PHYSICAL REVIEW VOLUME 74, NUMBER 1 JULY 1, 1948

The Angular Distribution of Pair-Produced Electrons and Bremsstrahlung*

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(Received March 8, 1948)

The Bethe-Heitler unscreened cross sections for pair production and bremsstrahlung have been integrated to give the angular distributions. For simplicity, results are restricted to large angles and high energies, but it is shown that extension to smaller angles requires that screening be considered at energies for which screening is important in the usual differential cross sections. The effect of the finite size of the nucleus is estimated and found to be large at the higher energies. It is shown that large-angle pair production will provide only a very small background for single meson production by quanta.

I. INTRODUCTION

The basic cross sections for pair production and bremsstrahlung have been derived by H. A. Bethe and W. Heitler.[1,2] They have also carried out the complete angular integrations, in pair production, e.g., over the angles of both electron and positron; for convenience, however, new variables are introduced mixing the two sets of angles so that the integration over one pair is not obtained. We have carried out such an integration and have obtained, through suitable approximation, usable formulas for the high energy large-angle distribution of pair-produced electrons and bremsstrahlung.

Apart from the intrinsic interest in these distributions and the opportunity they provide for a detailed experimental test of the consequences of radiation theory, the results are of use for estimating the background of electrons against which one must look for mesons produced by quanta.

II. DISTRIBUTION OF PAIR-PRODUCED ELECTRONS

The fundamental cross section for pair production[3] given by Bethe and Heitler is differential in the angles of electron and positron and in the energy division between them, and we shall refer to it as the triple differential cross section. The result on integration over both pairs of angles we call the single differential cross section. As a consequence of Born's approximation, electron and positron coordinates enter symmetrically in the triple differential cross section; it is immaterial therefore which particle's angles are integrated over and the distribution obtained applies to either. For definiteness we integrate on positron coordinates. The notation throughout is that of Bethe and Heitler, *viz.*, E_+, p_+, θ_+ for the positron's energy, momentum $\times$ c, and angle relative to the quantum's direction, with similar notation for the electron. k is the energy of the quantum.

The azimuthal integration is elementary and leads to eight integrals on the polar angle, of which the most complicated is the following:

$$\int_0^2 [(a+bx)x(2-x)dx/(x+\eta_+)^2X^{\frac{3}{2}}]. \qquad (1)$$

Here $x=1-\cos\theta_+$, $\eta_+=(E_+-p_+)/p_+$, X is a quadratic form in x, and a, b and the coefficients in X depend on E_+, E_-, p_+, p_-, and θ_-. The other integrals differ from (1) simply in the omission of various factors, e.g., in the denominator $x+\eta_+$ may occur to the first power or not at all, X may be raised to the $\frac{1}{2}$ power, etc. All have been evaluated exactly[4] with a resulting formula too complicated for convenient use. The high energy large-angle limit gives, however, our chief result, Eq. (2) below.

A second treatment[5] makes use of the large-angle high energy assumptions at the start and may be described with reference to the typical

* This work was supported in part by the Office of Naval Research.

[1] H. Bethe and W. Heitler, Proc. Roy. Soc. **146**, 90 (1934).

[2] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, 1936), §§17, 20.

[3] See reference 2, p. 196, Eq. (7).

[4] I wish to thank Mr. E. S. Lennox for an independent exact calculation of these integrals which made possible the elimination of several mistakes.

[5] This method was suggested by Professor H. A. Bethe.

integral (1). With μ the electron rest energy, at high energies $\eta_+ = \frac{1}{2}(\mu/E_+)^2$ and introducing $\delta \sim \mu/E_+$ we may break the integral $\int_0^2$ into $\int_0^\delta + \int_\delta^2$. In the first of these all factors in the integrand are put equal to their values at $x=0$, except $x/(x+\eta_+)^2$ which is easily integrated, and in the second η_+ is neglected compared to x. The latter approximation evidently requires only $(\mu/E_+) \ll 1$; the former demands that $(a+bx)/X^{\frac{3}{2}}$ does not vary rapidly for $x<\delta$. That this implies large E_- may be seen from Fig. 1. For the factor $(a+bx)/X^{\frac{3}{2}}$ is just the azimuthal integration of $1/q^4$, where $q=k-p_- -p_+$, as shown by the circle. And this integral does not vary rapidly with θ_+ for $\theta_+ \sim 0$ unless q is small there, i.e., unless θ_- or p_- is small.

We conclude, with reference to the triple differential formula, that when the rapid variation of $1/q^4$ and $1/(E_+ - p_+ \cos\theta_+)^2 = 1/E_+^2(x+\eta_+)^2$ can be separated, the integration is simple, and when not, it is complicated; the simple case is distinguished by simultaneous large θ_- and E_-.

The two methods agree in every detail, and we obtain for the probability per nucleus of electron production in solid angle $d\omega$ at angle θ (writing θ for θ_-) and in the energy interval dE_- at energy E_-, the following:

$$d\phi_- = \bar{\phi} d(E_-/k) d\omega_- (\mu/k)^2 (1/4\pi) S, \tag{2}$$

where $\bar{\phi} = (Z^2/137)(e^2/\mu)^2$, and S contains the angular variation.

$$S = (1+\gamma^2)\ln(2k/\mu) \cdot [\cot^2(\theta/2)/\sin^2(\theta/2)] + f(\gamma,\theta), \tag{2a}$$

$$f(\gamma,\theta) = (1+\gamma^2)\ln\frac{\gamma}{(1+\gamma)^2}\cdot\frac{\cot^2(\theta/2)}{\sin^2(\theta/2)} + \frac{\gamma(5-2\gamma)}{4\sin^4(\theta/2)} + \frac{\gamma^2 - 2\ln(1+\gamma)}{\sin^2(\theta/2)} + \frac{\gamma^2\left[\frac{\gamma}{\rho^2}[2\sin^2(\theta/2)+\gamma]-1\right]}{8\sin^6(\theta/2)} + \frac{(1+\gamma)[3\rho^2-\gamma(2+\gamma)]\ln(\rho+\gamma/\rho-\gamma)}{2\rho^3\sin^2(\theta/2)} \tag{2b}$$

with $\gamma = E_+/E_-$, and $\rho = |\mathbf{k}-\mathbf{p}_-|/E_- = [\gamma^2 + 4(1+\gamma)\sin^2(\theta/2)]^{\frac{1}{2}}$. We have separated in S the one term depending directly on k; the remainder, $f(\gamma,\theta)$, depends only on the ratio E_-/k. We have plotted f in Fig. 2 as a function of E_-/k for $\theta = 45°$, $90°$, $135°$, and $180°$. In addition, Fig. 3 shows S as a function of θ for representative E_-/k and $k = 500\mu = 255$ Mev.

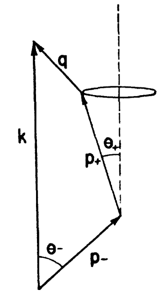

FIG. 1. Momentum diagram illustrating the discussion of the approximate integration.

Restrictions on the range of validity of (2) arise from two sources. The first is in the approximate integration, which, as shown by the reduction from the exact calculation, requires

$$E_-,\ E_+ \gg \mu, \quad \theta \gg \mu/E_-. \tag{3}$$

The second is the effect of screening by atomic electrons. Applying the considerations of Bethe and Heitler,[6] we find that the momentum transfer in a production process is large enough that screening may be neglected, provided

$$\theta \gg [(2\mu E_+/kE_-)(Z^{\frac{1}{3}}/137)]^{\frac{1}{2}}. \tag{4}$$

The limitation on θ by (4) and by (3) are of the same magnitude for

$$(2E_+E_-/k\mu) \sim 137Z^{-\frac{1}{3}}, \tag{5}$$

and this will be recognized as the boundary region where screening becomes important in the single differential cross section.[7]

Thus the extension of formula (2) to smaller angles by exact integration, which removes restrictions (3), is valid only for energies for which screening is not important in the single differential cross section. At the higher energies a small-angle formula requires introduction of the atom form factor, which is known only numerically; any but the simplest analytical approximation to the form factor would make it necessary to do all integrals numerically, and at best an analytical result would be much more complicated than (2).

[6] See reference 2, p. 168ff.
[7] See reference 2, p. 197, Eq. (11).

An exception to this generally unpleasant behavior at small angles is the case θ_- (θ in (2)) exactly zero. There the azimuthal dependence of the triple differential cross section disappears, and we get in the high energy, unscreened case simply

$$d\phi_-|_{\theta=0}=\bar{\phi}d\omega_-d\left(\frac{E_-}{k}\right)\frac{4}{\pi}\frac{E_-^2}{k^2}$$
$$\times\left(\frac{E_+^2+E_-^2}{\mu^2}\ln\frac{2E_+E_-}{\mu k}-\frac{(E_+-E_-)^2}{2\mu^2}\right). \quad (6)$$

The precise limitations on (6) are

$$E_-,\ E_+\gg\mu,\qquad (2E_+E_-/k\mu)\ll 137Z^{-\frac{1}{3}}. \quad (7)$$

We see that the forward production emphasizes high energy electrons (similarly positrons) by a factor E_-^2/k^2 modifying a formula otherwise symmetrical in E_- and $k-E_-$. This emphasis must occur since we have seen (Fig. 4) that at large angles low energy particles are relatively more numerous, while the single differential cross section is symmetrical about $E_-=k/2$.

The cross section at large angles is extremely small. For example, if $k=255$ Mev, $\theta=90°$, the cross section per unit solid angle for production of electrons of energy $\geq\frac{1}{2}k$ is from (2) $(Z^2/137)\times(1.6\times10^{-7})$ barn, for Pb 0.8×10^{-5} barn. At 45° and for electrons of energy $\geq 0.1k$, it becomes $\sim(Z^2/137)(10^{-4})$ barn, for Pb$\sim5\times10^{-3}$ barn.

For the large momentum changes here considered the modification of the Coulomb field resulting from the finite extent of the nucleus will give an additional very appreciable decrease in the cross section. The effect is estimated in Section IV below.

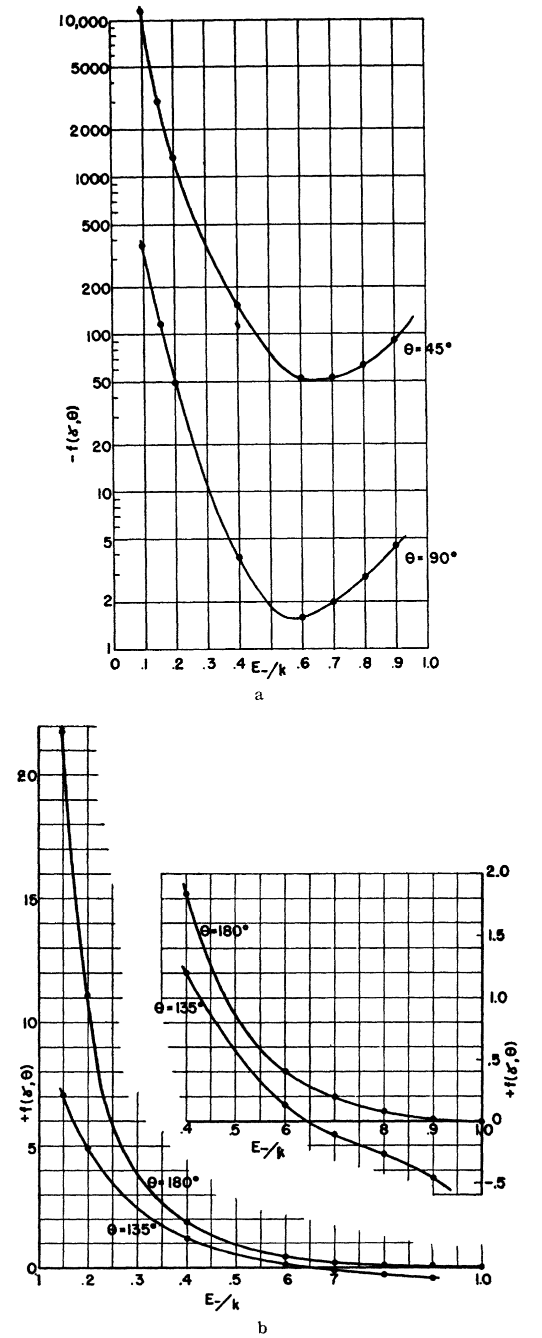

FIG. 2. The auxiliary function $f(\gamma,\theta)$, Eq. (2b) of the text. (a) $-f$ plotted semilogarithmically against E_-/k for $\theta=45°$, 90°. (b) $+f$ plotted linearly against E_-/k for $\theta=135°$, 180°. Insert: the same, with f-scale enlarged five times.

III. BREMSSTRAHLUNG DISTRIBUTION

Pair production and bremsstrahlung are just inverse processes, except that in pair production the initial electronic state is of negative energy. In the bremsstrahlung process, let E_0, p_0, θ_0 refer to the electron's initial energy, momentum $\times$ c, and angle relative to the quantum; E, p, θ describe its state after radiating, and k is still the quantum energy. Then, as Bethe and Heitler have shown,[8] the bremsstrahlung cross section is obtained from the pair production by writing E_0 and p_0 for E_-, p_-, $-E$ and $-p$ for E_+, p_+,

[8] See reference 2, §20.

and multiplying by $k^2dk/p_0^2dE_-$ to provide the correct energy density of final states with electron and quantum present rather than electron and positron. It will be noted that following the above prescription the conservation of energy $E_-+E_+=k$ becomes $E_0-E=k$, which is right, and q^2 becomes $(k+p-p_0)^2$, still correct for the square of the momentum transferred to the nucleus. Thus, except for certain square roots of perfect squares taken in deriving (2), which now (notably in the arguments of logarithms) require absolute value signs, the bremsstrahlung angular distribution is obtained from (2) simply by writing $-\gamma$ for γ and (a review of the work shows) $-\rho$ for ρ, where now $\gamma=+E/E_0$ while ρ retains its significance: $\rho=|\mathbf{p}_0-\mathbf{k}|/E_0=[\gamma^2+4(1-\gamma)\sin^2(\theta_0/2)]^{\frac{1}{2}}$. With these definitions and dropping the 0-subscript so that now θ represents the angle of emission of k relative to p_0, we get

$$kd\phi_k=E_0\bar{\phi}d(k/E_0)d\omega_k(\mu/E_0)^2(1/4\pi)T, \quad (8)$$

$$T=(1+\gamma^2)\ln 2E_0/\mu \cdot[\cot^2(\theta/2)/\sin^2(\theta/2)]+g(\theta,\gamma), \quad (8a)$$

$$g(\theta,\gamma)=(1+\gamma^2)\ln\frac{\gamma}{1-\gamma}\cdot\frac{\cot^2(\theta/2)}{\sin^2(\theta/2)} -\frac{\gamma(5+2\gamma)}{4\sin^4(\theta/2)}+\frac{\gamma^2-2\ln(1-\gamma)}{\sin^2(\theta/2)} -\frac{\gamma^2\left\{\frac{\gamma}{\rho^2}[2\sin^2(\theta/2)-\gamma]+1\right\}}{8\sin^6(\theta/2)} -\frac{(1-\gamma)[3\rho^2+\gamma(2-\gamma)]\ln(\rho+\gamma/\rho-\gamma)}{2\rho^3\sin^2(\theta/2)} \quad (8b)$$

$kd\phi_k$ is proportional to the intensity of radiation emitted in energy interval dk at k and in solid angle $d\omega_k$ at angle θ. The angular dependence has again been separated into a term depending on the absolute value of the incident energy (here E_0) and $g(\gamma,\theta)$ which depends only on ratios of energies.

Figure 4 gives $g(\gamma,\theta)$ for four angles and a range of k/E_0, and for $E_0=255$ Mev Fig. 5 shows T as a function of θ for representative quantum energies.

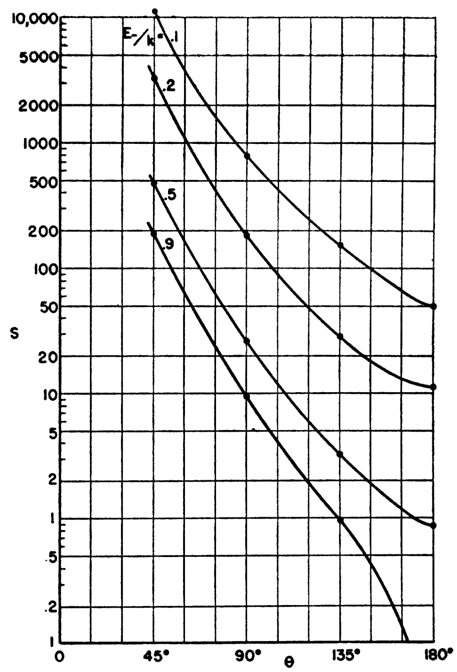

FIG. 3. S of Eqs. (2) and (2a) with $k=500\mu=255$ Mev. The numbers on the curves are E_-/k.

In complete analogy to (3) and (4), the restrictions on (8) are, from the approximate integration

$$E_0,\ E\gg\mu, \quad \theta\gg\mu/E_0, \quad (9)$$

and from screening

$$\theta\gg[(2\mu E/kE_0)(Z^{\frac{1}{3}}/137)]^{\frac{1}{2}}. \quad (10)$$

The cross section for radiation directly forward, corresponding to (6) above, is

$$kd\phi_k|_{\theta=0}=E_0\bar{\phi}d\omega_k d\left(\frac{k}{E_0}\right)\frac{4}{\pi} \times\left(\frac{E_0^2+E^2}{\mu^2}\ln\frac{2E_0E}{\mu k}-\frac{(E_0+E)^2}{2\mu^2}\right), \quad (11)$$

with restrictions

$$E_0,\ E\gg\mu \quad (2E_0E/\mu k)\ll 137Z^{-\frac{1}{3}}. \quad (12)$$

One finds on comparing (8) and (11) at energies where both are valid that in contrast to pair production there is no marked preferential radiation of high energy quanta forward. The relative number of quanta of different energies is almost independent of θ, except near 180° where hard quanta become relatively very scarce.

As in pair production, the cross section is further reduced due to the finite size of the nucleus, as discussed in the next section.

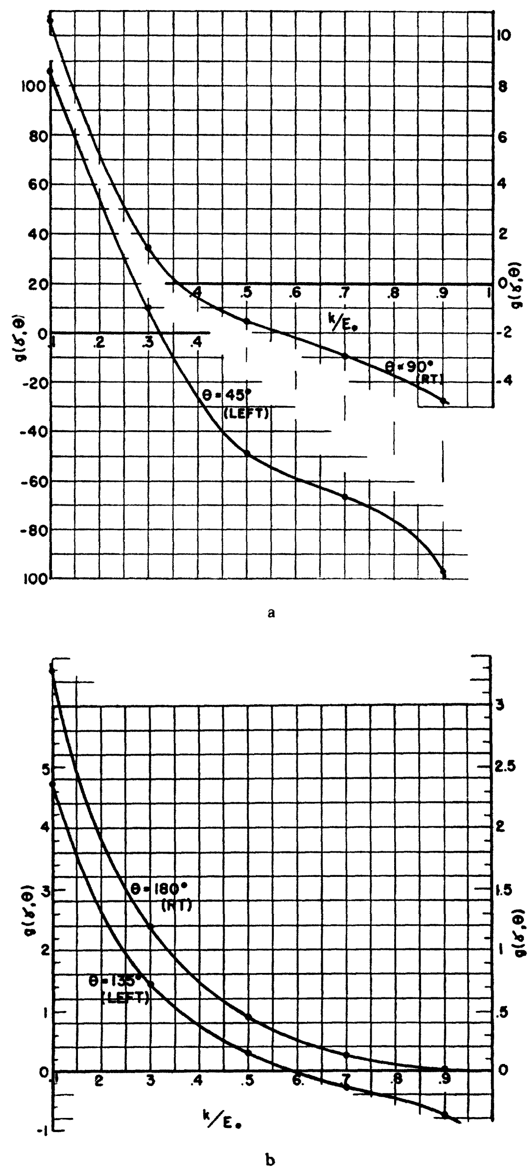

FIG. 4. The auxiliary function $g(\gamma,\theta)$, Eq. (8b) of the text. (a) g *vs.* k/E_0 for $\theta=45°$, 90°. (b) The same for $\theta=135°$, 180°.

It will be seen that while the bremsstrahlung distribution is simply obtained from the pair production, the third related distribution—of scattered radiating electrons—is not. A new integration is required, in which the angles relative to the quantum of both incident and scattered electron are varied, connected, however, by the requirement of fixed angle relative to each other.

IV. EFFECT OF THE FINITE EXTENT OF THE NUCLEUS

By confining ourselves to pair production and bremsstrahlung with large momentum transfer we need not consider screening by atomic electrons. But under this condition the effective impact parameter for both processes becomes of the order of the nuclear radius, and deviations from the Coulomb field on the inner side must be taken into account. For brevity we call the effect inner screening.

The original Bethe-Heitler cross sections are proportional to the square of the matrix element

$$V_q=\int V(\mathbf{r})\exp[i(\mathbf{q}\cdot\mathbf{r}/\hbar c)]d\tau$$
$$=4\pi(\hbar c/q)^2\int \rho(\mathbf{r})\exp[i(\mathbf{q}\cdot\mathbf{r}/\hbar c)]d\tau, \quad (13)$$

with $V(\mathbf{r})$ the potential and $\rho(\mathbf{r})$ the space density of electric charge. (The second well-known form follows from the first by writing $\exp[i(\mathbf{q}\cdot\mathbf{r}/\hbar c)] = -(\hbar c/q)^2\nabla^2\exp[i(\mathbf{q}\cdot\mathbf{r}/\hbar c)]$, and using Green's theorem and Poisson's equation in that order.) Equation (13) is of the form $V_q^{\text{coul}}\cdot F(q)$ where V_q^{coul} corresponds to a pure Coulomb field and gives the usual cross sections, and

$$F(q)=\frac{1}{Ze}\int \rho(\mathbf{r})\exp(i\mathbf{q}\cdot\mathbf{r}/\hbar c)d\tau \quad (14)$$

is the nuclear form factor. The effect of inner screening is thus to multiply the triple differential cross section by the square of (14).

Instead of attempting to integrate these modified cross sections to find the angular distributions with inner screening, we may obtain a fair approximation to the result by multiplying the distributions (2) and (8) by $|F(q)|^2$ at the q

which gives maximum contribution to the unscreened integrals. Unfortunately, as discussed in Section II, there are two such q's, the first when p_+ is parallel to k (in bremsstrahlung p parallel to k) and the second when q itself is a minimum. Calling them q_1 and q_2, respectively, one finds that in pair production $q_1 \simeq q_2$ for much greater ranges of the variables than in bremsstrahlung—in the latter case the approximate equality holds only for large θ and $E \ll E_0$. For this reason we limit ourselves in the following to the case of pair production.

From the definition of q_1 and q_2

$$q_1 = 2E_- \sin(\theta/2),$$
$$q_2 = (k^2 + p_-^2 - 2kp_- \cos\theta)^{\frac{1}{2}} - p_+.$$

In rough approximation, $q_2/q_1 = 1 - [1-(E_-/k)] \times [1-\sin(\theta/2)]$. At high k the contribution from around q_1 predominates, but only by virtue of a coefficient $\ln(k/\mu)$, and for quanta of a few hundred Mev or less it is best to use in (14) $\bar{q} = q_1 - (\frac{1}{3})(q_1 - q_2)$

$$\bar{q} = 2E_- \sin(\theta/2)\{1 - \tfrac{1}{3}[1-(E_-/k)] \times [1-\sin(\theta/2)]\}. \quad (15)$$

One may expect a relative error in $\bar{q}$

$$\delta\bar{q}/\bar{q} \sim \tfrac{1}{2}[1-(E_-/k)][1-\sin(\theta/2)]. \quad (16)$$

It is clear that for suitable angles and energies (16) can be made small, so that a given electron energy and angle correspond in (14) to a quite definite q. The nuclear charge distribution may thereby be explored, but since elastic electron scattering by nuclei gives the same information with less theoretical ambiguity and experimental difficulty we shall not consider this point in detail.

The effect on the pair-production cross section may be estimated sufficiently accurately by taking for the nucleus a uniform sphere of charge of radius R. Equation (14) gives

$$F(q) = (3/\alpha^2)[(\sin\alpha/\alpha) - \cos\alpha], \quad (17)$$

where $\alpha = Rq/\hbar c = R/\lambda\!\!\!^{-}$, $\lambda\!\!\!^{-}$ the wave-length $\div 2\pi$ corresponding to the momentum q/c. This $|F(q)|^2$ is plotted semilogarithmically in Fig. 6. To show what α's may be obtained we note that $\hbar c/E_- = 2.0 \times 10^{-13}$ cm for $E_- = 100$ Mev, hence

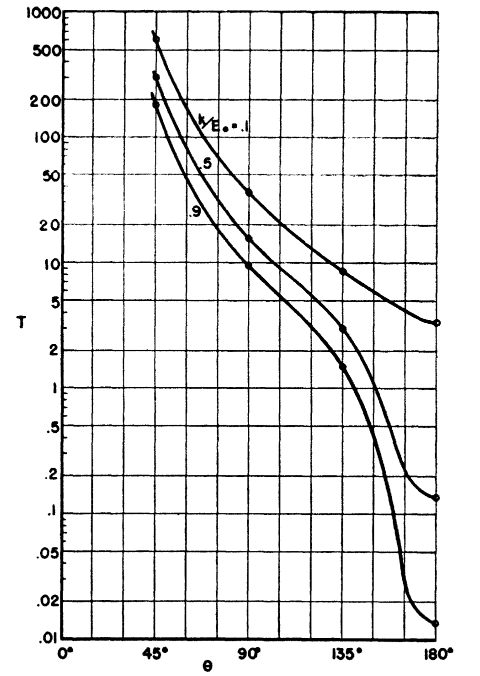

FIG. 5. T of Eqs. (8) and (8a), with $E_0 = 500\mu = 255$ Mev. The numbers on the curves are k/E_0.

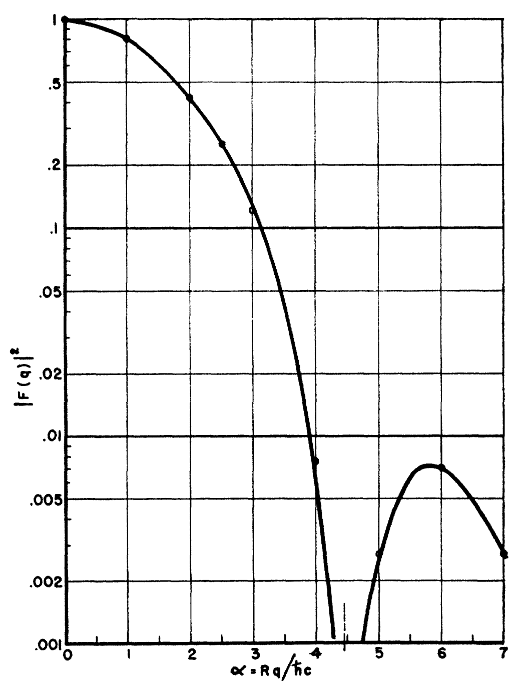

FIG. 6. $|F(q)|^2$ from Eq. (17), plotted semilogarithmically against $\alpha = Rq/\hbar c$, where R is the nuclear radius and q/c the momentum transfer.

for the Pb nucleus ($R=(1.5\times10^{-13}\text{ cm})A^{\frac{1}{3}}=8.9\times10^{-13}$ cm) α varies between 3.5 and 9 as θ ranges from 45° to 180°. Thus the effect is very large, and for high energy electrons a strong decrease over the cross section (2) is expected. Several remarks are appropriate here:

(a) The cross section for pair production with excitation of the nucleus must be expected to be of the same order as the "coherent" pair production when $|F(q)|^2\sim1/Z$, for a reduction factor of this amount corresponds to pair production in the fields of the individual protons.

(b) The second maximum in the $|F(q)|^2$ of Fig. 6 is due to the assumed sharp edge of the nucleus. Although the coherent production for such q's is half the incoherent or less, one may hope to observe a deviation from smooth decrease experimentally and correlate its magnitude with the indefiniteness of the nuclear boundary.

(c) The effect on the form factor of an increased proton density near the edge of the nucleus has been shown to be slight, and there seems small chance to detect such an increase by pair-production measurements.

(d) Since single meson production by quanta will not be coherent, for detection of mesons the decrease in large angle pair production caused by inner screening is pure gain.

V. MESON DETECTION

Estimates[9] of the total cross section per nucleon for single meson production by quanta of energy $\sim$250 Mev are 5×10^{-29} cm^2 or greater. For such energies the production will occur over quite a large solid angle, and we may take as a minimum for the cross section per unit solid angle at $90°\sim10^{-30}$ cm^2. The Pb nucleus thus provides a cross section of 10^{-28} cm^2 or more. We may compare this with the analogous pair-production cross section from Section II of 0.8×10^{-29} cm^2 per unit solid angle at 90°, for $k=255$ Mev, and for electrons of energy $\geq\frac{1}{2}k$. Adding in positrons of the same energy gives a cross section of a few $\times10^{-29}$ cm^2. (Electrons and positrons of lower energy can easily be screened from mesons by magnetic selection.) Considering inner screening this cross section drops to less than 10^{-30} cm^2. We conclude that pair production will not directly produce a background nearly as large as the expected meson yield.

VI. ACKNOWLEDGMENT

I wish to thank Professors Robert R. Wilson and H. A. Bethe for suggesting this work, and Professor Bethe for his continuing interest and very helpful advice.

[9] See e.g., L. W. Nordheim and G. Nordheim, Phys. Rev. **54**, 254 (1938).