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Contribution to the Theory of the Cherenkov Effect

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The rigorous solution is given for the field of a uniformly moving point charge which, coming from vacuum, enters at $t=0$ through a plane surface into an ideal, macroscopic, homogeneous dielectric medium ($\epsilon=$const.). The solution represents a characteristic transition radiation, as the velocity of motion approaches $c/(\epsilon)^{\frac{1}{2}}$ and furnishes information about the formation of the characteristic cone if $v>c/(\epsilon)^{\frac{1}{2}}$.

1. INTRODUCTION

THE radiation discovered by Cherenkov[1] and studied in more detail by other investigators[2] represents, according to Frank and Tamm,[3] a natural consequence of relativistic electrodynamics, analogous to phenomena known in acoustics in the case of projectiles moving with velocities higher than the velocity of sound. A very brilliant and detailed study, covering all aspects of the phenomenon, has been afterwards published by Tamm, resulting from discussions on the subject with L. Mandelstamm.[4] The present paper differs from Tamm's treatment both by its method and by its aim. While Tamm admits *a priori* the existence of the Cherenkov radiation, solving a problem with boundary conditions (outgoing waves), the purpose of this paper is to show that the problem can be reduced to one with given initial conditions (electron moving uniformly in vacuum) and leads subsequently, indeed, to radiation emission, i.e., to the solution admitted by Tamm, implying retarded potentials only.

The model adopted here, represents a point charge moving with uniform velocity, coming from infinity and entering at $t=0$ a semi-infinite ideal classical dielectric. A model of this type has the advantage of permitting rigorous solutions of Maxwell's equations for given initial conditions: a point charge moving in vacuum at $t=-\infty$ at infinite distance from the dielectric and leading at $t=+\infty$ to the quasistationary field of the point charge in the dielectric.

From the physical point of view, this model is subject to several restrictions, the most important of which are the macroscopic treatment of field and dielectric and the neglection of dispersion, $\epsilon=\epsilon(\nu)$. Both restrictions limit the validity of the obtained expressions, the first one to wave-lengths greater than the thickness of the surface film (here represented simply by a discontinuous jump of ϵ) the second one to spectral regions in which ϵ is sensibly constant. Information on the influence of dispersion is, however, available from Tamm's quoted paper and from an important paper of Fermi[5] who, considering the passage of charged particles through condensed dielectrics, obtained formulae which take into account dispersion and apply also to the phenomenon observed by Cherenkov.

2. THE STATIONARY SOLUTIONS

Since our problem is, essentially, a nonstationary one, its solution will necessarily contain a radiation field. Because of the fact that we can consider here uniformly moving charges, we can, however, separate from it a stationary part of the solution, which shall be considered first.

Assuming that our point charge moves along the z-axis, our solution depends only on the coordinates $\rho=(x^2+y^2)^{\frac{1}{2}}$ and z. We suppose that the charge moves in vacuum, $\epsilon=1$, for $z<0$, $t<0$, with constant velocity v. For $z>0$, $t>0$, our charge moves in a dielectric medium, $\epsilon>1$. We have to distinguish between the solution

[1] P. A. Cherenkov, C. R. Acad. Sci. U.S.S.R. **8**, 451 (1934).

[2] G. B. Collins and B. G. Reiling, Phys. Rev. **54**, 499 (1938); H. O. Wyckoff and J. E. Henderson, Phys. Rev. **64**, 1 (1943).

[3] I. Frank and Ig. Tamm, C. R. Acad. Sci. U.S.S.R., **14**, 109 (1937).

[4] Ig. Tamm, J. Phys. U.S.S.R., **1**, 439 (1939). The author desires to thank the Editorial Board of the Physical Review for having indicated to him this paper, which was not accessible to him during the present work and which, unfortunately, was not refered to in the accessible literature.

[5] E. Fermi, Phys. Rev. **57**, 485 (1940).

outside, $\Phi^{(0)}$, $z<0$, and *inside*, $\Phi^{(i)}$, $z>0$, the medium, both solutions satisfying the boundary conditions for $z=0$,

$$E_\rho{}^{(0)}=E_\rho{}^{(i)}; \quad D_z{}^{(0)}=D_z{}^{(i)}. \tag{1}$$

The solution Φ is determined by the static potential φ and the vector potential $A=A_z$, by means of which we express

$$\begin{aligned} &E_\rho=-\partial\varphi/\partial\rho; \quad D_\rho=\epsilon E_\rho \\ &E_z=-\partial\varphi/\partial z-(1/c)(\partial A/\partial t); \quad D_z=\epsilon E_z \\ &H_\varphi=-\partial A/\partial\rho; \quad B_\varphi=H_\varphi \end{aligned} \tag{2}$$

and satisfies the Lorentz condition

$$\partial A/\partial z+(\epsilon/c)(\partial\varphi/\partial t)=0. \tag{3}$$

The fact that the field of a point charge at rest, in front of the plane surface of a dielectric medium, can be represented by the superposition of the fields of several (real and virtual) point charges (electrical images), suggests that a similar superposition may be possible if we have to deal with uniformly moving point charges. We find, indeed, that Φ_1 and Φ_2 are solutions of our problem, with

$\Phi_1{}^{(0)}$:

$$\begin{aligned} \varphi^{(0)}&=e\left\{\frac{1}{[(1-v^2/c^2)\rho^2+(z-vt)^2]^{\frac{1}{2}}}-\frac{\epsilon-[1+(\epsilon-1)v^2/c^2]^{\frac{1}{2}}}{\epsilon+[1+(\epsilon-1)v^2/c^2]^{\frac{1}{2}}}\frac{1}{[(1-v^2/c^2)\rho^2+(z+vt)^2]^{\frac{1}{2}}}\right\}, \\ A^{(0)}&=e\frac{v}{c}\left\{\frac{1}{[(1-v^2/c^2)\rho^2+(z-vt)^2]^{\frac{1}{2}}}+\frac{\epsilon-[1+(\epsilon-1)v^2/c^2]^{\frac{1}{2}}}{\epsilon+[1+(\epsilon-1)v^2/c^2]^{\frac{1}{2}}}\frac{1}{[(1-v^2/c^2)\rho^2+(z+vt)^2]^{\frac{1}{2}}}\right\}. \end{aligned} \tag{4}$$

$\Phi_1{}^{(i)}$:

$$\begin{aligned} \varphi^{(i)}&=\frac{2e[1+(\epsilon-1)v^2/c^2]^{\frac{1}{2}}}{\epsilon+[1+(\epsilon-1)v^2/c^2]^{\frac{1}{2}}}\frac{1}{[(1-v^2/c^2)\rho^2+\{z[1+(\epsilon-1)v^2/c^2]^{\frac{1}{2}}-vt\}^2]^{\frac{1}{2}}}, \\ A^{(i)}&=\frac{2\epsilon ev/c}{\epsilon+[1+(\epsilon-1)v^2/c^2]^{\frac{1}{2}}}\frac{1}{[(1-v^2/c^2)\rho^2+\{z[1+(\epsilon-1)v^2/c^2]^{\frac{1}{2}}-vt\}^2]^{\frac{1}{2}}}, \end{aligned} \tag{5}$$

$\Phi_2{}^{(0)}$:

$$\begin{aligned} \varphi^{(0)}&=\frac{2e[1-(\epsilon-1)v^2/c^2]^{\frac{1}{2}}}{\epsilon[1-(\epsilon-1)v^2/c^2]^{\frac{1}{2}}+1}\frac{1}{[(1-\epsilon v^2/c^2)\rho^2+\{z[1-(\epsilon-1)v^2/c^2]^{\frac{1}{2}}-vt\}^2]^{\frac{1}{2}}}, \\ A^{(0)}&=\frac{2ev/c}{\epsilon[1-(\epsilon-1)v^2/c^2]^{\frac{1}{2}}+1}\frac{1}{[(1-\epsilon v^2/c^2)\rho^2+\{z[1-(\epsilon-1)v^2/c^2]^{\frac{1}{2}}-vt\}^2]^{\frac{1}{2}}}. \end{aligned} \tag{6}$$

$\Phi_2{}^{(i)}$:

$$\begin{aligned} \varphi^{(i)}&=\frac{e}{\epsilon}\left\{\frac{1}{[(1-\epsilon v^2/c^2)\rho^2+(z-vt)^2]^{\frac{1}{2}}}+\frac{\epsilon[1-(\epsilon-1)v^2/c^2]^{\frac{1}{2}}-1}{\epsilon[1-(\epsilon-1)v^2/c^2]^{\frac{1}{2}}+1}\frac{1}{[(1-\epsilon v^2/c^2)\rho^2+(z+vt)^2]^{\frac{1}{2}}}\right\}, \\ A^{(i)}&=e\frac{v}{c}\left\{\frac{1}{[(1-\epsilon v^2/c^2)\rho^2+(z-vt)^2]^{\frac{1}{2}}}-\frac{\epsilon[1-(\epsilon-1)v^2/c^2]^{\frac{1}{2}}-1}{\epsilon[1-(\epsilon-1)v^2/c^2]^{\frac{1}{2}}+1}\frac{1}{[(1-\epsilon v^2/c^2)\rho^2+(z+vt)^2]^{\frac{1}{2}}}\right\}. \end{aligned} \tag{7}$$

Φ_1 represents the stationary solution, while our point charge moves outside the dielectric medium, $t\leqslant 0$. It refers to one real charge e of velocity v and to two virtual charges

$$\begin{aligned} e'&=-e\frac{\epsilon-[1-(\epsilon-1)v^2/c^2]^{\frac{1}{2}}}{\epsilon+[1+(\epsilon-1)v^2/c^2]^{\frac{1}{2}}}, \\ e''&=\frac{2\epsilon e}{\epsilon+[1+(\epsilon-1)v^2/c^2]^{\frac{1}{2}}}, \\ v'&=-v, \\ v''&=\frac{v}{[1+(\epsilon-1)v^2/c^2]^{\frac{1}{2}}}\leqslant\frac{c}{(\epsilon)^{\frac{1}{2}}}. \end{aligned} \tag{8}$$

Φ_2 represents the stationary solution, while our point charge moves inside the dielectric medium, $t>0$. It refers to one real charge e of velocity v and to two virtual charges

$$\begin{aligned} e'&=e\frac{\epsilon[1-(\epsilon-1)v^2/c^2]^{\frac{1}{2}}-1}{\epsilon[1-(\epsilon-1)v^2/c^2]^{\frac{1}{2}}+1}, \\ e''&=\frac{2e}{\epsilon[1-(\epsilon-1)v^2/c^2]^{\frac{1}{2}}+1}, \\ v'&=-v, \\ v''&=\frac{v}{[1-(\epsilon-1)v^2/c^2]^{\frac{1}{2}}}. \end{aligned} \tag{9}$$

It is remarkable, that in (9) the velocity v'' becomes greater than the velocity of light c, as soon as $v>c/(\epsilon)^{\frac{1}{2}}$.

Equations (8) and (9) show that while our point charge penetrates into the medium, a rearrangement of the corresponding virtual charges takes place. This rearrangement of virtual charges gives rise to the transition radiation which becomes emitted during the considered process.

Φ_1 represents at $t=-\infty$, $z=-\infty$ a point charge e moving in vacuum with velocity v. It corresponds, therefore, to the correct initial conditions which we have to postulate. The influence of the virtual charges with increasing t accounts for the influence of the dielectric medium. We can, therefore, consider Φ_1 to be the correct solution for $t\leqslant 0$.

Φ_2 is not the correct solution for $t>0$, since, for $t=0$, it does not fit continuously with Φ_1. In order to assure continuity with time, we have to add to Φ_2 a radiation field, Φ_3, which is determined by the condition that for $t=0$ be

$$\Phi_3(0)=\Phi_1(0)-\Phi_2(0). \tag{10}$$

Satisfying the condition (10), we obtain the complete solution of our problem in the form

$$\begin{aligned}\Phi(t)&=\Phi_1(t) \text{ for } t\leqslant 0,\\ \Phi(t)&=\Phi_2(t)+\Phi_3(t) \text{ for } t\geqslant 0.\end{aligned} \tag{11}$$

3. THE RADIATION FIELD

The solution of our problem requires the determination of the radiation field Φ_3, from the condition (10). We shall proceed by representing Φ_3 by a Fourier series, determining its Fourier coefficients at $t=0$ by (10). The required solution is then obtained by summing up the Fourier series at any subsequent moment t. It is a necessary condition for (10), representing a radiation field, that the right-hand side field in (10) must not contain any real charges. It can be readily shown that this condition is satisfied.

Since the radiation field Φ_3 propagates in the dielectric medium with velocity $c/(\epsilon)^{\frac{1}{2}}$, we have to distinguish two cases:

(a) If $v<c/(\epsilon)^{\frac{1}{2}}$, the radiation field emitted in the moment $t=0$, when the charge enters the dielectric medium, will travel faster than the charge. After a sufficiently long time, the radiation field will be situated at a large distance from the charge and the field in the neighborhood of the charge will be sensibly the stationary field Φ_2. In this case, the influence of the surface of the dielectric medium leads mainly to the emission of a transition radiation, while the charge enters the medium.

(b) If $v>c/(\epsilon)^{\frac{1}{2}}$, the charge travels faster than the radiation field. This leads to the formation of the characteristic cone, already known from the elementary theory. The field is, then, given at any moment t and at any distance from the surface by $\Phi_2+\Phi_3$ and no separation into stationary and radiation part takes place.

Given the axial symmetry of our problem, we choose for the Fourier expansion an orthogonal system of cylindrical waves. Only waves not depending on the angle φ need to be taken into account. The following three sets of cylindrical waves satisfy the required conditions:

(a):

$$\begin{aligned}
E_\rho^{(0)}&=k_\rho J_1(k_\rho\rho)\cos k_z z\cdot e^{-ik_0ct},\\
E_\rho^{(i)}&=k_\rho J_1(k_\rho\rho)\cos \bar{k}_z z\cdot e^{-ik_0ct},\\
E_z^{(0)}&=-(k_\rho^2/k_z)J_0(k_\rho\rho)\sin k_z z\cdot e^{-ik_0ct},\\
E_z^{(i)}&=-(k_\rho^2/\bar{k}_z)J_0(k_\rho\rho)\sin \bar{k}_z z\cdot e^{-ik_0ct},\\
H_\varphi^{(0)}&=i(k_0k_\rho/k_z)J_1(k_\rho\rho)\sin k_z z\cdot e^{-ik_0ct},\\
H_\varphi^{(i)}&=i\epsilon(k_0k_\rho/\bar{k}_z)J_1(k_\rho\rho)\sin \bar{k}_z z\cdot e^{-ik_0ct},
\end{aligned} \tag{12}$$

$$N_a=\left[\frac{1}{\epsilon\pi}\frac{k_z}{\epsilon k_z+\bar{k}_z}\right]^{\frac{1}{2}}\frac{\bar{k}_z}{k_0k_\rho}.$$

. . . .

(b):

$$\begin{aligned}
E_\rho^{(0)}&=k_\rho J_1(k_\rho\rho)\sin k_z z\cdot e^{-ik_0ct},\\
E_\rho^{(i)}&=(k_\rho k_z/\epsilon \bar{k}_z)J_1(k_\rho\rho)\sin \bar{k}_z z\cdot e^{-ik_0ct},\\
E_z^{(0)}&=(k_\rho^2/k_z)J_0(k_\rho\rho)\cos k_z z\cdot e^{-ik_0ct},\\
E_z^{(i)}&=(k_\rho^2/\epsilon \bar{k}_z)J_0(k_\rho\rho)\cos \bar{k}_z z\cdot e^{-ik_0ct},\\
H_\varphi^{(0)}&=-i(k_0k_\rho/k_z)J_1(k_\rho\rho)\cos k_z z\cdot e^{-ik_0ct},\\
H_\varphi^{(i)}&=-i(k_0k_\rho/k_z)J_1(k_\rho\rho)\cos \bar{k}_z z\cdot e^{-ik_0ct},
\end{aligned} \tag{13}$$

$$N_b=\left[\frac{1}{\pi}\frac{\bar{k}_z}{\epsilon k_z+\bar{k}_z}\right]^{\frac{1}{2}}\frac{k_z}{k_0k_\rho}.$$

. . . .

(c):

$$E_\rho{}^{(0)} \pm k_\rho J_1(k_\rho\rho)\cdot e^{\kappa z}\cdot e^{-ik_0ct},$$

$$E_\rho{}^{(i)} = \frac{[\epsilon^2\kappa^2+\bar{k}_z{}^2]^{\frac{1}{2}}}{\epsilon\kappa} k_\rho J_1(k_\rho\rho) \times\cos(\bar{k}_z z+\bar{\delta})\cdot e^{-ik_0ct},$$

$$E_z{}^{(0)} = -(k_\rho{}^2/\kappa)J_0(k_\rho\rho)\cdot e^{\kappa z}\cdot e^{-ik_0ct},$$

$$E_z{}^{(i)} = -\frac{[\epsilon^2\kappa^2+\bar{k}_z{}^2]^{\frac{1}{2}}}{\epsilon\kappa}\frac{k_\rho{}^2}{\bar{k}_z} J_0(k_\rho\rho) \times\sin(\bar{k}_z z+\bar{\delta})\cdot e^{-ik_0ct}, \tag{14}$$

$$H_\varphi{}^{(0)} = i(k_0k_\rho/\kappa)J_1(k_\rho\rho)\cdot e^{\kappa z}\cdot e^{-ik_0ct},$$

$$H_\varphi{}^{(i)} = i\frac{[\epsilon^2\kappa^2+\bar{k}_z{}^2]^{\frac{1}{2}}}{\kappa}\frac{k_0k_\rho}{\bar{k}_z} J_1(k_\rho\rho) \times\sin(\bar{k}_z z+\bar{\delta})\cdot e^{-ik_0ct},$$

$$\text{tg}\bar{\delta} = \bar{k}_z/\epsilon\kappa, \quad N_z = \left[\frac{1}{\pi}\frac{\bar{k}_z{}^2}{\epsilon^2\kappa^2+\bar{k}_z{}^2}\right]^{\frac{1}{2}}\frac{\kappa}{k_0k_\rho}.$$

J_0 and J_1 denote Bessel functions of zero and first order. (k_ρ, k_z) and $(k_\rho, \bar{k}_z)$ are, respectively, the components of the wave vector in vacuum and in the dielectric medium,

$$k_\rho{}^2+k_z{}^2 = k_\rho{}^2-\kappa^2 = k_0{}^2, \quad k_\rho{}^2+\bar{k}_z{}^2 = \epsilon k_0{}^2,$$
$$k_z{}^2 = (1/\epsilon)\{\bar{k}_z{}^2-(\epsilon-1)\cdot k_\rho{}^2\} \text{ for } k_\rho/\bar{k}_z \leqslant 1/(\epsilon-1)^{\frac{1}{2}},$$
$$\kappa^2 = (1/\epsilon)\{(\epsilon-1)\cdot k_\rho{}^2-\bar{k}_z{}^2\} \text{ for } k_\rho/\bar{k}_z \geqslant 1/(\epsilon-1)^{\frac{1}{2}}.$$

The solutions (c), (14), refer to totally reflected waves in the medium.

The normalization factors N are chosen in order to satisfy the normalization

$$\int(\mathbf{E}_k{}^*\cdot\mathbf{D}_{k'}+\mathbf{H}_k{}^*\cdot\mathbf{B}_{k'})\cdot d\tau' = \delta_{kk'}; \quad d\tau' = \rho d\rho dz. \tag{15}$$

If we want each wave to represent one photon,

$$1/8\pi\int(\mathbf{E}_k{}^*\cdot\mathbf{D}_k+\mathbf{H}_k{}^*\cdot\mathbf{B}_k)\cdot d\tau' = (hc/2\pi)k_0, \tag{16}$$

we have to renormalize (12), (13) and (14) by a factor $[4hck_0]^{\frac{1}{2}}$.

In order to determine the Fourier coefficients for each of the three sets of cylindrical waves, a_k, b_k, c_k, we have to form for $t=0$, according to (10), the difference of the vector components D_ρ, D_z and B_φ corresponding to the two stationary solutions (4), (5) and (6), (7), to multiply them by the corresponding conjugate complex components of (12), (13) and (14) at $t=0$ and to integrate over

$$\int d\tau' = \int_0^\infty \rho d\rho\int_{-\infty}^{+\infty} dz.$$

The computation of these somewhat long expressions, which shall not be reproduced in detail, does not present major difficulties until it leads to integrals which, by means of elementary transformations can be reduced to

$$\int_0^\infty\int_0^\infty \frac{J_0(\eta)\cos\zeta}{(\alpha^2\eta^2+\zeta^2)^{\frac{1}{2}}}\eta d\eta d\zeta = \int_0^\infty\int_{-\infty}^0 \frac{J_0(\eta)\cos\zeta}{(\alpha^2\eta^2+\zeta^2)^{\frac{1}{2}}}\eta d\eta d\zeta = \frac{2}{\alpha^2+1},$$

$$\int_0^\infty\int_0^\infty \frac{J_0(\eta)\sin\zeta}{(\alpha^2\eta^2+\zeta^2)^{\frac{1}{2}}}\eta d\eta d\zeta = -\int_0^\infty\int_{-\infty}^0 \frac{J_0(\eta)\sin\zeta}{(\alpha^2\eta^2+\zeta^2)^{\frac{1}{2}}}\eta d\eta d\zeta = \frac{\alpha}{\alpha^2+1}, \tag{17}$$

$$\int_0^\infty\int_{-\infty}^0 \frac{J_0(\eta)e^\zeta}{(\alpha^2\eta^2+\zeta^2)^{\frac{1}{2}}}\eta d\eta d\zeta = \frac{\alpha-2}{\alpha^2-1},$$

$$\int_0^\infty\int_0^\infty \frac{J_0(\eta)\cos\zeta}{[-\alpha^2\eta^2+\zeta^2]^{\frac{1}{2}}}\eta d\eta d\zeta = \int_0^\infty\int_{-\infty}^0 \frac{J_0(\eta)\cos\zeta}{[-\alpha^2\eta^2+\zeta^2]^{\frac{1}{2}}}\eta d\eta d\zeta = \frac{2}{1-\alpha^2} - i\pi\delta(1-\alpha)+i\pi\delta(1+\alpha),$$

$$\int_0^\infty\int_0^\infty \frac{J_0(\eta)\sin\zeta}{[-\alpha^2\eta^2+\zeta^2]^{\frac{1}{2}}}\eta d\eta d\zeta = -\int_0^\infty\int_{-\infty}^0 \frac{J_0(\eta)\sin\zeta}{[-\alpha^2\eta^2+\zeta^2]^{\frac{1}{2}}}\eta d\eta d\zeta = \frac{i\alpha}{1-\alpha^2}+\pi\delta(1-\alpha)-\pi\delta(1+\alpha), \tag{18}$$

$$\int_0^\infty\int_{-\infty}^0 \frac{J_0(\eta)e^\zeta}{[-\alpha^2\eta^2+\zeta^2]^{\frac{1}{2}}}\eta d\eta d\zeta = \frac{2-i\alpha}{1+\alpha^2}.$$

TABLE I.

$$\frac{1}{N_a}a_\kappa = 2\epsilon e\frac{v^2}{c^2}\left\{\frac{2k_\rho^2}{k_\rho^2\{1+(\epsilon-1)v^2/c^2\}+\bar{k}_z^2(1-v^2/c^2)} - \frac{2k_\rho^2}{k_\rho^2+\bar{k}_z^2(1-\epsilon v^2/c^2)} + i\pi[\delta(1-[\epsilon(v^2/c^2)-1]^{\frac{1}{2}}\bar{k}_z/k_\rho) - \delta(1+[\epsilon(v^2/c^2)-1]^{\frac{1}{2}}\bar{k}_z/k_\rho)]\right\}$$

$$\frac{1}{N_b}b_\kappa = 2ie\frac{v}{c}\frac{k_0}{k_z}\left\{\frac{2\epsilon k_\rho^2}{k_\rho^2\{1+(\epsilon-1)v^2/c^2\}+\bar{k}_z^2(1-v^2/c^2)} - \frac{2k_\rho^2}{k_\rho^2+\bar{k}_z^2(1-\epsilon v^2/c^2)} + i\pi[\delta(1-[\epsilon(v^2/c^2)-1]^{\frac{1}{2}}\bar{k}_z/k_\rho) - \delta(1+[\epsilon(v^2/c^2)-1]^{\frac{1}{2}}\bar{k}_z/k_\rho)]\right\}$$

$$\frac{1}{N_c}c_\kappa = -4ie\frac{v}{c}\frac{k_0}{\kappa}\left\{\frac{\epsilon k_\rho^2}{k_\rho^2\{1+(\epsilon-1)v^2/c^2\}+\bar{k}_z^2(1-v^2/c^2)} - \frac{k_\rho^2}{k_\rho^2+\bar{k}_z^2(1-\epsilon v^2/c^2)}\right\} + 4\epsilon e\frac{v^2}{c^2}\left\{\frac{k_\rho^2}{k_\rho^2\{1+(\epsilon-1)v^2/c^2\}+\bar{k}_z^2(1-v^2/c^2)} - \frac{k_\rho^2}{k_\rho^2+\bar{k}_z^2(1-\epsilon v^2/c^2)}\right\}$$

Fourier coefficients in the case $v>c/(\epsilon)^{\frac{1}{2}}$, $1-(\epsilon-1)(v^2/c^2)>0$.
In the case $v<c/(\epsilon)^{\frac{1}{2}}$ we have to put, for real k_ρ and $\bar{k}_z$, $\delta(1\pm i[1-\epsilon(v^2/c^2)]^{\frac{1}{2}}(\bar{k}_z/k_\rho))=0$.
All square roots denote positive, real or imaginary, values.

Dirac's symbolic function δ has, in (18), a well determined sense. It stands for

$$2\pi\delta(1-\alpha) = -i\cdot\lim_{R\to\infty} e^{i(1-\alpha)R}/1-\alpha; \quad Jm(\alpha)\geqslant 0.$$

The evaluation of the integrals (17) and (18) cannot be performed by elementary means and represents the main difficulty for the solution of our problem. The values given above, have been obtained by using properties of Poisson integrals in the complex plane. Since the mathematical method involved does not present a major interest for the physical problem we have to deal with, it shall be treated in a separate paper.[6]

In the case $v<c/(\epsilon)^{\frac{1}{2}}$ only integrals of the type (17) occur. Integrals of the type (18) appear if $v>c/(\epsilon)^{\frac{1}{2}}$. We have, therefore, to deal with the Fourier coefficients in both cases separately.

Using the values of the integrals (17) and (18) we obtain the Fourier coefficients given in Table I.

4. THE TRANSITION RADIATION

In order to obtain the radiation field, we have to multiply each set of normalized cylindrical waves (12), (13), and (14) with the corresponding Fourier coefficient from Table I and to sum up the resulting series (Fourier integrals). The result of the summation cannot, however, be expressed in closed form. It can be shown, however, that we obtain, in general, a field of finite extension, propagating with velocity c in vacuum and $c/(\epsilon)^{\frac{1}{2}}$ in the dielectric medium. The fact, that the radiation field vanishes at finite distance from the surface after infinite time flows from the time dependence of (12), (13) and (14). Indeed, if t is sufficiently large, the mentioned time factor oscillates with k_0 more rapidly than the other terms, provided that ρ and z are finite, and makes, therefore, the Fourier integrals tend to zero. The initial field distribution, given by (10) shows, that the radiation field comes from the neighborhood of the coordinate origin, i.e., of the point where our charge enters the dielectric medium. The weak transition radiation could, therefore, become observable under favorable conditions, if we focus a lens on this point.

In the case of small velocities, the transition radiation is, according to Table I, insignificant. It increases, however, as soon as v approaches the critical value $c/(\epsilon)^{\frac{1}{2}}$. The main contribution, in this case, comes from the Fourier coefficients a_κ and b_κ in Table I, from the terms which contain the denominator $\{k_\rho^2+\bar{k}_z^2(1-\epsilon v^2/c^2)\}$. The number of photons emitted, due to these terms, is given by

$$dn \cong \frac{2(\epsilon)^{\frac{1}{2}}}{\pi^2}\frac{2\pi e^2}{hc}\frac{v^2}{c^2}\frac{\sin^3\Theta d\Theta}{\{1-\epsilon(v^2/c^2)\cos^2\Theta\}^2}\frac{d\lambda}{\lambda}. \quad (19)$$

The transition radiation has, according to (19), a strong maximum in the forward direction at

$$\Theta = \left[\frac{6}{5}\left(1-\epsilon\frac{v^2}{c^2}\right)\right]^{\frac{1}{2}}.$$

The transition radiation (19) corresponds to the radiation due to the acceleration of an elec-

[6] Mathematicæ Notæ. **VII**, 191, (1948).

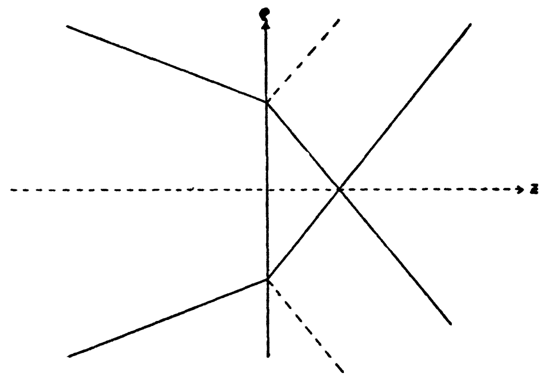

FIG. 1. The characteristic cone of the stationary solution.

tron moving in an infinite medium during a finite time, which has been determined by Tamm in his quoted paper.[7] Because of its different origin, at the surface of the medium, it differs from it both in intensity and in angular dependence, but it is of the same order of magnitude. Integrating over the angle, Tamm gives (in our notations) for the emitted energy the expression

$$dW=\frac{1}{\pi^2}\frac{2\pi e^2}{hc}\frac{1}{\epsilon(v/c)}\times\left\{\ln\frac{1+(\epsilon)^{\frac12}(v/c)}{1-(\epsilon)^{\frac12}(v/c)}-2(\epsilon)^{\frac12}(v/c)\right\}hd\nu$$

while (19) leads to

$$dW=h\nu dn\to\frac{1}{\pi^2}\frac{2\pi e^2}{hc}\left\{\tfrac12(1+\epsilon(v^2/c^2)\times\ln\frac{1+(\epsilon)^{\frac12}(v/c)}{1-(\epsilon)^{\frac12}(v/c)}-(\epsilon)^{\frac12}(v/c)\right\}hd\nu.$$

Because of the logarithmic dependence on $(\epsilon)^{\frac12}v/c$, dW increases very slowly while we approach the critical velocity $v\to c/(\epsilon)^{\frac12}$ and amounts to the order of about one visible photon per hundred incident electrons.

As we shall see in the next paragraph, the Fourier terms taken into consideration contribute in the case $v>c/(\epsilon)^{\frac12}$ to the formation of the characteristic cone, i.e., to a field propagation in one well-determined direction Θ. They do, then, no longer contribute to the transition radiation, though the remaining terms assure that even in this case the transition radiation does not disappear completely and becomes again important, if the velocity approaches c.

We conclude that the transition radiation shows a marked maximum in the forward direction for velocities slightly below the critical value of $c/(\epsilon)^{\frac12}$ and weakens considerably, as soon as v exceeds the critical value.

5. THE CHERENKOV EFFECT

The case $v>c/(\epsilon)^{\frac12}$ is characterized by the fact, that our solutions Φ_2 and Φ_3 show singularities of the field on the surface of a cone, which is given by:

Inside the medium:

$$\frac{\rho}{z}=\mathrm{tg}\vartheta^{(i)}=\frac{1}{(\epsilon(v^2/c^2)-1)^{\frac12}};$$
$$\frac{k_\rho}{k_z}=\mathrm{tg}\Theta^{(i)}=(\epsilon(v^2/c^2)-1)^{\frac12},\qquad(20)$$
$$\sin\vartheta^{(i)}=\cos\Theta^{(i)}=\frac{c}{(\epsilon)^{\frac12}v}.$$

Outside the medium:

$$\frac{\rho}{z}=\mathrm{tg}\vartheta^{(0)}=\left(\frac{1-(\epsilon-1)(v^2/c^2)}{\epsilon(v^2/c^2)-1}\right)^{\frac12};$$
$$\frac{k_\rho}{k_z}=\mathrm{tg}\Theta^{(0)}=\left(\frac{\epsilon(v^2/c^2)-1}{1-(\epsilon-1)(v^2/c^2)}\right)^{\frac12},\qquad(21)$$
$$\sin\vartheta^{(0)}=\cos\Theta^{(0)}=(c/v)(1-(\epsilon-1)(v^2/c^2))^{\frac12}.$$

ϑ denotes the angle of the singular cone, Θ the direction of its normal in which the singular cone propagates with, respectively, the velocity $c/(\epsilon)^{\frac12}$ (inside) and c (outside).

(21) represents the refracted cone of (20), due to the change of refraction index at the surface from $(\epsilon)^{\frac12}$ to 1. In the case

$$\epsilon>2;\quad 1-(\epsilon-1)\cdot\frac{v^2}{c^2}<0\qquad(22)$$

the cone (20) becomes totally reflected at the surface of the medium and no characteristic cone appears outside.

The characteristic singular cone of the stationary solution (6), (7) is given by Fig. 1. It will

[7] See reference (4), p. 453, Eq. (7.10).

be advantageous in the following discussion to distinguish between the advanced and the retarded part of the cone. Frank and Tamm have shown that the advanced part of the cone corresponds to the advanced potentials in Maxwell's theory while the retarded part is given by retarded potentials. Our stationary solution is, therefore, of the well known type

$$\Phi_2 = \tfrac{1}{2}\cdot(\Phi_{\text{ret}}+\Phi_{\text{adv}}).$$

In the domain outside the cone, the field becomes imaginary.

In order to study the radiation field $\Phi_3(t)$ we shall restrict our attention to the case

$$1-(\epsilon-1)(v^2/c^2)>0$$

i.e., to the case in which the Cherenkov radiation does not become totally reflected inside the medium. It will be sufficient to take into account the main terms of Table I, which are the same as the ones discussed in Section 4 plus the δ-functions. The remaining terms represent a supplementary transition radiation, which is of insignificant intensity, unless v approaches c.

The main terms of Table I lead to the following expression for the vector potential of the radiation field:

$$A_3{}^{(i)} = -\frac{2}{\pi}\frac{e(v/c)}{[\epsilon(v^2/c^2)-1]^{\frac{1}{2}}}\int_0^\infty J_0(k_\rho\rho)dk_\rho \times\int_{[(1-\epsilon)(\epsilon(v^2/c^2)-1)]^{\frac{1}{2}}}^\infty \left\{\frac{2}{1-\alpha^2}-i\pi\delta(1-\alpha)\right\} \times\left\{\frac{\epsilon(k_z/\bar{k}_z)}{\epsilon(k_z/\bar{k}_z)+1}\frac{v\bar{k}_z}{ck_0}\sin\bar{k}_z{}^2z+\frac{1}{\epsilon(k_z/\bar{k}_z)+1}\cos\bar{k}_z z\right\}e^{-ik_0ct}d\alpha + \text{small terms}, \quad (23)$$

with $\alpha=[\epsilon(v^2/c^2)-1]^{\frac{1}{2}}(\bar{k}_z/k_\rho)$.

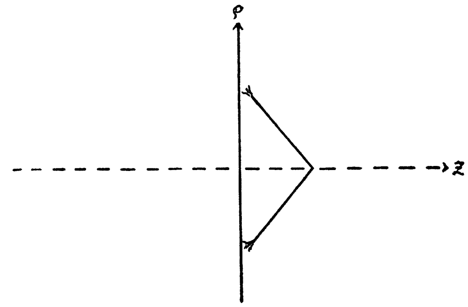

FIG. 2. The characteristic cone of the non-stationary solution for $t>0$ (formation of the Cherenkov cone).

In (23) the small contribution from the totally reflected waves, $\alpha=[(\epsilon-1)(\epsilon(v^2/c^2)-1)]^{\frac{1}{2}}$, has not been taken into account explicitly. The main contribution to the integral (23) comes from the neighborhood of the characteristic cone, $\alpha=1$. In general, (23) represents a radiation field, which shows a marked maximum of intensity in the direction of the characteristic cone. It can be evaluated rigorously at "large" distance from the surface,

$$\lim\frac{\alpha k_\rho z}{[\epsilon(v^2/c^2)-1]^{\frac{1}{2}}}\rightarrow\infty,\ \text{i.e.}\ z\gg(\lambda/2\pi)(v/c).$$

Since, from the physical point of view, we are not interested in very long wave-lengths, we may, therefore, say that the characteristic cone of the radiation field becomes formed behind the surface, after the electron has penetrated into the medium, within a distance of the order of one average optical wave length.

Making use of the integrals

$$\lim_{S\to\infty}\int_{1-\delta}^{1+\delta}\frac{\sin\alpha S}{1-\alpha}d\alpha=-\pi e^{-iS},$$

$$\lim_{S\to\infty}\int_{1-\delta}^{1+\delta}\frac{\cos\alpha S}{1-\alpha}d\alpha=i\pi e^{-iS},\text{*}$$

one finds easily

$$A_3{}^{(i)}\cong\frac{ie(v/c)}{\rho[\epsilon(v^2/c^2)-1]^{\frac{1}{2}}}\int_0^\infty J_0(\eta)\left\{\left[\exp i\eta\frac{z-vt}{\rho[\epsilon(v^2/c^2)-1]^{\frac{1}{2}}}\right]+\frac{\epsilon[1-(\epsilon-1)(v^2/c^2)]^{\frac{1}{2}}-1}{\epsilon[1-(\epsilon-1)(v^2/c^2)]^{\frac{1}{2}}+1}\left[\exp -i\eta\frac{z+vt}{\rho[\epsilon(v^2/c^2)-1]^{\frac{1}{2}}}\right]\right\}d\eta,$$

* The singularity on the real axis is understood to be avoided by deviating the path of integration into the *positive* imaginary half-plane of α.

and observing that

$$\int_0^\infty J_0(\eta)e^{i\beta\eta}d\eta=\begin{cases} +\dfrac{i}{(\beta^2-1)^{\frac{1}{2}}} & \\ \dfrac{1}{(1-\beta^2)^{\frac{1}{2}}} & \text{for } \begin{cases}\beta>1\\ -1<\beta<+1\\ \beta<-1\end{cases} \\ -\dfrac{i}{(\beta^2-1)^{\frac{1}{2}}} & \end{cases} \tag{24}$$

we obtain

$$A_3^{(i)}=\bar{A}_3^{(i)}+\bar{\bar{A}}_3^{(i)},$$

with

$$\bar{A}_3^{(i)}=\begin{cases} -\dfrac{ev/c}{[(1-\epsilon(v^2/c^2))\rho^2+(z-vt)^2]^{\frac{1}{2}}} & \\ +\dfrac{iev/c}{[(\epsilon(v^2/c^2)-1)\rho^2-(z-vt)^2]^{\frac{1}{2}}} & \text{for } \dfrac{z-vt}{\rho(\epsilon(v^2/c^2)-1)^{\frac{1}{2}}}\begin{cases}>+1\\ <+1\\ >-1\\ <-1\end{cases} \\ +\dfrac{ev/c}{[(1-\epsilon(v^2/c^2))\rho^2+(z-vt)^2]^{\frac{1}{2}}} & \end{cases}$$

and

$$\bar{\bar{A}}_3^{(i)}=\begin{cases} \dfrac{\epsilon[1-(\epsilon-1)(v^2/c^2)]^{\frac{1}{2}}-1}{\epsilon[1-(\epsilon-1)(v^2/c^2)]^{\frac{1}{2}}+1}\,\dfrac{ev/c}{[(1-\epsilon(v^2/c^2))\rho^2+(z+vt)^2]^{\frac{1}{2}}} & \\ \dfrac{\epsilon[1-(\epsilon-1)(v^2/c^2)]^{\frac{1}{2}}-1}{\epsilon[1-(\epsilon-1)(v^2/c^2)]^{\frac{1}{2}}+1}\,\dfrac{iev/c}{[(\epsilon(v^2/c^2)-1)\rho^2-(z+vt)^2]^{\frac{1}{2}}} & \end{cases} \text{for } \frac{z+vt}{\rho[\epsilon(v^2/c^2)-1]^{\frac{1}{2}}}\gtrless 1.$$

Comparing these expressions with the corresponding terms of the stationary solution $A_2^{(i)}$ from (7) and identifying their respective domains in Fig. 1 we find, finally, for the total solution for $t>0$ at sufficiently large distance from the surface inside the medium

$$A_2^{(i)}+A_3^{(i)}\cong\begin{cases} \dfrac{2e(v/c)}{[(1-\epsilon v^2/c^2)\rho^2+(z-vt)^2]^{\frac{1}{2}}} & \text{inside the retarded cone,} \\ 0 & \text{outside the retarded cone,}\end{cases} \tag{25}$$

(25) corresponds exactly to the solution of Tamm and Frank and is valid to the approximation to which the terms belonging to the transition radiation are negligible. Figure 2 shows the domain of the solution (25), which is identical with Cherenkov's cone.

6. CONCLUSIONS

The results of the developments given above are the following:

1. The existence of a transition radiation, of the order of one visible photon per hundred incident electrons, at electron velocities slightly below $c/(\epsilon)^{\frac{1}{2}}$, i.e., at the transition point to the Cherenkov effect (Section 4).

2. The justification and the limits of the solution given by Tamm and Frank for the Cherenkov radiation, in particular the justification for the use of retarded potentials (outgoing waves) only (Section 5).

3. The verification that the description of the phenomenon depends essentially on the behavior of non-uniform analytic functions (square roots). We may learn from this fact, that already the simple electrostatic solution, e/r, has to be written more explicitly as

$$\frac{e}{\pm(x^2+y^2+z^2)^{\frac{1}{2}}}$$

and that its singularity represents a branch point.