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The Theory of Magnetic Poles

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If one supposes that a particle with a single magnetic pole can exist and that it interacts with charged particles, the laws of quantum mechanics lead to the requirement that the electric charges shall be quantized—all charges must be integral multiples of a unit charge e connected with the pole strength g by the formula $eg=\frac{1}{2}\hbar c$. Since electric charges are known to be quantized and no reason for this has yet been proposed apart from the existence of magnetic poles, we have here a reason for taking magnetic poles seriously. The fact that they have not yet been observed may be ascribed to the large value of the quantum of pole.

In 1931 I gave a primitive theory which described the motion of a pole in the field of a charged particle whose motion is given, or the motion of a charged particle in the field of a pole whose motion is given. The present paper sets up a general theory of charged particles and poles in interaction through the medium of the electromagnetic field. The idea which makes this generalization possible consists in supposing each pole to be at the end of an unobservable string, which is the line along which the electromagnetic potentials are singular, and introducing dynamical coordinates and momenta to describe the motion of the strings. The whole theory then comes out by the application of standard methods. There are unsolved difficulties, concerned with the interaction of a point charge or a point pole with the field it produces itself, such as occur in all dynamical theories of fields and particles in interaction.

I. INTRODUCTION

THE field equations of electrodynamics are symmetrical between electric and magnetic forces. The symmetry between electricity and magnetism is, however, disturbed by the fact that a single electric charge may occur on a particle, while a single magnetic pole has not been observed to occur on a particle. In the present paper a theory will be developed in which a single magnetic pole can occur on a particle, and the dissymmetry between electricity and magnetism will consist only in the smallest pole which can occur, being much greater than the smallest charge. This will result in an enormous energy being needed to produce a particle with a single pole, which can very well explain why such particles have not been observed up to the present.[1]

There are several kinds of particles in experimental physics for which satisfactory theories do not yet exist, and one may wonder what is the value of postulating a quite new kind of particle for which there is no experimental evidence, and thus introducing a further complication into the study of elementary particles. The interest of the theory of magnetic poles is that it forms a natural generalization of the usual electrodynamics and *it leads to the quantization of electricity*. One can set up consistent equations in quantum mechanics for the interaction of a pole of strength g with an electric charge e, only provided

$$eg=\tfrac{1}{2}n\hbar c, \tag{1}$$

where n is an integer. Thus the mere existence of one pole of strength g would require all electric charges to be quantized in units of $\frac{1}{2}\hbar c/g$ and, similarly, the existence of one charge would require all poles to be quantized. The quantization of electricity is one of the most fundamental and striking features of atomic physics, and there seems to be no explanation for it apart from the theory of poles. This provides some grounds for believing in the existence of these poles.

I first put forward the idea of magnetic poles in 1931.[2] The theory I then proposed was very incomplete, as it provided only the equations of motion for a magnetic pole in the field of charged particles whose motion is given, or the equations of motion for a charged particle in the field of magnetic poles whose motion is given. The present development provides all the equations of

[1] F. Ehrenhaft [Phys. Rev. **67**, 63, 201 (1945)] has obtained some experimental results which he interprets in terms of particles with single magnetic poles. This is not a confirmation of the present theory, since Ehrenhaft does not use high energies and the theory does not lead one to expect single poles to occur under the conditions of Ehrenhaft's experiments.

[2] P. A. M. Dirac, Proc. Roy. Soc. **A133**, 60 (1931).

motion for magnetic poles and charged particles interacting with each other through the medium of the electromagnetic field in accordance with quantum mechanics, and is a complete dynamical theory, except for the usual difficulties of the appearance of divergent integrals in the solution of the wave equation, arising from the reaction on a particle of the field it produces itself, which difficulties are of the same nature in the present theory as in the usual electrodynamics.

II. THE CLASSICAL EQUATIONS OF MOTION

We shall work all the time with relativistic notation, using the four coordinates $x_\mu (\mu = 0, 1, 2, 3)$ to fix a point in space-time and taking the velocity of light to be unity. The electromagnetic field at any point forms a 6-vector $F_{\mu\nu} = -F_{\nu\mu}$. We shall need to use the notation of the dual $(F\dagger)_{\mu\nu}$ of a 6-vector $F_{\mu\nu}$, defined by

$$(F\dagger)_{01} = F_{23}, \quad (F\dagger)_{23} = -F_{01},$$

together with the equations obtained from these by cyclic permutation of 1, 2, 3. Note that

$$(F\dagger\dagger)_{\mu\nu} = -F_{\mu\nu} \tag{2}$$

and, with a second 6-vector $G_{\mu\nu}$,

$$(F\dagger)_{\mu\nu} G^{\mu\nu} = F_{\mu\nu} (G\dagger)^{\mu\nu}.$$

The ordinary Maxwell equations are

$$\partial F_{\mu\nu} / \partial x_\nu = -4\pi j_\mu, \tag{3}$$

where j_μ is the vector formed by the charge density and current, and

$$(F\dagger)_{\mu\nu} / \partial x_\nu = 0. \tag{4}$$

Equation (4) asserts that the divergence of the magnetic flux vanishes, and must be modified in a theory which allows single poles. The density of poles and the current of poles will form a vector k_μ which is the magnetic analog of j_μ, and Eq. (4) must be replaced by

$$(F\dagger)_{\mu\nu} / \partial x_\nu = -4\pi k_\mu. \tag{4'}$$

The world-line of a particle may be described by giving the four coordinates z_μ of a point on it as functions of the proper-time s measured along it,

$$z_\mu = z_\mu(s).$$

A particle with a point charge e gives rise to a contribution to j_μ which is infinite on the world-line and zero everywhere else. We may express it with the help of the δ-function, and then have for the charge-current vector at an arbitrary point x

$$j_\mu(x) = \sum\nolimits_e e \int (dz_\mu/ds) \delta_4(x-z) ds, \tag{5}$$

where the function δ_4 is defined by

$$\delta_4(x) = \delta(x_0)\delta(x_1)\delta(x_2)\delta(x_3),$$

and $\sum_e$ denotes the sum over all charged particles. Similarly, if the poles g are concentrated at points,

$$k_\mu(x) = \sum\nolimits_g g \int (dz_\mu/ds) \delta_4(x-z) ds, \tag{6}$$

where $\sum_g$ denotes the sum over all particles with poles. Equations (3), (4′), (5), and (6) fix the field when the motion of the particles and the incident radiation are known.

The motion of a charged particle is given by Lorentz's equation

$$m(d^2 z_\mu/ds^2) = e(dz^\nu/ds) F_{\mu\nu}(z).$$

We may assume the analogous equation for the motion of a pole

$$m(d^2 z_\mu/ds^2) = g(dz^\nu/ds)(F\dagger)_{\mu\nu}(z). \tag{7}$$

The field quantities $F_{\mu\nu}(z)$ occuring here are to be taken at the point z where the particle is situated and are there infinitely great and singular, so that these equations do not really have any meaning. It becomes necessary to make small changes in them to avoid the infinities. A method frequently used is to depart from the point charge model, which involves replacing the δ_4 function in (5) by a smoothed-out function approximating to it; and one could apply a similar procedure in (6) for the poles. But this method leads to an additional mass for the particles which does not transform according to the requirements of relativity. A possibly better method consists of introducing a limiting process, making a small change in the field equations in such a way that there is no additional mass for the particles in the limit. The resulting theory is not Lorentz invariant before the limit, but is Lorentz invariant in the limit. This method will be adopted here. It will require a slightly modified field function $F_{\mu\nu}^*$ to occur instead of $F_{\mu\nu}$ in Eqs. (3) and (7), so that we have the four equations of motion

$$\partial F_{\mu\nu}^*/\partial x_\nu = -4\pi \sum_e e \int (dz_\mu/ds)\delta_4(x-z)ds, \quad (8)$$

$$\partial(F\dagger)_{\mu\nu}/\partial x_\nu = -4\pi \sum_g g \int (dz_\mu/ds)\delta_4(x-z)ds, \quad (9)$$

$$m(d^2z_\mu/ds^2) = e(dz^\nu/ds)F_{\mu\nu}(z) \quad (10)$$

for charged particles, and

$$m(d^2z_\mu/ds^2) = g(dz^\nu/ds)(F\dagger)_{\mu\nu}^*(z) \quad (11)$$

for particles with poles. These equations, together with the equations that connect F and F^*, which are linear and will be given later, form the complete scheme of equations of motion. They are valid with arbitrary values for the e's and g's of the various particles.

III. THE ELECTROMAGNETIC POTENTIALS

To get a theory which can be transferred to quantum mechanics we need to put the equations of motion into the form of an action principle and for this purpose we require the electromagnetic potentials. The usual way of introducing them consists of putting

$$F_{\mu\nu} = \partial A_\nu/\partial x^\mu - \partial A_\mu/\partial x^\nu, \quad (12)$$

but this is no longer possible when there are magnetic poles, since Eq. (12) leads to Eq. (4) and thus contradicts (9). Therefore, it is necessary to modify (12).

If we consider one instant of time, Eq. (12) or (4) requires that the total magnetic flux crossing any closed surface at this time shall be zero. This is not true if there is a magnetic pole inside the closed surface. Equation (12) must then fail somewhere on the surface, and we may suppose that it fails at only one point. Equation (12) will fail at one point on every closed surface surrounding the pole, so that it will fail on a line of points, which we shall call a *string*, extending outward from the pole. The string may be any curved line, extending from the pole to infinity or to another pole of equal and opposite strength. Every pole must be at the end of such a string.

The variables needed to fix the positions of the strings will be treated as dynamical coordinates and momenta conjugate to them will be introduced later. These variables are needed for the dynamical theory, but they do not correspond to anything observable and their values in a specific problem are always arbitrary and do not influence physical phenomena. They may be called *unphysical* variables.

Unphysical variables have occurred previously in dynamical theory. For example, in ordinary electrodynamics the extra variables needed to describe the potentials when the field is fixed are unphysical variables. A more elementary example is provided by the azimuthal angle of a rotating body which is symmetrical about its axis of rotation. Unphysical variables can always be eliminated by a suitable transformation, but this may introduce such a lack of symmetry into the theory as to make it not worth while. (The unphysical variables describing the strings could be eliminated by imposing the condition that the strings must always extend in the direction of the x_1-axis from each pole to infinity. With the strings fixed in this way no variables would be needed to describe them, but the symmetry of the equations under three-dimensional rotations would be completely spoilt. The physical consequences of the theory would not be affected.)

Each string will trace out a two-dimensional sheet in space-time. These sheets will be the regions where Eq. (12) fails. Each sheet may be described by expressing a general point y_μ on it as a function of two parameters τ_0 and τ_1,

$$y_\mu = y_\mu(\tau_0, \tau_1).$$

Let us suppose for definiteness that each string extends to infinity. Then the parameters τ_0 and τ_1 may be arranged so that $\tau_1 = 0$ on the world-line of the pole and extends to infinity as one follows a string to infinity, and τ_0 goes from $-\infty$ to ∞ as one goes from infinite past to infinite future.

Equation (12) must be replaced by an equation of the form

$$F_{\mu\nu} = \partial A_\nu/\partial x^\mu - \partial A_\mu/\partial x^\nu + 4\pi \sum_g (G\dagger)_{\mu\nu}, \quad (13)$$

where each $(G\dagger)_{\mu\nu}$ is a field quantity which vanishes everywhere except on one of the sheets.

and the summation is taken over all the sheets, one of which is associated with each pole. Substituting (13) into (9) we find, remembering (2),

$$\partial G_{\mu\nu}/\partial x_\nu = g\int (dz_\mu/ds)\delta_4(x-z)ds. \tag{14}$$

This is the equation which determines $G_{\mu\nu}$.

It is easily verified that the solution of Eq. (14) is

$$G_{\mu\nu}(x) = g\int\int\left(\frac{\partial y_\mu}{\partial\tau_0}\frac{\partial y_\nu}{\partial\tau_1} - \frac{\partial y_\mu}{\partial\tau_1}\frac{\partial y_\nu}{\partial\tau_0}\right) \times\delta_4(x-y)d\tau_0 d\tau_1, \tag{15}$$

integrated over the whole sheet. In fact (15) gives directly

$$\begin{aligned}\frac{\partial G_{\mu\nu}}{\partial x_\nu} &= g\int\int\left(\frac{\partial y_\mu}{\partial\tau_0}\frac{\partial y_\nu}{\partial\tau_1} - \frac{\partial y_\mu}{\partial\tau_1}\frac{\partial y_\nu}{\partial\tau_0}\right)\frac{\partial\delta_4(x-y)}{\partial x_\nu}d\tau_0 d\tau_1,\\
&= -g\int\int\left(\frac{\partial y_\mu}{\partial\tau_0}\frac{\partial y_\nu}{\partial\tau_1} - \frac{\partial y_\mu}{\partial\tau_1}\frac{\partial y_\nu}{\partial\tau_0}\right)\times\frac{\partial\delta_4(x-y)}{\partial y_\nu}d\tau_0 d\tau_1,\\
&= -g\int\int\left(\frac{\partial y_\mu}{\partial\tau_0}\frac{\partial\delta_4(x-y)}{\partial\tau_1} - \frac{\partial y_\mu}{\partial\tau_1}\frac{\partial\delta_4(x-y)}{\partial\tau_0}\right)d\tau_0 d\tau_1.\end{aligned}$$

From Stokes' theorem, for any two functions U and V on the sheet,

$$\int\int\left(\frac{\partial U}{\partial\tau_0}\frac{\partial V}{\partial\tau_1} - \frac{\partial U}{\partial\tau_1}\frac{\partial V}{\partial\tau_0}\right)d\tau_0 d\tau_1 = \int U\left(\frac{\partial V}{\partial\tau_0}d\tau_0 + \frac{\partial V}{\partial\tau_1}d\tau_1\right), \tag{16}$$

the left-hand integral being taken over any area of the sheet and the right-hand integral along the rim of that area. Putting $U=\delta_4(x-y)$, $V=y_\mu$, and applying the theorem to the whole sheet, so that the only part of the rim not at infinity is the world-line of the pole, we get

$$\partial G_{\mu\nu}/\partial x_\nu = g\int \delta_4(x-y)[\partial y_\mu(\tau_0,0)/\partial\tau_0]d\tau_0,$$

which agrees with (14), since $y_\mu(\tau_0, 0) = z_\mu(s)$ with τ_0 some function of s.

With given world-lines for the particles, the solution of the field equations (8) and (9) for which there is no ingoing field is called the retarded field. It is connected by (13) with the retarded potentials. The retarded potentials consist of a contribution from each particle, depending only on the world-line of that particle and on the sheet attached to it in the case when the particle has a pole. These contributions may be conveniently expressed with the help of the Lorentz invariant function $J(x)$ defined by

$$\left.\begin{aligned}J(x) &= 2\delta(x_\mu x^\mu) \quad \text{for } x_0>0,\\ &=0 \quad\quad\quad\;\; \text{for } x_0<0,\end{aligned}\right\} \tag{17}$$

or by

$$J(x) = r^{-1}\delta(x_0-r),\quad r=(x_1^2+x_2^2+x_3^2)^{\frac{1}{2}}.$$

The function $\Delta(x)$ of Jordan and Pauli is connected with $J(x)$ by

$$\Delta(x) = J(x) - J(-x). \tag{18}$$

It is easily verified that

$$\Box J(x) = 4\pi\delta_4(x). \tag{19}$$

(One can check this result in the neighborhood of the origin by expressing the integral of $\Box J(x)$ over a small four-dimensional volume around the origin as a three-dimensional surface integral over the boundary of that volume.)

The contribution of a charged particle to the retarded potentials is, according to the Lienard-Wiechert formula,

$$A_\nu{}^*(x)_{re} = e\int_{-\infty}^{\infty} J(x-z)(dz_\nu/ds)ds. \tag{20}$$

(A^* is put on the left here instead of A since the field F^* occurs in (8)). The corresponding formula for the contribution of a pole is

$$A_\nu(x)_{re} = g\epsilon_{\nu\lambda\rho\sigma}\int\int\frac{\partial y^\lambda}{\partial\tau_0}\frac{\partial y^\rho}{\partial\tau_1}\frac{\partial J(x-y)}{\partial x_\sigma}d\tau_0 d\tau_1 \tag{21}$$

integrated over all the sheet, where $\epsilon_{\nu\lambda\rho\sigma}$ is the antisymmetrical tensor of the fourth rank with $\epsilon_{0123}=1$. To verify (21), we note that it leads to

$$\epsilon^{\mu\nu\alpha\beta}\frac{\partial A_{\nu re}}{\partial x^\mu} = g\epsilon^{\mu\nu\alpha\beta}\epsilon_{\nu\lambda\rho\sigma}\int\int\frac{\partial y^\lambda}{\partial\tau_0}\frac{\partial y^\rho}{\partial\tau_1}\frac{\partial^2 J(x-y)}{\partial y^\mu\partial y_\sigma}d\tau_0 d\tau_1.$$

Using

$$\epsilon^{\mu\nu\alpha\beta}\epsilon_{\nu\lambda\rho\sigma}=\delta_\lambda{}^\mu\delta_\rho{}^\alpha\delta_\sigma{}^\beta+\delta_\rho{}^\mu\delta_\sigma{}^\alpha\delta_\lambda{}^\beta+\delta_\sigma{}^\mu\delta_\lambda{}^\alpha\delta_\rho{}^\beta-(\alpha\beta),$$

where $-(\alpha\beta)$ means that we must subtract all the preceding terms with α and β interchanged, we get

$$\epsilon^{\mu\nu\alpha\beta}\frac{\partial A_{\nu re}}{\partial x^\mu}=g\int\int\left\{\frac{\partial^2 J(x-y)}{\partial\tau_0\partial y_\beta}\frac{\partial y^\alpha}{\partial\tau_1}+\frac{\partial^2 J(x-y)}{\partial\tau_1\partial y_\alpha}\frac{\partial y^\beta}{\partial\tau_0}+\frac{\partial y^\alpha}{\partial\tau_0}\frac{\partial y^\beta}{\partial\tau_1}\Box J(x-y)\right\}d\tau_0 d\tau_1-(\alpha\beta)$$

$$=g\int\left[\frac{\partial J(x-y)}{\partial y_\beta}\right]_{y=z}\frac{dz^\alpha}{ds}ds-(\alpha\beta)+4\pi G^{\alpha\beta},$$

with the help of Stokes' theorem (16), and (19) and (15). According to (13), this gives the retarded field

$$(F\dagger)_{re}{}^{\alpha\beta}=\epsilon^{\mu\nu\alpha\beta}\partial A_{\nu re}/\partial x^\mu-4\pi G^{\alpha\beta},$$

$$=-g\int\frac{\partial J(x-z)}{\partial x_\beta}\frac{dz^\alpha}{ds}ds-(\alpha\beta),$$

$$=-\partial B_{re}{}^\alpha/\partial x_\beta+\partial B_{re}{}^\beta/\partial x_\alpha, \qquad (22)$$

with

$$B_{re}{}^\alpha=g\int J(x-z)(dz^\alpha/ds)ds. \qquad (23)$$

This is seen to be the correct value for the retarded field produced by a pole, from the analogy of (23) to the Lienard-Wiechert potential (20).

In the usual electrodynamics the potentials are restricted by the condition

$$\partial A_\nu/\partial x_\nu=0 \quad \text{or} \quad \partial A_\nu{}^*/\partial x_\nu=0. \qquad (24)$$

This condition can be retained in the present theory, as it is satisfied by the retarded potentials (20), (21). The two forms of (24) are equivalent because of the linear connection between the starred and unstarred field (see Eq. (29)).

IV. THE ACTION PRINCIPLE

The action integral of ordinary electrodynamics may be expressed as a sum of the three terms, $I_1+I_2+I_3$, where I_1 is the action integral for the particles alone,

$$I_1=\textstyle\sum_e m\displaystyle\int ds,$$

I_2 is the action integral for the field alone,

$$I_2=(16\pi)^{-1}\int F_{\mu\nu}F^{\mu\nu}d^4x,\ (d^4x=dx_0dx_1dx_2dx_3),$$

and I_3 is the contribution of the interaction of the charges with the field,

$$I_3=\textstyle\sum_e e\displaystyle\int A^\nu(z)(dz_\nu/ds)ds. \qquad (25)$$

The $F_{\mu\nu}$ in I_2 are to be regarded as functions of the potentials.

The same action integral will do in the present theory, provided the sum in I_1 is extended to include the particles with poles as well as those with charges,

$$I_1=\textstyle\sum_{e+g}m\displaystyle\int ds. \qquad (26)$$

No further term is needed to give the interaction between the poles and the field, this interaction being taken into account in I_2, in which $F_{\mu\nu}$ is now to be regarded as a function of the potentials and the string variables $y_\mu(\tau_0,\tau_1)$ given by (13) and (15).

In order to avoid infinities in the equations of motion arising from the infinite fields produced by point charges and poles, we shall make a small modification in the field equations, by replacing I_2 by

$$I_2'=(16\pi)^{-1}\int\int F_{\mu\nu}(x)F^{\mu\nu}(x')\gamma(x-x')d^4xd^4x',$$

where $\gamma(x)$ is a function which approximates to the function $\delta_4(x)$, and is made to tend to $\delta_4(x)$ in the limit. We shall assume that

$$\gamma(-x)=\gamma(x), \qquad (27)$$

and shall assume other properties for $\gamma(x)$ as they are needed, but the precise form of $\gamma(x)$ will be left arbitrary. We may write I_2' as

$$I_2'=(16\pi)^{-1}\int F_{\mu\nu}{}^*(x)F^{\mu\nu}(x)d^4x, \qquad (28)$$

using the notation that for any field quantity $U(x)$,

$$U^*(x)=\int U(x')\gamma(x-x')d^4x'. \qquad (29)$$

It will now be verified that the variation of

$$I=I_1+I_2'+I_3$$

leads to the correct equations of motion. The variation of I_1 is well known and gives

$$\delta I_1=-\sum_{e+g}m\int(d^2z_\mu/ds^2)\delta z^\mu ds. \tag{30}$$

The variation of I_3 may be carried out the same as in ordinary electrodynamics and gives

$$\delta I_3=\sum_e e\int\{[(\partial A_\nu/\partial x^\mu)-(\partial A_\mu/\partial x^\nu)]_{x=z}\delta z^\mu +(\delta A_\nu)_{x=z}\}(dz^\nu/ds)ds. \tag{31}$$

The variation of I_2' gives, using (27)

$$\delta I_2'=(8\pi)^{-1}\int\int F_{\mu\nu}(x')\delta F^{\mu\nu}(x)\gamma(x-x')d^4xd^4x'$$

$$=(8\pi)^{-1}\int F_{\mu\nu}{}^*(x)\delta F^{\mu\nu}(x)d^4x.$$

Substituting for $F^{\mu\nu}$ its value given by (13), we get

$$\delta I_2'=-(4\pi)^{-1}\int F_{\mu\nu}{}^*(\partial\delta A^\mu/\partial x_\nu)d^4x +\tfrac{1}{2}\sum_g\int F_{\mu\nu}{}^*\delta(G\dagger)^{\mu\nu}d^4x \tag{32}$$

$$=(4\pi)^{-1}\int(\partial F_{\mu\nu}{}^*/\partial x_\nu)\delta A^\mu d^4x +\tfrac{1}{2}\sum_g\int(F\dagger)_{\mu\nu}{}^*\delta G^{\mu\nu}d^4x. \tag{33}$$

Using (15), the second term here becomes

$$\sum_g g\int(F\dagger)_{\mu\nu}{}^*d^4x\,\delta\int\int\frac{\partial y^\mu}{\partial\tau_0}\frac{\partial y^\nu}{\partial\tau_1}\delta_4(x-y)d\tau_0d\tau_1$$

$$=\sum_g g\int(F\dagger)_{\mu\nu}{}^*d^4x\int\int\left\{\delta\left(\frac{\partial y^\mu}{\partial\tau_0}\frac{\partial y^\nu}{\partial\tau_1}\right)\delta_4(x-y) +\frac{\partial y^\mu}{\partial\tau_0}\frac{\partial y^\nu}{\partial\tau_1}\frac{\partial\delta_4(x-y)}{\partial y^\rho}\delta y^\rho\right\}d\tau_0d\tau_1$$

$$=\sum_g g\int\int\left\{(F\dagger)_{\mu\nu}{}^*(y)\left(\frac{\partial\delta y^\mu}{\partial\tau_0}\frac{\partial y^\nu}{\partial\tau_1}-\frac{\partial\delta y^\mu}{\partial\tau_1}\frac{\partial y^\nu}{\partial\tau_0}\right) +\frac{\partial(F\dagger)_{\mu\nu}{}^*(y)}{\partial y^\rho}\frac{\partial y^\mu}{\partial\tau_0}\frac{\partial y^\nu}{\partial\tau_1}\delta y^\rho\right\}d\tau_0d\tau_1$$

$$=\sum_g g\int\int\left\{\frac{\partial((F\dagger)_{\mu\nu}{}^*\delta y^\mu)}{\partial\tau_0}\frac{\partial y^\nu}{\partial\tau_1} -\frac{\partial((F\dagger)_{\mu\nu}{}^*\delta y^\mu)}{\partial\tau_1}\frac{\partial y^\nu}{\partial\tau_0}-\frac{\partial(F\dagger)_{\mu\nu}{}^*}{\partial y^\rho}\left(\frac{\partial y^\rho}{\partial\tau_0}\frac{\partial y^\nu}{\partial\tau_1}\delta y^\mu -\frac{\partial y^\rho}{\partial\tau_1}\frac{\partial y^\nu}{\partial\tau_0}\delta y^\mu-\frac{\partial y^\mu}{\partial\tau_0}\frac{\partial y^\nu}{\partial\tau_1}\delta y^\rho\right)\right\}d\tau_0d\tau_1$$

$$=\sum_g g\int(F\dagger)_{\mu\nu}{}^*(z)\delta z^\mu\frac{dz^\nu}{ds}ds -\sum_g g\int\int\left(\frac{\partial(F\dagger)_{\mu\nu}{}^*}{\partial y^\rho}+\frac{\partial(F\dagger)_{\nu\rho}{}^*}{\partial y^\mu} +\frac{\partial(F\dagger)_{\rho\mu}}{\partial y^\nu}\right)\frac{\partial y^\rho}{\partial\tau_0}\frac{\partial y^\nu}{\partial\tau_1}\delta y^\mu d\tau_0d\tau_1 \tag{34}$$

by a further application of Stokes' theorem (16). The total variation δI is given by the sum of (30), (31), (34) and the first term in (33).

By equating to zero the coefficient of $\delta A^\mu(x)$ in δI, we get precisely Eq. (8). By equating to zero the coefficient of δz^μ for a charged particle we get

$$m(d^2z_\mu/ds^2) =e[(\partial A_\nu/\partial x_\mu)-(\partial A_\mu/\partial x^\nu)]_{x=z}(dz^\nu/ds).$$

This agrees with the equation of motion (10) provided the charged particle does not lie on any of the strings, so that $G_{\mu\nu}(z)=0$. By equating to zero the coefficient of δz^μ for a pole, we get precisely (11). Equation (9) is a consequence of Eqs. (13) and (15), which express $F_{\mu\nu}$ in terms of the potentials and string variables. Thus all the equations of motion (8), (9), (10), and (11) follow from the action principle $\delta I=0$, provided we impose the condition that *a string must never pass through a charged particle.*

By equating to zero the coefficient of the variation δy^μ in a string variable, we get

$$\partial(F\dagger)_{\mu\nu}{}^*/\partial y^\rho+\partial(F\dagger)_{\nu\rho}{}^*/\partial y^\mu+\partial(F\dagger)_{\rho\mu}{}^*/\partial y^\nu=0$$

or

$$\partial F_{\mu\nu}{}^*/\partial y_\nu=0,$$

holding at all points on the sheet. From (8) this is automatically satisfied, provided the string never passes through a charged particle. Thus the action principle leads to no equations of motion for the string variables, in conformity with the unphysical nature of these variables.

The action integral I is a correct one and may be used as basis for a theory of electrodynamics, but it leads to some inconvenience in the Hamiltonian formulation of the equations of motion, since it makes the momentum conjugate to A_0 vanish identically. This inconvenience may be avoided by a method due to Fermi, which consists in adding on a further term to the action integral

$$I_4 = (8\pi)^{-1}\int (\partial A_\nu{}^*/\partial x_\mu)(\partial A_\mu/\partial x_\nu)d^4x. \quad (35)$$

This gives

$$\delta I_4 = (4\pi)^{-1}\int (\partial A_\nu{}^*/\partial x_\mu)(\partial \delta A_\mu/\partial x_\nu)d^4x \quad (36)$$

$$= -(4\pi)^{-1}\int (\partial^2 A_\nu{}^*/\partial x_\nu \partial x^\mu)\delta A^\mu d^4x, \quad (37)$$

and leads to a further term $-\partial^2 A_\nu{}^*/\partial x_\nu \partial x^\mu$ on the left-hand side of Eq. (8). This further term in (8) does not affect the equations of motion, because it vanishes when one uses the supplementary condition (24), but in the Hamiltonian formulation it is necessary to distinguish between those equations that hold only in virtue of supplementary conditions and those that are independent of supplementary conditions. Therefore, we must leave this term in (8) to have an equation of the latter kind. Equation (8) may now be written, with the help of (13) and (15),

$$\Box A_\mu{}^*(x) = 4\pi \sum_e e\int (dz_\mu/ds)\delta_4(x-z)ds + 4\pi \sum_g \partial(G\dagger)_{\mu\nu}{}^*/\partial x_\nu. \quad (38)$$

V. THE METHOD OF PASSING TO THE HAMILTONIAN FORMULATION

When one has the equations of motion of a dynamical system in the form of an action principle, one must put them into the Hamiltonian form as the next step in the process of quantization. The general procedure for doing this is to take the action integral previous to a certain time t and to form its variation allowing t to vary. This variation δI appears as a linear function of δt and of the variations δq in the dynamical coordinates at time t, the other terms in δI cancelling when one uses the equations of motion. One introduces the total variation in the final q's

$$\Delta q = \delta q + \dot{q}\delta t,$$

and expresses δI in terms of the Δq's and δt. One puts this equal to

$$\delta I = \sum p_r \Delta q_r - W\delta t, \quad (39)$$

(or the corresponding expression with an integral instead of a sum) and so defines the momenta p_r and the energy W. The p_r and W appear as functions of the coordinates q_r and velocities $\dot{q}_r$, and since the number of variables in the set p_r, W is one greater than the number of velocities $\dot{q}_r$, there must be a relation between the p_r, W and the coordinates, of the form

$$W - H(pq) = 0. \quad (40)$$

The p's and $-W$ are the partial derivatives of I with respect to the q's and t, so (40) gives a differential equation satisfied by I, called the *Hamilton-Jacobi equation*. From this equation one can pass to the wave equation of quantum mechanics by the application of certain rules. There may be more than one equation connecting the p's, q's, and W, in which case there are more than one Hamilton-Jacobi equation, leading to more than one wave equation.

To make the above procedure relativistic, one must take the action integral over space-time previous to a certain three-dimensional space-like surface S extending to infinity. One must form its variation, making a general variation in S as well as in the dynamical coordinates previous to S, and express the result in terms of the total variation in various dynamical quantities on S. This will again give an equation of the type (39) and one can again define the coefficients in it as momenta and set up the Hamilton-Jacobi equation.

There are various ways of modifying this procedure, which may be convenient for particular problems. Instead of stopping the action integral sharply at one definite time or at one definite three-dimensional space-like surface S, one may stop different terms in it at different times. One can picture the stopping of the action integral by supposing the dynamical system to go out of existence in some unnatural way and taking the total action before it goes out of existence. To stop different terms in the action integral at different times one must picture different parts of the dynamical system going out of existence at different times. After some parts have gone out of existence, the remaining parts continue to move in accordance with the equations of motion which follow from the surviving terms in the action integral, until they in turn go out of existence. The various ways of stopping the action integral lead to different Hamilton-Jacobi equations (40), which are equally valid and differ one from another by contact transformations.

A convenient way of stopping the action integral when one has particles interacting with a field is first to suppose the particles go out of existence at points in space-time lying outside each other's light cones, and then to stop the field at a considerably later time. One varies this stopped action integral, making variations in the points z_μ in space-time where the particles go out of existence and also in the surface S_F where the field goes out of existence. By equating to zero that part of the variation of the stopped action integral which is not connected with boundary variations, one gets the same equations of motion for the particles before they go out of existence as one had with the unstopped action integral, and one gets field equations which continue to govern the field after the particles have gone out of existence. Owing to the variations Δz_μ in the points z_μ occurring in regions of space-time completely immersed in the field, one gets equations which are more convenient to handle than those of the usual method in which one supposes the particles and the field to go out of existence together.

With the new electrodynamics let us suppose all the particles, and also the strings attached to the poles, go out of existence at a three-dimensional space-like surface S_P and the electromagnetic field goes out of existence at a much later surface S_F. This means that the integrals I_1, I_3 given by (26), (25) are to be stopped when the world-lines reach S_P and the integral over the sheet in (15) is to be stopped when the sheet reaches S_P, while I_2', I_4 given by (28), (35) are to be stopped at the boundary S_F. The stopping of these integrals will not affect the equations of motion for particles and field previous to S_P, namely (10), (11), (38), and further (38) will continue to hold through S_P and afterwards, until S_F.

Let us assume that the connection (29) between a field quantity U and U^* is such that the value of either of them at a point x is determined by the values of the other at points in space-time near x. Thus, if either of them vanishes in a certain region of space-time, the other will also vanish in that region, except possibly at points near the boundary.

Since $G_{\mu\nu}$ vanishes everywhere except on the sheets, $G_{\mu\nu}^*$ must now vanish in the region between S_P and S_F, with the exception of points near where the strings go out of existence. In this region we also have the first sum on the right-hand side of (38) vanishing, since the integrals are stopped at S_P, and hence we can infer from (38),

$$\square A_\mu^*(x)=0. \tag{41}$$

By a similar argument we can infer

$$\square A_\mu(x)=0 \tag{42}$$

in the region between S_P and S_F, with the exception of points near where the charged particles go out of existence.

In the regions where (42) and (41) hold we can make Fourier resolutions of $A_\mu(x)$ and $A_\mu^*(x)$ thus:

$$A_\mu(x)=\sum_{k_0}\int A_{k\mu}e^{i(kx)}k_0^{-1}d^3k, \tag{43}$$

$$A_\mu^*(x)=\sum_{k_0}\int A_{k\mu}^*e^{i(kx)}k_0^{-1}d^3k, \tag{44}$$

where

$$\begin{aligned}(kx)&=k_0x_0-k_1x_1-k_2x_2-k_3x_3,\\ d^3k&=dk_1dk_2dk_3,\\ k_0&=\pm(k_1^2+k_2^2+k_3^2)^{\frac{1}{2}},\end{aligned}$$

and $\sum_{k_0}$ means the sum over both values of k_0 for given k_1, k_2, k_3. The factor k_0^{-1} is introduced because $k_0^{-1}d^3k$ is Lorentz invariant. The condition that $A_\mu(x)$, $A_\mu^*(x)$ are real gives

$$A_{-k\mu}=-\bar{A}_{k\mu},\quad A_{-k\mu}^*=-\bar{A}_{k\mu}^*. \tag{45}$$

Let the Fourier resolution of the function $\gamma(x)$ be

$$\gamma(x)=(2\pi)^{-4}\int\gamma_l e^{i(lx)}d^4l$$

with

$$\gamma_{-l}=\bar{\gamma}_l.$$

The condition $\gamma(-x)=\gamma(x)$ gives

$$\gamma_{-l}=\gamma_l, \tag{46}$$

so γ_l is real. We now find by straightforward integration that

$$A_{k\mu}{}^*=\gamma_k A_{k\mu}. \tag{47}$$

We shall need to have the Fourier resolution (43) holding at each point z where a charged particle goes out of existence and the Fourier resolution (44) holding at each point y where a string goes out of existence. It seems probable that this can be arranged to be so by a suitable choice of the function γ, provided a point y is never very close to a point z. Let us assume that a field quantity $U(x)$ is determined by $U^*(x')$ at points x' lying close to x and outside the light cone from x. Then $A_\mu(z)$ is determined by $A_\mu{}^*(x')$ at points x' for which the Fourier resolution (44) is valid, so the Fourier resolution of $A_\mu(z)$ will be valid. Similarly, the Fourier resolution of $A_\mu{}^*(y)$ will be valid if $U^*(x)$ is determined by $U(x')$ at points x' lying close to x and outside the light cone of x.

The supplementary condition (24) gets modified in the region between S_P and S_F. With the integrals in (20) and (21) stopped at S_P, we have, writing z' for $z(s')$,

$$\begin{aligned}\frac{\partial A_\nu{}^*}{\partial x_\nu}&=\sum_e e\int_{-\infty}^{s}\frac{\partial J(x-z')}{\partial x_\nu}\frac{dz_\nu'}{ds'}ds',\\ &=-\sum_e e\int_{-\infty}^{s}\frac{\partial J(x-z')}{\partial z_\nu'}\frac{dz_\nu'}{ds'}ds',\\ &=-\sum_e eJ(x-z).\end{aligned} \tag{48}$$

This quantity differs from zero when x is on the future light cone of any point z where a charged particle goes out of existence. Equations (41) and (48) show that the potentials $A_\mu{}^*$ give a Wentzel type of field[8] between S_P and S_F.

VI. THE HAMILTONIAN FORMULATION

Let us form the variation of the action integral bounded as above, allowing S_P to vary but not S_F, and let us evaluate the terms in δI connected with the boundaries. The terms arising from δI_1 and δI_3 are, as in ordinary electrodynamics.

$$\sum_{e+g} m(dz_\mu/ds)\Delta z^\mu+\sum_e eA_\mu(z)\Delta z^\mu, \tag{49}$$

where the Δz^μ are the total changes of the coordinates of the point where a particle goes out of existence. In forming $\delta I_2'$ we can no longer use (32), but must use instead

$$\begin{aligned}\delta I_2'=&-(8\pi)^{-1}\int_{-\infty}^{S_F}\{F_{\mu\nu}{}^*(\partial\delta A^\mu/\partial x_\nu)\\ &+F_{\mu\nu}(\partial\delta A^{\mu*}/\partial x_\nu)\}d^4x+\tfrac{1}{4}\sum_g\int_{-\infty}^{S_F}\{F_{\mu\nu}{}^*\delta(G\dagger)^{\mu\nu}\\ &+F_{\mu\nu}\delta(G\dagger)^{\mu\nu*}\}d^4x.\end{aligned} \tag{50}$$

The second term here is equal to

$$\tfrac{1}{2}\sum_g\int_{-\infty}^{\infty}F_{\mu\nu}{}^*\delta(G\dagger)^{\mu\nu}d^4x,$$

provided S_F is sufficiently far from S_P, so that $\gamma(x-x')=0$ for x earlier than S_P and x' later than S_F. We may now use the calculation which led to (34), with the integrals over the sheets extending only over the parts of the sheets previous to S_P, and we then get extra terms, coming from the application of Stokes' theorem, of the form of line integrals along the lines where the sheets meet S_P. By arranging the parametrization of the sheets so that the line where a sheet meets S_P is given by $\tau_0=$constant, and the line where the varied sheet meets the varied S_P is given by $\tau_0=$same constant, these line integrals take the form

$$\sum_g g\int_0^\infty (F\dagger)_{\mu\nu}{}^*\delta y^\mu(dy^\nu/d\tau_1)d\tau_1. \tag{51}$$

The lines of integration are the positions of the strings when they go out of existence. In forming δI_4 we can no longer use (36), but must use instead

$$\begin{aligned}\delta I_4=(8\pi)^{-1}\int_{-\infty}^{S_F}&\{(\partial A_\nu{}^*/\partial x^\mu)(\partial\delta A^\mu/\partial x_\nu)\\ &+(\partial A_\nu/\partial x^\mu)(\partial\delta A^{\mu*}/\partial x_\nu)\}d^4x.\end{aligned} \tag{52}$$

[8] The properties of this field are given, for example, in P. Dirac, Annales de l'Inst. Henri Poincaré 9, 23 (1939).

This quantity is of the same form as the first term of the right-hand side of (50), and the two together give, on integration by parts, a boundary term of the form of an integral over the three-dimensional surface S_F, which may be written

$$(8\pi)^{-1}\int \{(\partial A_\mu{}^*/\partial x^\nu)\delta A^\mu + (\partial A_\mu/\partial x^\nu)\delta A^{\mu *}\} dS^\nu, \quad (53)$$

dS^ν being an element of this surface. The other terms in δI all cancel when one uses the equations of motion, provided S_F is not very close to S_P, so we are left with δI equal to the sum of (49), (51) and (53).

With this expression for δI, we cannot directly introduce the momenta in accordance with formula (39), since the A^μ, $A^{\mu *}$ whose variations occur in (53) are not independent, and since we have not varied S_F. A convenient way of proceeding is to pass to the Fourier components of the potentials, for which we may use the Fourier resolutions given by (43) and (44), as we are concerned in expression (53) with the potentials on the surface S_F. Let us take a varied motion which satisfies the equation of motion, so that the Fourier resolutions (43), (44) are valid on S_F also for the varied motion. Then expression (53) becomes, with the help of (47),

$$(8\pi)^{-1} i \sum_{k_0 k_0'} \int\int\int k_\nu(\gamma_k+\gamma_{k'}) \times A_{k\mu}\delta A_{k'}{}^\mu e^{i(k+k',\,x)} k_0^{-1} d^3k k_0'^{-1} d^3k' dS^\nu.$$

If we take the surface S_F to be $x_0=$constant for simplicity (any space-like surface must give the same final result), this becomes, on integrating with respect to x_1, x_2 and x_3,

$$\pi^2 i \sum_{k_0 k_0'} \int\int (\gamma_k+\gamma_{k'}) A_{k\mu}\delta A_{k'}{}^\mu \times \exp[i(k_0+k_0')x_0]\delta_3(k+k') d^3k k_0'^{-1} d^3k',$$

where $\delta_3(k)$ means $\delta(k_1)\delta(k_2)\delta(k_3)$. The factor $\delta_3(k+k')$ here shows that the integrand vanishes except when $k_r'=-k_r(r=1, 2, 3)$, which implies $k_0'=\pm k_0$. Thus the expression reduces, with the help of (46), to

$$-2\pi^2 i \sum_{k_0}\int \gamma_k A_{k\mu}\delta A_{-k}{}^\mu k_0^{-1} d^3k + \pi^2 i \sum_{k_0}\int (\gamma_k+\gamma_{k_0,\,-k_r}) A_{k\mu}\delta A_{k_0,\,-k_r}{}^\mu \times \exp[2ik_0x_0] k_0^{-1} d^3k.$$

The second term here may be written as a perfect differential,

$$\pi^2 i\delta \sum_{k_0}\int \gamma_k A_{k\mu} A_{k_0,\,-k_r}{}^\mu \exp[2ik_0x_0] k_0^{-1} d^3k,$$

and may therefore be discarded. The first term may be written, if we now restrict k_0 to be >0 and use (45) and (46),

$$2\pi^2 i\int \gamma_k(A_{k\mu}\delta\bar{A}_k{}^\mu - \bar{A}_{k\mu}\delta A_k{}^\mu) k_0^{-1} d^3k = 2\pi^2 i\delta\int \gamma_k A_{k\mu}\bar{A}_k{}^\mu k_0^{-1} d^3k - 4\pi^2 i\int \gamma_k \bar{A}_{k\mu}\delta A_k{}^\mu k_0^{-1} d^3k. \quad (54)$$

The first term in (54) is a perfect differential and may be discarded. We thus get the final result that δI is equal to, apart from a perfect differential, the sum of (49), (51), and the second term of (54).

We take as dynamical coordinates the coordinates z_μ of the particles when they go out of existence, the coordinates $y_\mu(\tau_1)$ of points on the strings when they go out of existence (providing a one-dimensional continuum of coordinates for each pole and each value of μ), and the Fourier components $A_{k\mu}$, with $k_0>0$, of the potentials after the particles and strings have gone out of existence. The coefficients of the variations of these coordinates in the expression for δI given by the sum of (49), (51), and the second term of (54) will be the conjugate momenta. Thus the momenta of a charged particle are

$$p_\mu = m dz_\mu/ds + eA_\mu(z), \quad (55)$$

those of a particle with a pole are

$$p_\mu = m dz_\mu/ds, \quad (56)$$

the momenta conjugate to the string variables

$y^\mu(\tau_1)$—let us call them $\beta^\mu(\tau_1)$—are

$$\beta_\mu(\tau_1)=g(F\dagger)_{\mu\nu}{}^* dy^\nu/d\tau_1, \quad (57)$$

and the momentum conjugate to $A_k{}^\mu$ is

$$-4\pi^2 i\gamma_k \bar{A}_{k\mu}k_0{}^{-1}. \quad (58)$$

The string momenta $\beta_\mu(\tau_1)$ form a one-dimensional continuum of variables, corresponding to the one-dimensional continuum of coordinates $y^\mu(\tau_1)$, and the field momenta (58) form a three-dimensional continuum, corresponding to the three dimensional continuum of field coordinates.

We may introduce Poisson brackets in the usual way. For the coordinates and momenta of each particle we have

$$[p_\mu, z_\nu]=g_{\mu\nu}. \quad (59)$$

For the coordinates and momenta of a string we have

$$[\beta_\mu(\tau_1), y_\nu(\tau_1{}')]=g_{\mu\nu}\delta(\tau_1-\tau_1{}'), \quad (60)$$

and for the field variables we have, according to (58),

$$[\bar{A}_{k\mu}, A_{k'\nu}]=i(4\pi^2)^{-1}g_{\mu\nu}\gamma_k{}^{-1}k_0\delta_3(k-k'). \quad (61)$$

The other P.B.'s all vanish.

In the limit when $\gamma(x)\rightarrow\delta_4(x)$, we have $\gamma_k\rightarrow 1$ and Eq. (61) gives the usual P.B. relation for the Fourier amplitudes of the elctromagnetic potentials. If we take $\gamma_k{}^{-1}=\cos(k\lambda)$, where λ is a small four-vector satisfying $\lambda^2>0$, and make $\lambda\rightarrow 0$, we get a limiting procedure which has already been used in electrodynamics, classical and quantum, and which gets over some of the difficulties connected with the infinite fields caused by point particles. This value for γ_k might be suitable in the present theory, but I have not investigated whether it would be compatible with all the requirements of the function $\gamma(x)$.

From Eqs. (55) and (56) we can eliminate the velocities dz_μ/ds and get

$$\{p_\mu-eA_\mu(z)\}\{p^\mu-eA^\mu(z)\}-m^2=0 \quad (62)$$

for each charged particle and

$$p_\mu p^\mu-m^2=0 \quad (63)$$

for each particle with a pole. These equations should be joined with (57) or

$$\beta_\mu(\tau_1)-g(F\dagger)_{\mu\nu}{}^*(y)dy^\nu/d\tau_1=0. \quad (64)$$

With the $A_\mu(z)$ in (62) and the $(F\dagger)_{\mu\nu}{}^*(y)$ in (64) expressed in terms of the Fourier components $A_{k\mu}$, $\bar{A}_{k\mu}$ (the validity of this was discussed near the end of the preceding section), Eqs. (62), (63), and (64) are equations involving only dynamical coordinates and momenta. They are differential equations satisfied by the action integral I, when the momenta are looked upon as derivatives of I, and they are the Hamilton-Jacobi equations of the present theory. Since they are known to have a solution, namely I itself, we can infer from the theory of differential equations that the P.B.'s of their left-hand sides all vanish, as may also be verified directly from (59), (60), and (61).

The supplementary conditions (48) should be brought in at this stage and treated as further Hamilton-Jacobi equations. The various equations (48) obtained by taking different field points x are not independent of the equations of motion or of one another, and we get a complete independent set of equations from them by making a Fourier resolution in the region between S_P and S_F. In this region we may, from (18), replace $J(x-z)$ by $\Delta(x-z)$, whose Fourier components are given by

$$\Delta(x-z)=-i(4\pi^2)^{-1}\sum k_0\int e^{i(k,\,x-z)}k_0{}^{-1}d^3k, \quad (65)$$

so the Fourier resolution of (48) in this region gives, with $k_0>0$,

$$k^\nu\gamma_k A_{k\nu}-(4\pi^2)^{-1}\sum_e ee^{-i(kz)}=0, \quad (66)$$

$$k^\nu\gamma_k\bar{A}_{k\nu}-(4\pi^2)^{-1}\sum_e ee^{i(kz)}=0. \quad (67)$$

These equations involve only dynamical coordinates and momenta, so they are of the right type to form Hamilton-Jacobi equations. One can easily verify that they and the previous Hamilton-Jacobi equations (62), (63), and (64) form a consistent set of differential equations for I, by verifying that the P.B.'s of their left-hand sides all vanish.

VII. QUANTIZATION

From the foregoing Hamiltonian formulation of classical electrodynamics one can pass over to quantum electrodynamics by applying the usual rules. One replaces the dynamical coordinates and momenta of the classical theory by operators satisfying commutation relations corresponding to the P.B. relations (59), (60), and (61), and one replaces the Hamilton-Jacobi equations by the wave equations which one gets by equating to zero the left-hand sides of the Hamilton-Jacobi equations (now involving operators for the dynamical variables) applied to the wave function ψ. The wave equations obtained in this way will be consistent with one another, since the operators on ψ in their left-hand sides commute, as may be inferred from the vanishing of the P.B.'s of the left-hand sides of the Hamilton-Jacobi equations.

This straightforward quantization leads to wave equations of the Klein-Gordon type for all particles, corresponding to their having no spins. For dealing with electrons one should replace these wave equations by the wave equations corresponding to spin $\frac{1}{2}h$. We have no information concerning the spins of the poles, and may assume provisionally that they also have the spin $\frac{1}{2}h$, as this gives the simplest relativistic theory. The change from zero spin to spin $\frac{1}{2}h$ does not affect the mutual consistency of the wave equations.

We now have the following scheme of wave equations, expressed in terms of a set of the usual spin matrices α_1, α_2, α_3, α_m for each particle:

$$\{p_0 - eA_0(z) - \alpha_r[p_r - eA_r(z)] - \alpha_m m\}\psi = 0; \quad (68)$$

for each charged particle,

$$\{p_0 - \alpha_r p_r - \alpha_m m\}\psi = 0; \quad (69)$$

for each particle with a pole,

$$\{\beta_\mu(\tau_1) - g(F\dagger)_{\mu\nu}{}^*(y)dy^\nu/d\tau_1\}\psi = 0; \quad (70)$$

for each string, and

$$\left.\begin{array}{l}\{4\pi^2 k^\nu \gamma_k A_{k\nu} - \sum_e ee^{-i(kz)}\}\psi = 0,\\ \{4\pi^2 k^\nu \gamma_k \bar{A}_{k\nu} - \sum_e ee^{i(kz)}\}\psi = 0,\end{array}\right\} \quad (71)$$

for the field variables. The wave function ψ may be taken to be a function of the particle variables z_μ, suitable spin variables for each particle, the string variables $y_\mu(\tau_1)$ with $0 < \tau_1 < \infty$, and the field variables $A_{k\nu}$. It is defined only when all the points z_μ, $y_\mu(\tau_1)$ lie outside each other's light cones.

Equation (69) suggests at first sight that the electromagnetic field does not act on the poles. However, it acts on the strings, as shown by (70), and since the poles are constrained to be at the ends of the strings, the field does affect the motion of the poles. That it affects them in the right way can be inferred from analogy with the classical theory, in which the poles move according to Eq. (11).

VIII. THE UNIT CHARGE AND POLE

The action integral I of the classical theory may be considered as a function of the points in space-time z_μ where the particles go out of existence, of the lines $y_\mu(\tau_1)$ in space-time where the strings go out of existence, and of suitable field variables, and is defined only provided the strings do not pass through any points z_μ where charged particles go out of existence. It is, however, not a single-valued function of these variables, as may be seen in the following way.

Let us make a continuous change in the variables in I according to the following procedure. We keep all the particle points z_μ fixed, and also all the strings except one. This one we vary continuously, keeping it always in the three-dimensional surface S_P, and loop it around one of the points z_μ where a charged particle is situated just before going out of existence and bring it back to its original position. At the same time the potentials $A_\mu(x)$ are varied continuously, to keep Eqs. (13), (15) always satisfied with fixed values for the field $F_{\mu\nu}(x)$, and are brought back to their original values together with the string. We have here a continuous deformation of the variables in I which brings them all back to their starting values, and this deformation cannot be continuously shrunk up to no deformation at all, because we cannot make a string pass through a charged particle. The string will sweep out a closed two-dimensional surface, σ say, lying in S_P and enclosing the point z_μ where the charge is situated, and this surface σ cannot be continuously shrunk up to zero, since it must not pass through the charge. We may therefore expect I to vary under this deformation process, and can easily calculate its variation DI as follows.

A small variation of a string and of the potentials, with the particle points z_μ fixed, leads to a variation of I given by the sum of the right-hand sides of (50) and (52). Under the closed deformation process described above, the first term on the right-hand side of (50) will give zero, since the $F_{\mu\nu}$, $F_{\mu\nu}{}^*$ are kept fixed and the A^μ, $A^{\mu *}$ are brought back to their original values. The right-hand side of (52) will also give zero, since it gives the total variation in I_4 and I_4 is brought back to its original value. We are left with the second term on the right-hand side of (50), which is equal to expression (51) and gives for the closed deformation process

$$DI = g\int (F\dagger)_{\mu\nu}{}^* d\sigma^{\mu\nu},$$

where $d\sigma^{\mu\nu}$ is an element of the two-dimensional surface swept out by the string. The integral here is, according to (8), just the total electric flux passing out through the closed surface σ, and is thus 4π times the charge e enclosed by the surface. Thus

$$DI = 4\pi g e.$$

We may loop any string around any charge any number of times, so the total uncertainty in I is the sum

$$4\pi \sum_{ge} m_{ge} g e, \tag{72}$$

summed for all the charges e and the poles g, with an arbitrary integral coefficient m_{ge} for each term.

The phenomenon of an action integral which is not single-valued occurs frequently in mechanics. It occurs most simply with the dynamical system consisting of a rigid body rotating about a fixed axis, for which the action integral is just the angular momentum multiplied by the azimuthal angle, so that the uncertainty in the action integral is 2π times the angular momentum. The rule of quantization of Bohr's theory is given by putting the uncertainty in the action integral equal to an integral multiple of h. Applying this rule to the uncertainty (72), we get

$$4\pi g e = nh, \tag{73}$$

where n is an integer, for each pole g and charge e. This result is the same as (1), with c, the velocity of light, put equal to unity.

The result (73) may also be obtained from the quantum electrodynamics of Section VII without Bohr's rule of quantization, by using the condition that the wave function must be single-valued. The commutation relation (60) shows that $\beta_\mu(\tau_1)$ is $i\hbar$ times the operator of functional differentiation with respect to $y_\mu(\tau_1)$, so that the wave equation (70) is

$$i\hbar\partial\psi/\partial y_\mu(\tau_1) = g(F\dagger)_{\mu\nu}{}^*(y)(dy^\nu/d\tau_1)\psi. \tag{74}$$

This equation shows how ψ varies when the position of a string is varied. If a string is displaced and sweeps out a two-dimensional surface σ, Eq. (74) shows that ψ gets multiplied by

$$\exp\left[-ig\int (F\dagger)_{\mu\nu}{}^* d\sigma^{\mu\nu}/\hbar\right], \tag{75}$$

provided the $(F\dagger)_{\mu\nu}{}^*$ occurring at different points of the integrand here all commute. (One can easily arrange to satisfy this condition accurately, in the case when σ lies in a flat three-dimensional space-like surface S_P, by a suitable choice of the function γ, and in the case of a general S_P the lack of commutation tends to zero as $\gamma(x)\to\delta_4(x)$ and does not invalidate the calculation.) Let us now apply the procedure we had before of looping the string around one of the charges and bringing it back to its original position. Since ψ is single-valued it must return to its original value, and so the factor (75) must be unity. This requires

$$g\int (F\dagger)_{\mu\nu}{}^* d\sigma^{\mu\nu}/\hbar = 2\pi n$$

with n an integer, which gives again the condition (73).

We come to the important conclusion that *the quantization of the equations of motion of charged particles and particles with poles is possible only provided the charges and poles are integral multiples of a unit charge e_0 and a unit pole g_0 satisfying*

$$e_0 g_0 = \tfrac{1}{2}\hbar c. \tag{76}$$

The theory does not fix the value of e_0 or g_0, but only gives their product.

IX. DISCUSSION

The foregoing work provides a general theory of particles with electric charges and magnetic poles in interaction with the electromagnetic field. It is not a perfect theory, because the interaction of a particle with its own field is not dealt with satisfactorily. This is shown up by the continual use in the theory of a function $\gamma(x)$ which has not been precisely specified—only certain desirable properties for it having been given. Even if a satisfactory function γ can be given, the difficulties will not all be solved because there will still be infinities appearing in the wave function when one tries to solve the wave equation. However, these difficulties occur in the ordinary electrodynamics of electrons without any poles, and if a solution of them can be found for ordinary electrodynamics, it will probably apply also for the more general electrodynamics with poles. Thus the occurrence of these difficulties does not provide an argument against the existence of magnetic poles.

The question arises as to whether an elementary particle can have both a charge and a pole. The classical equations of motion given in Section II can be immediately extended to this case, but the Hamiltonian theory meets with some difficulties connected with the precise form of γ. It does not seem possible to answer the question reliably until a satisfactory treatment of the interaction of a particle with its own field is obtained.

The theory developed in the present paper is essentially symmetrical between electric charges and magnetic poles. There is a considerable apparent difference between the treatment of charges and poles, which shows itself up in the first place through the introduction of potentials according to (13). However, one could work equally well with the rôles of the charges and poles interchanged. One would then have strings attached to the charges, and would work with potentials B_μ defined by

$$(F\dagger)_{\mu\nu}=\partial B_\nu/\partial x^\mu-\partial B_\mu/\partial x^\nu+4\pi\sum_e (G\dagger)_{\mu\nu},$$

with the $G_{\mu\nu}$ vanishing except on the sheets traced out by the new strings, instead of (13). The final result would be an equivalent quantum electrodynamics, referred to a different representation.

Although there is symmetry between charges and poles from the point of view of general theory, there is a difference in practice on account of the different numerical values for the quantum of charge and the quantum of pole. If we take the experimental value for the fine-structure constant,

$$e_0{}^2=(1/137)\hbar c,$$

we can infer the value of g_0,

$$g_0{}^2=(137/4)\hbar c.$$

Thus g_0 is much larger than e_0. It corresponds to a fine-structure constant 137/4. The forces of radiation damping must be very important for the motion of poles with an appreciable acceleration.

The great difference between the numerical values of e_0 and g_0 explains why electric charges are easily produced and not magnetic poles. Two one-quantum poles of opposite sign attract one another with a force $(137/2)^2$ times as great as that between two one-quantum charges at the same distance. It must therefore be very difficult to separate poles of opposite sign. To get an estimate of the energy needed for this purpose, we might suppose that elementary particles with poles form an important constituent of protons and have a mass μ of the order, say, half the proton mass. The binding energy of two of these particles cannot be calculated accurately without a more reliable theory of radiation damping than exists at present, but one might expect from Sommerfeld's formula for the energy levels of hydrogen with relativistic effects that this binding energy would be of the order of μc^2, or say 5×10^8 electron volts. One should look for particles with poles in atomic processes where energies of this order are available. They would appear as heavily ionizing particles and would be distinguishable from ordinary charged particles by the property that the ionization they produce would not increase towards the end of their range, but would remain roughly constant.