PHYSICAL REVIEW VOLUME 74, NUMBER 7 OCTOBER 1, 1948

Elastic Scattering of High Energy Nucleons by Deuterons

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(Received June 21, 1948)

The elastic scattering cross section of deuterons for neutrons and protons of 90 and 350 Mev has been calculated, using the non-relativistic Born approximation. Two different approaches give roughly the same result: a total neutron-deuteron elastic cross section of about half the neutron-proton cross section at the same energy but sharply confined to small angles. The half-width at 90 Mev is about 15° in the laboratory system. The proton-deuteron elastic scattering is similar to the neutron-deuteron for large angular deflections, but at 90 Mev a difference due to the Coulomb repulsion should be noticeable within 10°. It is concluded that the assumption of a simple additivity of nucleon-nucleon cross sections for light elements is unjustified.

I. INTRODUCTION

THE scattering by deuterons of protons and neutrons of energies less than 14 Mev has been studied extensively, both experimentally and theoretically, the most complete theoretical treatment having been given by Buckingham and Massey.[1] Recently a total cross section at 90 Mev has been obtained with the Berkeley cyclotron,[2] and it should be possible in the near future to determine the angular distribution as well as the division between elastic and inelastic processes. Thus it is desirable to see what can be said theoretically about these high energy collisions.

The high energy problem differs from the low in three important features: (a) Collisions in which the deuteron disintegrates are quite probable. (b) Because the De Broglie wavelength of the incident particle is short compared to the range of the nuclear force, high angular momenta contribute strongly. Thus the computation of individual phase shifts by straightforward integration of the Schrödinger equation is impracticable. (c) The relative energy in the collision is sufficiently great that it is now reasonable to use the Born approximation. The latter will not give completely satisfactory results, as has been shown by experience with neutron-proton scattering,[3] but the order of magnitude should be correct.

The simplest theoretical guess that can be made is that for an incident particle of energy large compared to the deuteron binding, one may neglect the latter and treat the scattering as the sum of that due to a proton and an independent neutron. This idea implies negligible elastic scattering. It will be shown in this paper that the elastic scattering is never negligible, even at the highest energies, and that therefore the assumption of additivity is unjustified.

The inelastic scattering is much less easily treated since it involves the continuum wave functions of the deuteron. This problem is being attacked, however, and it is hoped that some theoretical results will be forthcoming soon.

Since the completion of the major portion of this work a paper by Wu and Ashkin[4] on the same subject has appeared. In spite of the fact that in both papers the Born approximation is used, there are important differences in the choice of the nuclear forces and of the deuteron wave function. In particular the results presented in this paper are largely independent of the detailed mechanism of the forces and can be deduced at least in part from direct data of scattering of nucleons on nucleons. Finally, the choice by Wu and Ashkin of gaussian instead of exponential wave functions seems to have affected their results to a considerable extent.

II. A SIMPLIFIED APPROACH TO THE PROBLEM

A. Theory

It is desirable, in considering the nucleon-deuteron problem, to separate as much as possible those features which have to do with the

* National Research Council Predoctoral Fellow.
[1] R. A. Buckingham and H. S. W. Massey, Proc. Roy. Soc. **A179**, 123 (1942).
[2] Cook, McMillen, Peterson, and Sewell, Phys. Rev. **72**, 1264 (1947).
[3] M. Camac and H. Bethe, Phys. Rev. **73**, 191 (1948).
[4] Ta-You Wu and J. Ashkin, Phys. Rev. **73**, 986 (1948).

nature of the nuclear forces from those which reflect the presence of two of the three particles in a bound state. This is clearly not possible in an exact sense, but an understanding of the importance of the binding can only be obtained by first making such a separation. The results of a straightforward but more complicated calculation later will then be more easily interpretable.

Let us therefore think of the elastic scattering probability as the product of two separate factors: (a) The chance that the incident nucleon may suffer a collision with either of the two deuteron particles, suddenly transferring a momentum $\mathbf{\Delta p}$ to that particle. (b) The probability of finding the two deuteron particles still in their ground state after this dislocation. The first factor can be estimated from the known neutron-proton scattering, with a guess as to the corresponding neutron-neutron or proton-proton cross section. The second factor might be called the "sticking" factor, $S(\Delta p)$, and will be our principal concern in this section. If the elastic cross section is to be negligible in comparison to the total, it will be due to the smallness of $S(\Delta p)$.

To put this idea in a more definite form, the following hypothetical situation is considered: A deuteron is bombarded by a third particle of the same mass as a neutron or proton but inequivalent to either and with no spin. The third particle has an ordinary interaction with the proton, $V_p(r)$, and with the neutron, $V_n(r)$. All complications due to symmetry, spin, exchange forces, etc., are deliberately ignored. Let us also assume that the Born approximation is applicable and calculate the elastic scattering cross section.

Designate the coordinate of the neutron by $\mathbf{r}_n$, that of the proton by $\mathbf{r}_p$, and that of the incident particle by $\mathbf{r}_i$. More convenient variables are obtained by the transformation:

$$R=\tfrac{1}{3}(\mathbf{r}_n+\mathbf{r}_p+\mathbf{r}_i),\quad \mathbf{r}=\mathbf{r}_n-\mathbf{r}_p,\quad \mathbf{x}=\mathbf{r}_i-\tfrac{1}{2}(\mathbf{r}_n+\mathbf{r}_p).$$

The Hamiltonian of the problem is

$$\begin{aligned}H&=\frac{p_n^2}{2m}+\frac{p_p^2}{2m}+\frac{p_i^2}{2m}+V_{np}(|\mathbf{r}_n-\mathbf{r}_p|)\\&\quad+V_n(|\mathbf{r}_n-\mathbf{r}_i|)+V_p(|\mathbf{r}_p-\mathbf{r}_i|)\\&=\frac{P^2}{2(3m)}+\frac{p_r^2}{2(m/2)}+\frac{p_x^2}{2(2/3)(m)}\\&\quad+V_{np}(r)+V_n(|\mathbf{x}-\mathbf{r}/2|)+V_p(|\mathbf{x}+\mathbf{r}/2|),\end{aligned}$$

where m is the mass of the proton or neutron and $(\mathbf{p}_n, \mathbf{r}_n)$, $(\mathbf{p}_p, \mathbf{r}_p)$, $(\mathbf{p}_i, \mathbf{x}_i)$, $(\mathbf{P}, \mathbf{R})$, $(\mathbf{p}_r, \mathbf{r})$, $(\mathbf{p}_x, \mathbf{x})$, are canonically conjugate pairs of variables. $V_{np}(r)$ is the interaction between neutron and proton.

The Hamiltonian is split into two parts,

$$H=H_0+\mathcal{H},$$

where $\mathcal{H}=V_n(|\mathbf{x}-\mathbf{r}/2|)+V_p(|\mathbf{x}+\mathbf{r}/2|)$ and H_0 is the remainder, and use is made of the time dependent perturbation theory to calculate the transition probability between eigenstates of the operator H_0, due to the perturbation $\mathcal{H}$. Since the eigenstates form a continuum, we may apply the well-known formula:

$$\lambda=2\pi/\hbar|\mathcal{H}_{fo}|^2\rho_E,$$

where λ is the probability per unit time of a transition from the state o into the continuum of states near f, $\mathcal{H}_{fo}$ is the matrix element of $\mathcal{H}$ between states o and f, and ρ_E is the number of final states per unit energy interval. Energy must of course be conserved.

The initial state, in the center-of-gravity system, can be written as the product of a plane wave, and the ground state of the deuteron, $\psi_0(r)$.

$$\Psi_0(\mathbf{x},\mathbf{r})=\Omega^{-\frac{1}{2}}\exp(i\mathbf{p}_0\cdot\mathbf{x})\psi_0(r).$$

This represents the deuteron and the third particle approaching each other with relative velocity v, total momentum zero, in a box of volume Ω. The relative momentum and velocity are related by $\hbar\mathbf{p}_0=\frac{2}{3}m\mathbf{v}$.

The final state is a similar expression with $\mathbf{p}_0$ replaced by $\mathbf{p}_f$. Energy conservation requires that $|\mathbf{p}_f|=|\mathbf{p}_0|$. The required matrix element is

$$\begin{aligned}\mathcal{H}_{fo}&=\int\int d\mathbf{x}d\mathbf{r}\Psi_f^*(\mathbf{x},\mathbf{r})\mathcal{H}\Psi_o(\mathbf{x},\mathbf{r})\\&=\Omega^{-1}\int\int d\mathbf{x}d\mathbf{r}\exp(i(\mathbf{p}_f-\mathbf{p}_0)\cdot\mathbf{x})\\&\quad\times[V_n(|\mathbf{x}-\mathbf{r}/2|)+V_p(|\mathbf{x}+\mathbf{r}/2)]\psi_0^*(r)\psi_0(r).\end{aligned}$$

Consider first the term involving V_n and make the change of variables $\mathbf{y}=\mathbf{x}-\mathbf{r}/2$, $\mathbf{r}=\mathbf{r}$. It may then be written, with the abbreviation,

$$\mathbf{p}_f-\mathbf{p}_0=\mathbf{\Delta p},$$

$$\Omega^{-1}\int\int d\mathbf{y}d\mathbf{r}\exp(i\mathbf{\Delta p}\cdot(\mathbf{y}+\mathbf{r}/2))\psi_0^2(r)V_n(y),$$

or

$$\Omega^{-1}\left\{\int d\mathbf{y}\exp(i\boldsymbol{\Delta}\mathbf{p}\cdot\mathbf{y})V_n(y)\right\}$$
$$\times\left\{\int d\mathbf{r}\exp(i(\boldsymbol{\Delta}\mathbf{p}/2)\cdot\mathbf{r})\psi_0^2(r)\right\}.$$

Similarly, the part containing V_p may be written

$$\Omega^{-1}\left\{\int d\mathbf{y}\exp(i\boldsymbol{\Delta}\mathbf{p}\cdot\mathbf{y})V_p(y)\right\}$$
$$\times\left\{\int d\mathbf{r}\exp(i(\boldsymbol{\Delta}\mathbf{p}/2)\cdot\mathbf{r})\psi_0^2(r)\right\}.$$

In anticipation of a later result the following quantities are defined:

$$S^{\frac{1}{2}}(\Delta p)=\int d\mathbf{r}\exp(i(\boldsymbol{\Delta}\mathbf{p}/2)\cdot\mathbf{r})\psi_0^2(r), \tag{1}$$

$$\mathcal{V}_n(\boldsymbol{\Delta}\mathbf{p})=\int d\mathbf{y}\exp(i\boldsymbol{\Delta}\mathbf{p}\cdot\mathbf{y})V_n(y),$$
$$\mathcal{V}_p(\boldsymbol{\Delta}\mathbf{p})=\int d\mathbf{y}\exp(i\boldsymbol{\Delta}\mathbf{p}\cdot\mathbf{y})V_p(y), \tag{2}$$

and the two terms are collected together again.

$$\mathcal{H}_{fo}=\Omega^{-1}\{\mathcal{V}_n(\boldsymbol{\Delta}\mathbf{p})+\mathcal{V}_p(\boldsymbol{\Delta}\mathbf{p})\}S^{\frac{1}{2}}(\Delta p).$$

The cross section, $\sigma_{el}d\omega$, for scattering in the direction $\mathbf{p}_f$ per unit solid angle, is related to the corresponding transition probability by

$$\lambda=\Omega^{-1}v\sigma_{el}d\omega.$$

The number of final states per unit energy is

$$\rho_E=(1/8\pi^3)\Omega p_f^2(dp_f/dE_f)d\omega,$$

and since $dE_f/dp_f=hv$,

$$\rho_E=(1/8\pi^3hv)\Omega p_f^2d\omega.$$

The formula for the cross section thus becomes

$$\sigma_{el}d\omega=\lambda\Omega/v=(2\pi/\hbar)|\mathcal{H}_{fo}|^2(\Omega\rho_E/v)$$
$$=\frac{(2m/3)^2}{4\pi^2\hbar^4}|\mathcal{V}_n(\boldsymbol{\Delta}\mathbf{p})+\mathcal{V}_p(\boldsymbol{\Delta}\mathbf{p})|^2S(\Delta p)d\omega, \tag{3}$$

since $p_f^2/v^2=(2m/3)^2\hbar^{-2}$.

This result contains the essential feature we are looking for. The factor $S(\Delta p)$ is a function only of the deuteron configuration. It approaches zero for very weak binding and unity for very strong binding. It is always unity for zero momentum transfer. The rest of the expression, on the other hand, has nothing to do with the deuteron, depending only on the character of the forces between the projectile and the individual proton and neutron. It may be looked upon as the collision probability suggested at the beginning of this section, while $S(\Delta p)$ is the "sticking" probability.

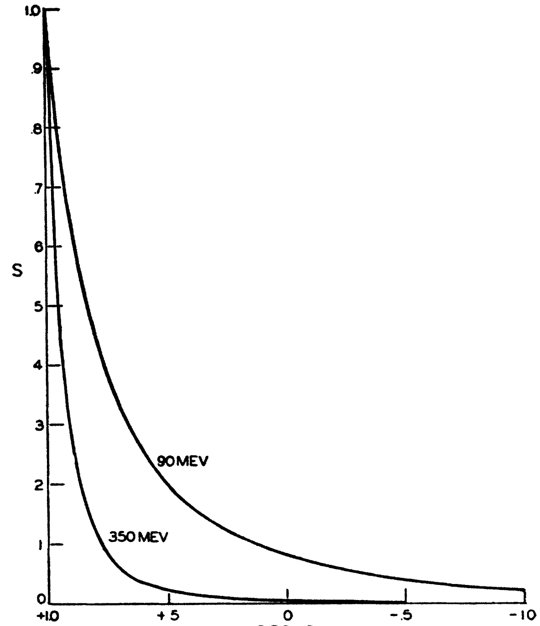

FIG. 1. The "sticking factor" S, as a function of the cosine of the scattering angle in the center-of-gravity system.

From the definition (1) of $S(\Delta p)$ one observes that it decreases monotonically with increasing magnitudes of momentum transfer. In terms of angular distribution at a given energy, this means that forward scattering is favored. In Fig. 1, S is plotted as a function of the cosine of the scattering angle in the center-of-gravity system for incident laboratory energies of 90 and 350 Mev. The wave function used in the computation (see Section III B) corresponds to a central Yukawa potential, but clearly any reasonable choice of wave function will give about the same result.

B. Estimate of Cross Section

The average value of the sticking factor over all solid angles is quite small, 0.17 at 90 Mev.

and 0.05 at 350 Mev. However, the effective average is always greater, since the simple nucleon-nucleon scattering also favors small momentum transfers. This fact, in combination with the constructive interference between neutron and proton scatterings, evident in Eq. (3), and the reduced mass effect, which contributes a factor 16/9, may give an elastic scattering cross section of the same order of magnitude as the simple neutron-proton scattering at the same energy. The ratio of elastic deuteron scattering to simple nucleon-nucleon scattering should not diminish appreciably, even at the highest energies, because momentum transfers much larger than $\hbar/a$, where a is the range of the nuclear force, will always be improbable. We now attempt to make a quantitative estimate of this ratio at 90 Mev.

We propose to use the formula (3), and insert for $\mathcal{V}_n(\mathbf{\Delta p})$ and $\mathcal{V}_p(\mathbf{\Delta p})$ matrix elements which in a corresponding simplified theory of the elementary nucleon-nucleon scattering would give the actually observed cross sections. In this way we avoid any reference to the nuclear force involved, which only confuses the issue here. In Section III a complete calculation will reveal the mistakes which this type of estimate can make.

For the sake of argument, let us assume the incident particle is a neutron. A calculation of the cross section for its scattering on a free proton, with the same simplifications as above,

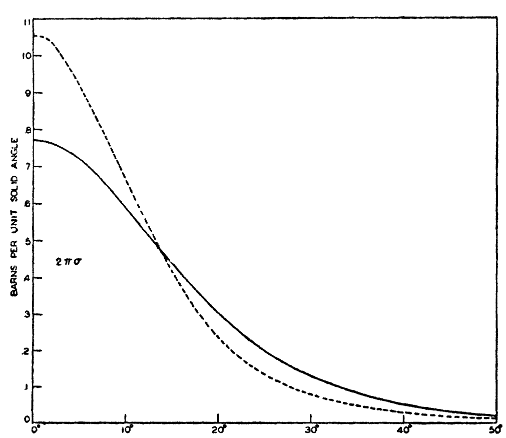

FIG. 2. Differential elastic scattering cross section for neutrons on deuterons, referred to the laboratory system of coordinates. The solid curve represents the results of Section II and the broken curve that of Section III.

yields a formula:

$$\sigma_{np}d\omega'=\frac{(m/2)^2}{4\pi^2\hbar^4}|\mathcal{V}_p(\mathbf{\Delta p}')|^2d\omega',$$

where ω', $\mathbf{\Delta p}'$ refer to the center-of-gravity system of the two-particle problem and $\mathcal{V}_p(\mathbf{\Delta p}')$ has again the definition (2). At 90 Mev, the scattering has been observed to be roughly symmetric about 90° and to have a total value of 0.083 B.[2,5] We are clearly interested in only the forward part of the scattering here, which can be empirically fitted by choosing

$$\mathcal{V}_p(\mathbf{\Delta p})=\frac{2\pi\mathcal{V}_0}{q}\left[\frac{1}{q^2+(\Delta p)^2}\right], \tag{4}$$

where $\mathcal{V}_0=226$ Mev and $1/q=0.578\times10^{-13}$ cm.

The corresponding neutron-neutron or proton-proton matrix element cannot be obtained so directly since there is as yet no experimental information at high energies. If we assume that the force is the same, one might guess at an expression of the form of (4) with an additional factor of about one-half in front. The argument for this is as follows: The experimentally observed symmetry implies that only even angular momenta are scattered or only the singlet spin states of identical particles. This cuts down the cross section by a factor of one-quarter due to the statistics and farther because the singlet potential is weaker than the triplet. A survey of recent calculations[3,6] would indicate the latter factor to be about one-half. Finally, we must remember the factor of two which generally appears for identical particles with spatial symmetry. The net result is a factor $\frac{1}{4}\times\frac{1}{2}\times2=\frac{1}{4}$ in the cross section or one-half in the matrix element. The effect of the Coulomb force which is present when the projectile is a proton will be considered in Section III C.

Substitution of these estimates into formula (3) leads to

$$\sigma_{el}d\omega=\frac{m^2\mathcal{V}_0^2}{\hbar^4q^2}\left[\frac{1}{q^2+(\Delta p)^2}\right]S(\Delta p)d\omega, \tag{5}$$

a result which is to be taken seriously only for

[5] J. Hadley, Kelly, Leith, Segrè, Wiegand, and York, Phys. Rev. **73**, 1114 (1948).

[6] J. Ashkin and Ta-You Wu, Phys. Rev. **73**, 973 (1948).

rather small angles. These small angles, of course, will include most of the total cross section.

This formula yields a total cross section of 0.045 B at 90 Mev, about half the n–p cross section, and the angular distribution shown by the solid curve in Fig. 2. It may be in error by as much as a factor of two, but the state of knowledge about nuclear forces being what it is, is probably as good an estimate as can be made at this time. The neglect of possible exchange forces is not so bad as it seems, since they tend to break up the deuteron. An interesting effect not taken account of in this crude picture is an exchange of the two identical particles, leaving the deuteron intact. This will appear in the formally more correct calculation of Section III.

III. THE COMPLETE CALCULATION

A. Choice of Nuclear Forces

In this section a calculation of the elastic scattering is carried out which includes considerations of symmetry and spin but which of necessity must be based on some assumption as to the nature of the forces. It is clear from Section II that a force which gives an erroneous angular distribution for simple nucleon-nucleon scattering is likely to produce an even more misleading result for nucleon-deuteron scattering. Purely ordinary forces with a reasonable range, for example, give a strong forward maximum at high energies. When this is multiplied by the "sticking" factor, the result is an elastic cross section at 90 Mev of two to three times the observed *total* cross section. This violent dependence on the nature of the force has already been shown by Wu and Ashkin.[4] There seems no point therefore in considering a type of force which does not fairly well represent the actual high energy neutron-proton angular distribution.

The simplest acceptable force has been studied extensively by Serber's group at Berkeley.[7] It is of the type $\frac{1}{2}(1+P)V(r)$, where P is the space exchange operator and $V(r)$ is a central potential. Actually, the Berkeley study has included tensor forces, but the deuteron problem becomes unmanageable from a practical point of view if a tensor force is considered. We therefore shall take for the potential, between like as well as unlike nucleons,

$$V^{S,T}(r)=\tfrac{1}{2}(1+P)V_0^{S,T}\frac{\exp(-\mu r)}{\mu r}, \qquad (6)$$

where $V_0^T=-67.8$ Mev and $V_0^S=-46.5$ Mev, corresponding to triplet and singlet states, respectively, and $\mu^{-1}=1.18\times10^{-13}$ cm. The range and singlet depth are adjusted to fit the low energy proton-proton scattering,[8] and the triplet depth is fixed by the binding energy of the deuteron.[9] It has been shown at Berkeley that such a potential represents satisfactorily both the magnitude and the angular distribution of the 90-Mev neutron-proton scattering.

Since the Born approximation is to be used, it is advisable to consider how good the latter is when applied to the neutron-proton problem. It turns out to give a total cross section within 10 percent of the exact value at 90 Mev when used with the above potential. The angular distribution has the correct symmetry but too deep a minimum at 90° in the center-of-gravity system. The exact calculation gives a factor of 3 between the intensities at 180° and at 90°, while the Born approximation gives a factor 5.7. It is therefore expected that the deuteron elastic cross section, calculated in this way, may be too large by as much as 30 percent. At 350 Mev, the Born approximation can probably be used without reservation, but relativistic corrections will be serious.

B. Neutron-Deuteron Cross Section

The neutron case is considered first in order to avoid the Coulomb force complication. The calculation uses the same time-dependent perturbation theory as in Section II, generalized for the presence of two identical particles. Let the space and spin coordinates of the two neutrons be $(\mathbf{r}_1, \sigma_1)$, $(\mathbf{r}_3, \sigma_3)$ and that of the proton $(\mathbf{r}_2, \sigma_2)$. The symbol ξ_i denotes both space and spin variables of the ith particle. The Hamiltonian is

$$H=\frac{p_1^2}{2m}+\frac{p_2^2}{2m}+\frac{p_3^2}{2m}+V(\xi_1,\xi_2)+V(\xi_1,\xi_3)+V(\xi_2,\xi_3),$$

[7] R. Serber, private communication.

[8] L. E. Hoisington, S. S. Share, and G. Breit, Phys. Rev. **56**, 884 (1939).

[9] R. G. Sachs and M. Goeppert-Mayer, Phys. Rev. **53**, 991 (1938).

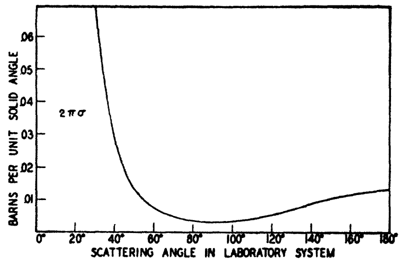

FIG. 3. Large-angle elastic scattering of 90-Mev nucleons by deuterons in the laboratory system.

where $\mathbf{p}_i$ is the momentum of particle i and $V(\xi_i, \xi_j)$ is the potential acting between particles i and j. We choose for our complete set of eigenfunctions, antisymmetric in particles 1 and 3,

$$\Psi_n(\xi_1, \xi_2, \xi_3) = 1/\sqrt{2}(1-P_{13}{}^T)\Phi_n(\xi_1, \xi_2, \xi_3),$$

where the $\Phi_n(\xi_1, \xi_2, \xi_3)$ belong to the operator $p_1{}^2/2m+p_2{}^2/2m+p_3{}^2/2m+V(\xi_1, \xi_2)$ and where $P_{13}{}^T$ interchanges all coordinates of the particles 1 and 3. The cross-section formula, as in Section II, may be written

$$\sigma_{el}d\omega = \frac{(\frac{2}{3}m)^2\Omega^2}{4\pi^2\hbar^4}|\mathcal{H}_{fo}|^2 d\omega,$$

where now $\mathcal{H}_{fo}$ must be understood to mean

$$\mathcal{H}_{fo} = \sum_{\xi_1, \xi_2, \xi_3} [1/\sqrt{2}(1-P_{13}{}^T)\Phi_f{}^*(\xi_1, \xi_2, \xi_3)] \times[1/\sqrt{2}(1-P_{13}{}^T) \times(V(\xi_1, \xi_3)+V(\xi_2, \xi_3))\Phi_o(\xi_1, \xi_2, \xi_3)]. \quad (7)$$

This result can be obtained in a manner exactly analogous to that used to get the usual transition probability formula.[10] The sum over ξ_1, ξ_2, ξ_3 is meant to include integration over the continuous variables.

After making the usual change of variables, $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$, $\mathbf{R}=\frac{1}{3}(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3)$, $\mathbf{x}=\mathbf{r}_3-\frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)$, we obtain the initial state in the center-of-gravity system by choosing

$$\Phi_0 = \Omega^{-\frac{1}{2}}\exp(i\mathbf{p}_0\cdot\mathbf{x})\psi_0(r)\chi_{1\cdots 6}(\sigma_1, \sigma_2, \sigma_3),$$

and the final state by choosing a similar function with $\mathbf{p}_0$ replaced by $\mathbf{p}_f$. All symbols in these expressions have the same meaning as in Section II.

[10] See, for example, W. Heitler, *Quantum Theory of Radiation* (Oxford University Press, London, 1944), second edition, p. 88.

The spin functions $\chi_{1\cdots 6}(\sigma_1, \sigma_2, \sigma_3)$ are taken as follows:

$$\left.\begin{aligned}\chi_1&=a_1a_2a_3,\\ \chi_2&=3^{-\frac{1}{2}}(a_1a_2b_3+a_1b_2a_3+b_1a_2a_3),\\ \chi_3&=3^{-\frac{1}{2}}(b_1b_2a_3+a_1b_2b_3+b_1a_2b_3),\\ \chi_4&=b_1b_2b_3,\end{aligned}\right\}\text{quartet states.}$$

$$\left.\begin{aligned}\chi_5&=6^{-\frac{1}{2}}(2a_1a_2b_3-a_1b_2a_3-b_1a_2a_3),\\ \chi_6&=6^{-\frac{1}{2}}(2b_1b_2a_3-a_1b_2b_3-b_1a_2b_3),\end{aligned}\right\}\text{doublet states.}$$

a_i represents z component of spin $+\frac{1}{2}$ for the ith particle and b_i represents z component $-\frac{1}{2}$.

The force chosen does not mix spin and angular momentum, so the final spin state must be the same as the initial. It is also obvious that the matrix element does not depend on the z component of the total spin. Therefore, there are only two different matrix elements to consider, one for the quartet states and one for the doublet. If expression (7) is written out in full and the spin products performed, we find for the quartet states:

$$\mathcal{H}_{fo}{}^Q = \frac{1}{2\Omega}\{I_1+I_2-2I_3\}, \quad (8)$$

and for the doublet:

$$\mathcal{H}_{fo}{}^D = \frac{1}{2\Omega}\{\tfrac{1}{4}(1+9\eta)(I_1+I_2)+I_3\}, \quad (9)$$

where

$$I_1 = \left\{\int d\mathbf{y}\exp(-i\Delta\mathbf{p}\cdot\mathbf{y})V_0{}^T\frac{\exp(-\mu y)}{\mu y}\right\} \times\left\{\int d\mathbf{r}\psi_0{}^*(r)\psi_0(r)\exp(i\Delta\mathbf{p}/2\cdot\mathbf{r})\right\},$$

$$I_2 = \int d\mathbf{y}\exp(-i(\mathbf{p}_f+\mathbf{p}_0/2)\cdot\mathbf{y})V_0{}^T\frac{\exp(-\mu y)}{\mu y} \times\int d\mathbf{r}\psi_0{}^*(r)\psi_0(|\mathbf{y}+\mathbf{r}|)\exp(-i\Delta\mathbf{p}/2\cdot\mathbf{r}),$$

$$I_3 = \left\{\int d\mathbf{y}\exp(-i(\mathbf{p}_f+\mathbf{p}_0/2)\cdot\mathbf{y})\psi_0(y)\right\} \times\left\{\int d\mathbf{r}\exp(i(\mathbf{p}_f/2+\mathbf{p}_0)\cdot\mathbf{r})\psi_0{}^*(r)V_0{}^T\frac{\exp(-\mu r)}{\mu r}\right\}$$

and $\eta = V_0{}^S/V_0{}^T$.

In the evaluation of the integrals I_1, I_2, I_3, the bound state of the deuteron $\psi_0(r)$, corresponding to the Yukawa potential, $V_0{}^T(\exp(-\mu r)/\mu r)$, must be used. This function can be approximated with great accuracy by the difference of two exponentials, divided by r.

$$\psi_0(r) \cong A \frac{\exp(-\alpha r) - \exp(-\beta r)}{r},$$

where

$$A^2 = \frac{\alpha\beta(\alpha+\beta)}{2\pi(\alpha-\beta)^2}, \quad \alpha = \frac{(mB)^{\frac{1}{2}}}{\hbar}, \quad \beta = 5.476\alpha.$$

B is the binding energy of the deuteron, 2.18 Mev. This function has a maximum deviation from the exact function as obtained by numerical integration of 3 percent. Outside the range of the force, the two practically coincide. As a variational trial function it gives a fraction 0.999 of the binding energy.

It should be noted that I_1 is precisely the type of expression obtained for the matrix element in the simplified calculation. Of the three terms it is the most important since both of its factors have a strong maximum in the forward direction. I_3 has a weaker maximum to the back, and I_2 has a weak maximum to the back and a stronger one in front. These two terms represent the complications due to exchange forces and to the identity of the two neutrons.

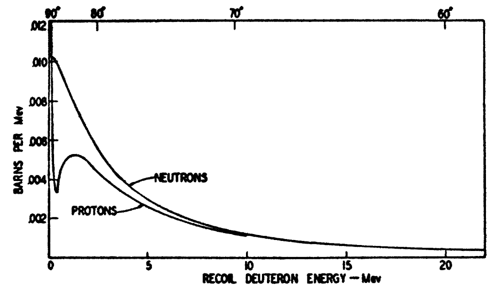

FIG. 5. Spectrum of recoil deuterons for incident nucleons of 90 Mev. The recoil angles in the laboratory system are indicated at the top of the figure.

With the above approximate wave function, I_1 can be completely evaluated analytically.

$$I_1 = \frac{4\pi V_0^T}{\mu} \frac{1}{\mu^2 + (\Delta p)^2} S^{\frac{1}{2}}(\Delta p),$$

where

$$S^{\frac{1}{2}}(\Delta p) = \frac{8\pi A^2}{\Delta p} \{ \tan^{-1}\Delta p/4\alpha + \tan^{-1}\Delta p/4\beta - 2\tan^{-1}\Delta p/2(\alpha+\beta) \}.$$

This is the square root of the sticking factor plotted in Fig. 1.

It is also possible to write I_3 as a fairly simple function

$$I_3 = -\frac{16\pi^2\hbar^2 A^2}{m} \frac{(\beta^2-\alpha^2)^2}{(\alpha^2+Q^2)(\beta^2+Q^2)^2},$$

where

$$Q = |\mathbf{p}_1 + \mathbf{p}_0/2|,$$

but I_2, in which the variables of integration do not separate, can only be partially reduced. One is left finally with a single very complicated integral which can either be done numerically or, for a few special angles, by contour integration.

Having evaluated the quartet and doublet matrix elements, the total cross section is obtained by the weighted average:

$$\sigma_{el} d\omega = \frac{(\frac{2}{3}m)^2\Omega^2}{4\pi^2\hbar^4} [\tfrac{2}{3}|\mathcal{H}_{fo}{}^Q|^2 + \tfrac{1}{3}|\mathcal{H}_{fo}{}^D|^2] d\omega.$$

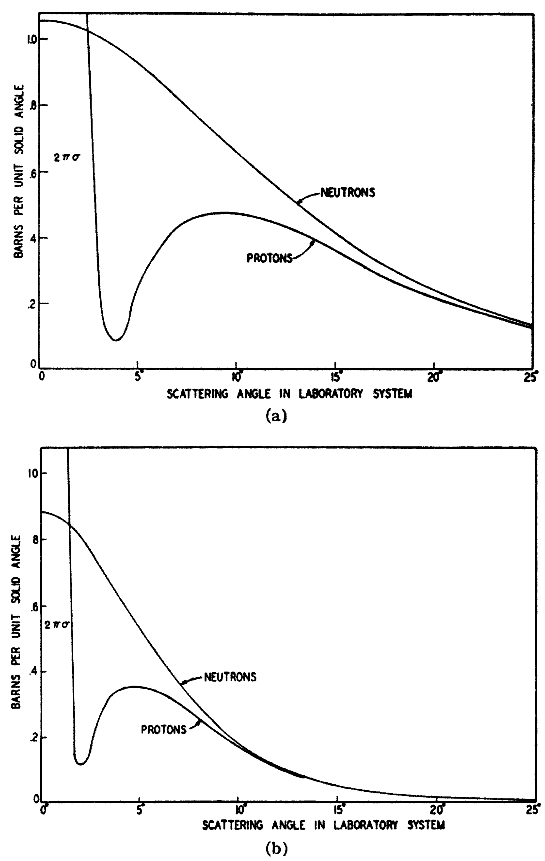

FIG. 4. Comparison of neutron-deuteron with proton-deuteron elastic scattering for small angles in the laboratory system. (a) 90-Mev incident particles, (b) 350-Mev incident particles.

The result is plotted as the dotted curve in Fig. 2 for 90 Mev. The total cross section of 0.059 B is slightly larger than given by the crude calculation, and the angular distribution is considerably steeper in the forward direction. Both differences are possibly attributable to the Born approximation. The one genuinely new feature is the weak backward maximum, shown in Fig. 3, a consequence of the exchange effects.

At 350 Mev the total cross section turns out to be 0.010 barn, but most of the change is due simply to the decrease in the nucleon-nucleon cross section, which is roughly inversely proportional to the energy. The angular distribution is shoved even further forward, of course, as shown in Fig. 4b.

C. Proton-Deuteron Cross Section

When the incident particle is a proton one must add to the perturbing energy the Coulomb potential between the two identical particles. The Born approximation should still be applicable, since at 90 Mev the proton velocity is half that of light. All results of part B then remain valid if one adds to both doublet and quartet matrix elements the terms

$$\Omega^{-1}\{I_1^C - I_2^C\},$$

where I_1^C and I_2^C are obtained from I_1 and I_2 by replacing, in the coefficients, $-V_0^T/\mu$ by e^2 and setting $\mu=0$ elsewhere. Since $\mu e^2/V_0^T=-0.018$, the additional terms are of negligible effect except for very small scattering angles and there only I_1^C need be considered. One may easily estimate the order of magnitude of the largest angle strongly affected. I_1^C becomes comparable to I_1 when $e^2/4p_0^2\sin^2\theta/2$ is no longer small compared to

$$\frac{V_0^T}{\mu}\frac{1}{\mu^2+4p_o^2\sin^2\theta/2}.$$

This is equivalent to $\theta\sim(\mu e^2/V_0^T)^{\frac{1}{2}}(\mu/p_0)$. At 90 Mev, this is an angle of 5° and at 350 Mev an angle of 2.5°, both in the center-of-gravity system. The corresponding laboratory angles are two-thirds of these.

In Figs. 4a and b the small-angle scattering of neutrons and protons is compared, showing in detail the above qualitative feature. Actually a difference between the neutron and proton scattering at 90 Mev should already be noticeable at 10° in the laboratory system. As is usual when the Coulomb force is involved, it is necessary to set a lower limit on the scattering angle if a finite total cross section is to be obtained.

IV. SUMMARY AND DISCUSSION

We have found that at high energies the cross section for the elastic scattering of neutrons on deuterons should be roughly half the neutron-proton cross section, the ratio not being a strong function of the energy. The reason for this is that in the individual nucleon-nucleon collisions momentum transfers corresponding to energies greater than 10 or 15 times the binding energy of the deuteron are unlikely, due to the finite range of the forces. Interference and the reduced mass effect then combine to give the fairly large total.

Nearly all the scattering should be confined to small angles, the angular half-width being about 15° at 90 Mev and varying inversely as the square root of the energy. There probably is a very weak backward maximum, as a result of exchange effects, which is unlikely to be observed.

The proton scattering at 90 Mev should be found to deviate appreciably from that of neutrons if the small-angle intensity can be measured at all. The Coulomb force ought to show itself quite strongly at 10°.

From an experimental standpoint, it may be easier to measure the energy spectrum of the recoil deuterons rather than the actual scattered nucleon intensity. In Fig. 5, accordingly, we have plotted the recoil energy spectrum to be expected from the calculation of Section III, for both neutron and proton projectiles at 90 Mev. The simpler considerations of Section I would predict a somewhat less steep distribution, with a half-width of about 4 Mev for the deuterons.

ACKNOWLEDGMENTS

The author wishes to express his gratitude to E. Fermi, who suggested the problem and gave frequent advice and encouragement. Many thanks are also due M. L. Goldberger who contributed invaluably to all phases of the investigation.