}),$$
$$\delta(\mathbf{r}-\mathbf{r}_0) = \sum_\nu Z_\nu(\mathbf{r}) Z_\nu(\mathbf{r}_0), \tag{27}$$

where

$$\bar{y}_\nu = \int_{\mathfrak{p}} \bar{y}(\mathbf{r}) Z_\nu(\mathbf{r}) d\mathbf{r}.$$

The index ν is an abbreviation for the three indices (λ, μ, ν) in case the pile is three dimensional, and the characteristic functions depend on three coordinates.

Since

$$\int_\infty Z_\nu(\mathbf{r}') P(|\mathbf{r}-\mathbf{r}'|) d\mathbf{r}' = \bar{P}(\kappa_\nu^2) Z_\nu(\mathbf{r}), \tag{28}$$

the integral terms in (23a) reduce to

$$\int_\infty \bar{y}(\mathbf{r}') P(|\mathbf{r}-\mathbf{r}'|) d\mathbf{r}' = \sum_\nu \bar{y}_\nu \bar{P}(\kappa_\nu^2) Z_\nu(\mathbf{r}). \tag{29}$$

After substituting the expansion (27) for $\bar{y}(\mathbf{r})$ in (23a) and (23b), and equating coefficients of Z_ν, there results

$$\bar{y}(\mathbf{r}) = \frac{S_a}{\sigma_p} \sum_\nu \frac{n_0(\mathbf{r}_0) Z_\nu(\mathbf{r}_0) Z_\nu(\mathbf{r})}{\frac{k}{p}\left\{(1-\beta)\bar{P}(\kappa_\nu^2) + \sum_j \frac{\beta_j}{1+i\omega\tau_j} \bar{P}_j(\kappa_\nu^2)\right\} - L^2\kappa_\nu^2 - i\omega\tau}, \tag{30}$$

where $\tau = 1/v\sigma_p$ is the lifetime against capture of a thermal neutron in the reactor, and $L = (D/\sigma_p)^{\frac{1}{2}}$ is the diffusion length of the thermal neutrons. With this value for $\bar{y}$, we find, finally,

$$n(\mathbf{r}, t) = \bar{n}(\mathbf{r}) + y(\mathbf{r}) e^{i\omega t} = \bar{n}(\mathbf{r}) + \frac{S_a}{\sigma_p} (\boldsymbol{\varrho} \cdot \nabla) n_0(\mathbf{r}_0)$$
$$\times \sum_\nu \frac{Z_\nu(\mathbf{r}_0) Z_\nu(\mathbf{r}) e^{i\omega t}}{\frac{k}{p}\left\{(1-\beta)\bar{P}(\kappa_\nu^2) + \sum_j \frac{\beta_j}{1+i\omega\tau_j} \bar{P}_j(\kappa_\nu^2)\right\} - L^2\kappa_\nu^2 - 1 - i\omega\tau}, \tag{31a}$$

$$c_j(\mathbf{r}, t) = \bar{c}_j(\mathbf{r}) + w_j(\mathbf{r}) e^{i\omega t} = \bar{c}_j(\mathbf{r}) + \frac{k}{p} \frac{\beta_j \tau_j \sigma_p y(\mathbf{r})}{1+i\omega\tau_j} e^{i\omega t}. \tag{31b}$$

The real parts of Eqs. (31a) and (31b) constitute the formal solution of the problem. We now discuss the physical properties of the solution in various cases.

a. Absorber of Fluctuating Strength Distributed Uniformly throughout Pile

In this case

$$\sigma_a(\mathbf{r}, t) = \frac{S_a}{V_p}(1 + \alpha e^{i\omega t}), \tag{32}$$

where V_p is the pile volume and α measures the relative amplitude of the oscillation. The oscillatory part of the neutron density can be obtained from $\bar{y}(\mathbf{r})$ (which, except for the factor $e^{i\omega t}$, is the solution for a stationary point absorber whose strength oscillates with amplitude S_a) by integrating with respect to $\mathbf{r}_0$ over the whole pile and multiplying by $\alpha e^{i\omega t}$. Since $n_0(\mathbf{r}_0)$ is proportional to $Z_0(\mathbf{r}_0)$, all terms except the fundamental disappear in the integration, and the result is

$$n(\mathbf{r}, t) = \bar{n}(\mathbf{r}) + \frac{\alpha S_a}{V_p \sigma_p} \frac{n_0(\mathbf{r}) e^{i\omega t}}{\frac{k}{p}\left\{(1-\beta)\bar{P}(\kappa_0^2) + \sum_i \frac{\beta_j \bar{P}_j(\kappa_0^2)}{1+i\omega\tau_j}\right\} - L^2\kappa_0^2 - 1 - i\omega\tau}. \tag{33}$$

If there are no delayed neutrons, the fluctuating part of the response is

$$n(\mathbf{r}, t) - \bar{n}(\mathbf{r}) = \frac{\alpha S_a}{V_p \sigma_p} \frac{n_0(\mathbf{r}) e^{i\omega t}}{\frac{k}{p}\bar{P}(\kappa_0^2) - L^2\kappa_0^2 - 1 - i\omega\tau}. \tag{34}$$

Since the pile was critical before the absorber was introduced, Eq. (8) holds:

$$\frac{k}{p}\bar{P}(\kappa_0^2) - L^2\kappa_0^2 - 1 = 0,$$

hence,

$$\frac{n(\mathbf{r}, t) - \bar{n}(\mathbf{r})}{n_0(\mathbf{r})} = -\frac{\alpha S_a e^{i\omega t}}{V_p \sigma_p i\omega\tau}.$$

The amplitude of the neutron response falls off with increasing frequency; it lags behind the absorber oscillation by 90°. This phase lag arises because it is the rate of change of the pile intensity, not the intensity itself, which is determined by the instantaneous value of k. If the delayed neutrons are taken into account, the phase lag is a little less than 90°.

b. Localized Absorber, Slow Oscillation ($\omega\tau_j \ll 1$)

If the absorber moves back and forth slowly, $\omega\tau_j \ll 1$, and (31a) becomes

$$n(\mathbf{r}, t) \approx \bar{n}(\mathbf{r}) + \frac{S_a}{\sigma_p}(\boldsymbol{\varrho}\cdot\nabla) n_0(\mathbf{r}_0) \sum_\nu \frac{Z_\nu(\mathbf{r}_0) Z_\nu(\mathbf{r}) e^{i\omega t}}{\frac{k}{p}\bar{P}_{\mathrm{Av}}(\kappa_\nu^2) - L^2\kappa_\nu^2 - 1 - i\omega[\tau + \sum_i \beta_j \tau_j \bar{P}_j(\kappa_\nu^2)]}. \tag{35}$$

Since the pile is critical,

$$\frac{k}{p}\bar{P}_{\mathrm{Av}}(\kappa_0^2) - L^2\kappa_0^2 - 1 = 0,$$

and the ratio of the fundamental to the νth harmonic in the series (35) is

$$\frac{Z_0(\mathbf{r}_0) Z_0(\mathbf{r})}{Z_\nu(\mathbf{r}_0) Z_\nu(\mathbf{r})} \left[\frac{\frac{k}{p}\bar{P}_{\mathrm{Av}}(\kappa_\nu^2) - L^2\kappa_\nu^2 - 1 - i\omega[\tau + \sum_i \beta_j \tau_j \bar{P}_j(\kappa_\nu^2)]}{-i\omega[\tau + \sum_i \beta_j \tau_j \bar{P}_j(\kappa_0^2)]}\right]. \tag{36}$$

This ratio becomes infinite as $\omega \to 0$; i.e., the fundamental predominates as the oscillation becomes slower. In other words, if the oscillation is very slow the pile intensity fluctuates as a whole, the neutron flux at any point having the same phase as at any other point.

The slow oscillation method in which the over-all pile response to a moving absorber is detected has been used in Langsdorf's oscillator for cross-section measurements.

c. Localized Absorber, Fast Oscillation ($\omega\tau_j \gg 1$), or Detector Close to Absorber

If the oscillation is fast, or if the detector is close to the absorber ($\mathbf{r}_0 \approx \mathbf{r}'$), the fundamental no longer predominates, and the series (31a) converges poorly. Since the higher harmonics all have different phases (because the coefficients in the series are complex), the phase of the neutron intensity oscillation will change from point to point. Close to the absorber the intensity will be in phase with the motion of the absorber since the local neutron depression caused by the absorber will be the major part of the intensity fluctuation. Far from the absorber the neutron intensity will tend more and more toward the phase of the fundamental. The disturbance set up by the oscillating absorber is therefore wave-like: the absorber sends out damped waves of neutron intensity which are reflected at the boundary. As will be shown below, the wave-length of the traveling disturbance is short at high frequency and long at low frequency. It is for this reason that the disturbance has the same phase everywhere at low frequencies, and has a varying phase at high frequency.

These qualitative considerations can be made exact by transforming the poorly converging series (31a) into a form which displays the wave character of the disturbance. In order to effect this transformation it will be convenient to specialize the shape of the pile; we therefore suppose the pile to be a rectangular parallelopiped of sides a, b, c. The normalized characteristic functions are

$$Z_{lmn}(\mathbf{r}) = \left(\frac{8}{abc}\right)^{\frac{1}{2}} \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{n\pi z}{c}, \tag{37}$$

and

$$\kappa^2{}_{lmn} = \frac{l^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2} + \frac{n^2\pi^2}{c^2}. \tag{38}$$

The intensity, as given by (31a), is therefore

$$n(\mathbf{r},t) = \bar{n}(\mathbf{r}) + \frac{S_a}{8\sigma_p abc}(\boldsymbol{\varrho}\cdot\nabla)n_0(\mathbf{r}_0)e^{i\omega t}\sum_{l,m,n}$$
$$\times\frac{\exp\left(i\pi\left\{l\left(\frac{x-x_0}{a}\right)+m\left(\frac{y-y_0}{b}\right)+n\left(\frac{z-z_0}{c}\right)\right\}\right)-\exp\left(i\pi\left\{l\left(\frac{x+x_0}{a}\right)+m\left(\frac{y+y_0}{b}\right)+n\left(\frac{z+z_0}{c}\right)\right\}\right)}{\frac{k}{p}\left\{(1-\beta)\bar{P}(\kappa^2{}_{lmn})+\sum_j\frac{\beta_j\bar{P}_j(\kappa^2{}_{lmn})}{1+i\omega\tau_j}\right\}-L^2\kappa^2{}_{lmn}-1-i\omega\tau}, \tag{39}$$

the sums being taken from $-\infty$ to $+\infty$.

We now apply Poisson's summation formula[6]

$$\sum_{l,m,n}\varphi(l,m,n) = \sum_{\lambda,\mu,\nu}\int\int\int_{-\infty}^{\infty}\varphi(u,v,w)e^{-2\pi i(\lambda u+\mu v+\nu w)}dudvdw, \tag{40}$$

to the series in (39). The series then becomes, after making the transformation,

$$\sum_{l,m,n}\frac{\exp\left(i\pi\left\{l\left(\frac{x-x_0}{a}\right)+m\left(\frac{y-y_0}{b}\right)+n\left(\frac{z-z_0}{c}\right)\right\}\right)-\exp\left(i\pi\left\{l\left(\frac{x+x_0}{a}\right)+m\left(\frac{y+y_0}{b}\right)+n\left(\frac{z+z_0}{c}\right)\right\}\right)}{k\bar{Q}(\kappa^2{}_{lmn})-L^2\kappa^2{}_{lmn}-1-i\omega\tau}$$
$$=\frac{abc}{\pi^3}\sum_{\lambda,\mu,\nu}\int\int\int_{-\infty}^{\infty}\frac{\exp(i\boldsymbol{\xi}\cdot(\mathbf{R}^- - \boldsymbol{\Lambda}))-\exp(i\boldsymbol{\xi}\cdot(\mathbf{R}^+ - \boldsymbol{\Lambda}))}{k\bar{Q}(\xi^2)-L^2\xi^2-1-i\omega\tau}d\boldsymbol{\xi}, \tag{41}$$

[6] Courant-Hilbert, *Methoden der Mathematischen Physik* (Verlag Julius Springer, Berlin, 1931), Vol. I, p. 65.

where

$$\boldsymbol{\xi}=\frac{\pi u}{a}\mathbf{i}+\frac{\pi v}{b}\mathbf{j}+\frac{\pi w}{c}\mathbf{k},\quad d\boldsymbol{\xi}=\frac{\pi^3}{abc}dudvdw,$$

$$\mathbf{R}^{\pm}=(x\pm x_0)\mathbf{i}+(y\pm y_0)\mathbf{j}+(z\pm z_0)\mathbf{k},$$

$$\boldsymbol{\Lambda}=2\lambda a\mathbf{i}+2\mu b\mathbf{j}+2\nu c\mathbf{k},$$

$$\bar{Q}(\xi^2)=\frac{1}{p}\left\{(1-\beta)\bar{P}(\xi^2)+\sum_i\frac{\beta_i\bar{P}_i(\xi^2)}{1+i\omega\tau}\right\}.$$

To evaluate the integral in (41) we shift to spherical coordinates and obtain

$$\iiint_{-\infty}^{\infty}\frac{\exp[i\boldsymbol{\xi}\cdot(\mathbf{R}-\boldsymbol{\Lambda})]}{k\bar{Q}(\xi^2)-L^2\xi^2-1-i\omega\tau}d\boldsymbol{\xi}=\frac{4\pi}{|\mathbf{R}-\boldsymbol{\Lambda}|}\int_0^{\infty}\frac{\sin\xi|\mathbf{R}-\boldsymbol{\Lambda}|}{k\bar{Q}(\xi^2)-L^2\xi^2-1-i\omega\tau}\xi d\xi$$
$$=\frac{4\pi}{2i|\mathbf{R}-\boldsymbol{\Lambda}|}\int_{-\infty}^{\infty}\frac{\exp(i|\mathbf{R}-\boldsymbol{\Lambda}|\xi)}{k\bar{Q}(\xi^2)-L^2\xi^2-1-i\omega\tau}\xi d\xi. \quad (42)$$

To compute this integral we follow a semicircular contour which embraces the entire positive half-plane. If B_s are the zeros (assumed simple) with positive imaginary part of

$$k\bar{Q}(\xi^2)-L^2\xi^2-1-i\omega\tau=0, \quad (43)$$

then the value of the integral is

$$\int_{-\infty}^{\infty}\frac{\exp(i|\mathbf{R}-\boldsymbol{\Lambda}|\xi)}{k\bar{Q}(\xi^2)-L^2\xi^2-1-i\omega\tau}\xi d\xi=\pi i\sum_s\frac{\exp(iB_s|\mathbf{R}-\boldsymbol{\Lambda}|)}{k\bar{Q}'(B_s^2)-L^2}, \quad (44)$$

$$\bar{Q}'(B_s^2)\equiv\frac{d}{d(\xi^2)}\bar{Q}(\xi^2)|_{\xi=B_s}.$$

Hence

$$\frac{abc}{\pi^3}\iiint_{-\infty}^{\infty}\frac{\exp[i\boldsymbol{\xi}\cdot(\mathbf{R}-\boldsymbol{\Lambda})]}{k\bar{Q}(\xi^2)-L^2\xi^2-1-i\omega\tau}d\boldsymbol{\xi}=\frac{2abc}{\pi|\mathbf{R}-\boldsymbol{\Lambda}|}\sum_s\frac{\exp(iB_s|\mathbf{R}-\boldsymbol{\Lambda}|)}{k\bar{Q}'(B_s^2)-L^2}. \quad (45)$$

Substituting this into (41), and then putting the resulting series into (39), we finally obtain the transformed series:

$$n(\mathbf{r},t)-\bar{n}(\mathbf{r})=\frac{S_a}{\sigma_p}(\boldsymbol{\varrho}\cdot\nabla)n_0(\mathbf{r}_0)e^{i\omega t}\sum_{\lambda,\mu,\nu}\sum_s\frac{1}{k\bar{Q}'(B_s^2)-L^2}\left\{\frac{\exp(iB_sR^-_{\lambda\mu\nu})}{4\pi R^-_{\lambda\mu\nu}}-\frac{\exp(iB_sR^+_{\lambda\mu\nu})}{4\pi R^+_{\lambda\mu\nu}}\right\}, \quad (46)$$

where

$$R^{\pm}_{\lambda\mu\nu}=((x\pm x_0-2\lambda a)^2+(y\pm y_0-2\mu b)^2+(z\pm z_0-2\nu c)^2)^{\frac{1}{2}}. \quad (47)$$

Equation (46) can be described as a sequence of positive and negative neutron sources. The positive sources are situated at the points

$$(-x_0+2\lambda a)\mathbf{i}+(-y_0+2\mu b)\mathbf{j}+(-z_0+2\nu c)\mathbf{k},$$

and negative sources at the points

$$(x_0+2\lambda a)\mathbf{i}+(y_0+2\mu b)\mathbf{j}+(z_0+2\nu c)\mathbf{k}.$$

Thus (46) constitutes a sequence of images which are so placed that the boundary conditions are satisfied on the pile surface.

Each term in the series (46) is of the form

$$\frac{S_a}{\sigma_p}\frac{(\boldsymbol{\varrho}\cdot\nabla)n_0(\mathbf{r}_0)}{k\bar{Q}'(B_s^2)-L^2}\frac{\exp[i(\omega t+B_sR)]}{4\pi R}=\frac{S_a}{\sigma_p}\frac{1}{k\bar{Q}'(B_s^2)-L^2}\left\{\frac{\exp[i(\omega t+B_sR)]}{4\pi R}\boldsymbol{\varrho}\cdot\nabla n_0(\mathbf{r}_0)+n_0(\mathbf{r}_0)\boldsymbol{\varrho}\cdot\nabla\frac{\exp[i(\omega t+B_sR)]}{4\pi R}\right\}. \quad (48)$$

According to (48), the disturbance represented by each term in (46) is a spherical traveling wave of complex wave-length $1/B_s$. The traveling wave, in the approximation in which $|\boldsymbol{\varrho}|$, the amplitude of oscillation, is small compared to $|\mathbf{r}-\mathbf{r}_0|$, consists of two parts: a spherically symmetric wave emitted by a point sink of strength

$$S_a(\boldsymbol{\varrho}\cdot\nabla)n_0(\mathbf{r}_0),$$

and a non-spherically symmetric wave emitted by a dipole of strength

$$S_a n_0(\mathbf{r}_0)|\boldsymbol{\varrho}|.$$

The spherically symmetric wave falls off as $\exp(iB_sR_{\lambda\mu\nu})/R_{\lambda\mu\nu}$ while the dipole wave falls off as $\exp(iB_sR_{\lambda\mu\nu})/R_{\lambda\mu\nu}{}^2$ hence, far from the absorber only the spherically symmetric wave prevails.

Actually waves corresponding to multipoles of all orders are generated by a periodically moving absorber. The appearance of only the first two multipoles in (48) is a result of the approximation $|\boldsymbol{\varrho}|\ll|\mathbf{r}-\mathbf{r}_0|$, which was the justification for keeping only two terms in the Taylor expansion (22) of the neutron flux. In principle, of course, there is no difficulty in extending the expansion to higher order in $|\boldsymbol{\varrho}|/|\mathbf{r}-\mathbf{r}_0|$; this would lead, in the final answer, to multipole "radiation" of correspondingly higher order.

Corresponding to each different root B_s, the wave sources emit a wave. Each wave has an amplitude and phase which depends on the value of its B_s. The number of roots B_s, and therefore the number of different waves emitted by each source, depends on the slowing-down function—for a general kernel $P(r)$, (43) will be a transcendental equation and there will be an infinite number of roots B_s.

The waves with $s>0$ are significant close to the oscillator; as s increases they fall off more and more rapidly with $|\mathbf{r}-\mathbf{r}_0|$ because the real parts of iB_s increase with s. In order to see more clearly the physical significance of these transient waves, we compute the fluctuating part of the slowing-down density. This is obtained according to (3) and (5) by multiplying each term in (46) by $(k'/p)(1-\beta)\sigma_p\bar{P}(B_s^2)$; i.e.,

$$q(\mathbf{r},t)-\bar{q}(\mathbf{r})=\frac{k'}{p}(1-\beta)\sigma_p\int_\infty[n(\mathbf{r}',t)-\bar{n}(\mathbf{r})]P(|\mathbf{r}-\mathbf{r}'|)d\mathbf{r}'$$
$$=\frac{k'}{p}(1-\beta)S_a(\boldsymbol{\varrho}\cdot\nabla)n_0(\mathbf{r}_0)e^{i\omega t}\sum_{\lambda,\mu,\nu}\sum_s\frac{\bar{P}(B_s^2)}{k\bar{Q}'(B_s^2)-L^2}\left\{\frac{\exp(iB_sR^-{}_{\lambda\mu\nu})}{4\pi R^-{}_{\lambda\mu\nu}}-\frac{\exp(iB_sR^+{}_{\lambda\mu\nu})}{4\pi R^+{}_{\lambda\mu\nu}}\right\}. \quad (49)$$

Because of the extra factor $\bar{P}(B_s^2)$ in each term of (49) as compared with (46), the slowing-down density is not proportional to the slow neutron flux close to the oscillator. In fact, the spectrum of epithermal neutrons changes continuously from its shape close to the oscillator, where the epithermal neutrons have no singularity while the thermal ones do, to its shape far from the oscillator, where the neutrons of all energies have similar spatial distributions. Thus the transients ($s>0$) are necessary to describe the variation in neutron spectrum which exists in a chain reactor near an absorber which captures only thermal neutrons.

4. PROPERTIES OF THE NEUTRON WAVES

Far from the absorber, and far from the pile surface, only the $s=\lambda=\mu=\nu=0$ spherically symmetric wave persists. From (48) this is

$$n(\mathbf{r},t)-\bar{n}(\mathbf{r})=\frac{S_a}{\sigma_p}\frac{1}{[k\bar{Q}'(B_0^2)-L^2]}\times\frac{e^{i(\omega t+B_0R)}}{4\pi R}(\boldsymbol{\varrho}\cdot\nabla)n(\mathbf{r}_o), \quad (50)$$

R being used to denote $R^{-}{}_{000}$.

The value of B_0 is calculable explicitly provided the pile is large (i.e., $k\approx 1$) and the oscillation is fast compared to the delayed neutron periods but slow compared to the pile period:

$$\omega\tau \ll 1, \quad \omega\tau_j \gg 1. \quad (51)$$

Since τ_j is of the order of 10 seconds or more, while τ is of the order of 10^{-3} to 10^{-4} sec., an angular frequency ω of 1 to 100 sec.$^{-1}$ will satisfy these conditions. To compute B_0 under these assumptions we return to the definition of $\bar{Q}(\xi^2)$:

$$\bar{Q}(\xi^2)=\frac{1}{p}\left\{(1-\beta)\bar{P}(\xi^2)+\sum_j\frac{\beta_j}{1+i\omega\tau_j}\bar{P}_j(\xi^2)\right\}$$

$$=\frac{4\pi}{p}\left\{(1-\beta)\int_0^\infty P(r)\frac{\sin\xi r}{\xi r}r^2dr+\sum_j\frac{\beta_j}{1+i\omega\tau_j}\int_0^\infty P_j(r)\frac{\sin\xi r}{\xi r}r^2dr\right\}.$$

We expand $(\sin\xi r/\xi r)$ in Taylor's series and integrate term by term; the result, correct to terms of order ξ^2, is:

$$\bar{Q}(\xi^2)=1-\beta+\sum_j\frac{\beta_j}{1+i\omega\tau_j}-\frac{\xi^2}{6}\langle r^2(\omega)\rangle_{Av}+O(\xi^4), \quad (52)$$

or, if the condition (51) is fulfilled,

$$\bar{Q}(\xi^2)\approx(1-\beta)-\frac{\xi^2}{6}\langle r^2(0)\rangle_{Av}+O(\xi^4), \quad (53)$$

where

$$\langle r^2(\omega)\rangle_{Av}=\frac{4\pi}{p}\left[\int_0^\infty P(r)r^4dr+\sum_j\frac{\beta_j}{1+i\omega\tau_j}\int_0^\infty P_j(r)r^4dr\right],$$

and $O(\xi^4)$ is an abbreviation for terms of order ξ^4 and above.

When $\omega=0$,

$$\langle r^2(0)\rangle_{Av}=\frac{4\pi}{p}\left[\int_0^\infty P(r)r^4dr+\sum_j\beta_j\int_0^\infty P_j(r)r^4dr\right];$$

i.e., $\langle r^2(0)\rangle_{Av}$ is the mean square distance which a fission neutron travels while slowing down. In general, for $\omega\neq 0$, $\langle r^2(\omega)\rangle_{Av}$ is a complex number.

We now substitute (53) into the characteristic Eq. (43):

$$M^2\xi^2-k(1-\beta)+1+i\omega\tau-kO(\xi^4)=0, \quad (54)$$

where

$$M^2=\frac{k\langle r^2(0)\rangle_{Av}}{6}+L^2 \quad (55)$$

is called the "migration area." Equation (54) can be solved by successive approximations provided

$$\frac{kO(B_0^4)}{M^2B_0^2}\ll 1,$$

B_0^2 being the value of ξ^2 which solves (54). In first approximation,

$$B_0^2=\frac{k(1-\beta)-1-i\omega\tau}{M^2}, \quad (56)$$

while the first and most important term in $O(B_0^4)$ is

$$O(B_0^4)\approx\frac{1}{120}B_0^4\langle r^4(0)\rangle_{Av};$$

i.e.,

$$\frac{kO(B_0^4)}{M^2B_0^2}\approx\frac{1}{120}[k(1-\beta)-1-i\omega\tau]k\frac{\langle r^4(0)\rangle_{Av}}{M^4}<\frac{3}{10}\frac{k(1-\beta)-1-i\omega\tau}{k}\frac{\langle r^4(0)\rangle_{Av}}{[\langle r^2(0)\rangle_{Av}]^2}.$$

Now for any physically plausible kernel $\langle r^4(0)\rangle_{Av}/[\langle r^2(0)\rangle_{Av}]^2$ is of order unity; hence since $k-1\ll 1$ and $\omega\tau\ll 1$, the required ratio is small in absolute value, and the successive approximation is justified. The remaining discussion will be based on the value (56) for B_0^2.

Since

$$k\bar{Q}(\xi^2)\approx k(1-\beta)-k\frac{\xi^2}{6}\langle r^2(0)\rangle_{\mathrm{Av}},$$

the quantity in the denominator of (50) is

$$k\bar{Q}'(B_0{}^2)-L^2\approx -M^2,$$

and therefore

$$n(\mathbf{r},t)-\bar{n}(\mathbf{r})\approx\frac{-S_a}{\sigma_p M^2}\frac{e^{i(\omega t+B_0R)}}{4\pi R}(\boldsymbol{\rho}\cdot\nabla)n(\mathbf{r}_0). \quad (57)$$

In order to calculate the velocity, wave-length, and attenuation length (distance for wave amplitude to damp by factor e), it is necessary to compute B_0 explicitly. From (56),

$$B_0=\frac{-1}{M}\left(\frac{[X^2+(\omega\tau)^2]^{\frac{1}{2}}+X}{2}\right)^{\frac{1}{2}}+\frac{i}{M}\left(\frac{[X^2+(\omega\tau)^2]^{\frac{1}{2}}-X}{2}\right)^{\frac{1}{2}}, \quad (58)$$

where

$$X\equiv k(1-\beta)-1.$$

Upon substituting (58) into (57), we observe that the resulting damped wave has a velocity

$$\frac{M\omega}{\left(\frac{[X^2+(\omega\tau)^2]^{\frac{1}{2}}+X}{2}\right)^{\frac{1}{2}}}, \quad (59)$$

a wave-length

$$\frac{2\pi M}{\left(\frac{[X^2+(\omega\tau)^2]^{\frac{1}{2}}+X}{2}\right)^{\frac{1}{2}}}, \quad (60)$$

and an attenuation length

$$\frac{M}{\left(\frac{[X^2+(\omega\tau)^2]^{\frac{1}{2}}-X}{2}\right)^{\frac{1}{2}}}. \quad (61)$$

If $X=0$, the medium is neither a net absorber nor producer of prompt neutrons. In this case

$$v=M\left(\frac{2\omega}{\tau}\right)^{\frac{1}{2}}, \quad (62)$$

$$\lambda=2\pi M\left(\frac{2}{\omega\tau}\right)^{\frac{1}{2}}, \quad (63)$$

$$l=M\left(\frac{2}{\omega\tau}\right)^{\frac{1}{2}}. \quad (64)$$

The waves characterized by (62), (63), and (64) are completely analogous to the classical thermal waves set up in a conducting medium in which the ratio of conductivity to specific heat per unit volume is M^2/τ, and the angular frequency of the impressed oscillating heat source is ω.

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