From the above data the ratio of reaction cross sections for the production of Kr^{77} and Kr^{79} by alpha-particle bombardment of selenium is determined to be 1.4.

IV. THE 4.6-HOUR Kr⁸⁵ ISOTOPE

The 4.6-hour Kr^{85} period was reported by Snell[1] as a result of deuteron bombardment of krypton. The Kr^{85} period was not obtained by Clancy[3] as a result of alpha-particle bombardment of selenium. In the latter bombardments, only a 114-minute krypton period, assigned to the excited level of stable Kr^{83}, and a 33-hour krypton period, assigned to $Kr^{79, 81}$, were observed. The Kr^{85} period was then considered by Clancy to be long or fairly short, and a 4-hour krypton activity obtained from deuteron bombardment of krypton was presumed to be caused by Kr^{87}. The 4.6-hour Kr^{85} activity has been reported by Seelmann-Eggebert and Born[5] as a result of uranium fission by decay of the 3-minute Br^{85}

siotope, and also as a result of $Sr(n, \alpha)$ and $Rb(n, p)$ reactions.[6]

A 4.4 ± 0.2-hour half-life of β^--activity appears in Fig. 2, which shows the decay curves of separated β^- and β^+ krypton activity from alpha-particle bombardment of Hilger selenium. This curve indicates that the Kr^{85} isotope is produced by alpha-particle bombardment of selenium.

Figure 4 shows the Se, Br, Kr part of the periodic table. The new Kr^{77} isotope and new reactions found are noted by heavy lines.

ACKNOWLEDGMENTS

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[5] W. Seelmann-Eggebert and H. J. Born, Naturwiss. 31, 59 (1943).

[6] H. J. Born and W. Seelmann-Eggebert, Naturwiss. 31, 86 (1943).

On the Angular Distribution in Nuclear Reactions and Coincidence Measurements

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Theorems concerning the general form of the angular distribution of products of nuclear reactions and distintegrations are derived. These theorems are based only on the invariance properties of the physical process under space rotation and under inversion. The following examples are studied in detail: (i) angular correlation between the electron and the neutrino in β-decay; (ii) angular correlation between a β-ray and a γ-ray emitted in succession by a nucleus; and (iii) angular correlation between two γ-rays emitted in succession by a nucleus.

INTRODUCTION

IN the calculation of the angular distribution in nuclear reactions and of the angular correlation in processes involving β- and γ-decay it often happens that many terms cancel out at the end of a laborious computation. The consistency of the occurrence of such cancellation leads one to suspect that some general reasons quite independent of the particular form of interaction are at work. In this paper we shall

show that this is indeed the case. In fact, the general form of the angular distribution in many cases can be obtained directly from the theorems derived in this paper.

For nuclear reactions between spinless particles the existence of a limitation on the complexity of the angular distribution for fixed orbital angular momentum of the incoming particles is well known. That the same result holds *with the spin* taken into consideration (for un-

polarized incoming beam) was first pointed out by Critchfield and Teller.[1] A proof of this statement was recently given by Eisner, Sachs, and Wolfenstein.[2] We shall in this paper formulate a new proof that lends itself easily to generalization to the case in which the particles involved have relativistic velocities.

It will be shown in general that in studying the angular correlation between two particles, as long as one of them has a wave-length long compared to the size of the nucleus, the process can be classified into different orders and for a process of given order the general form of the angular correlation is essentially known. In case both of the particles have long wave-lengths, particularly simple conclusions may be reached, as in the case of β-neutrino correlation in β-decay.

Experimentally the angular correlations β-neutrino and $\gamma-\gamma$ have been studied by many authors. Various calculations of these correlations based on different kinds of interactions have also been made. These will be separately discussed in the different sections.

NUCLEAR REACTION

Consider the following reaction,

$$A+P\rightarrow B+Q, \qquad (1)$$
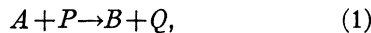
and suppose both the target nucleus A and the bombarding beam of particles P are unpolarized. The complexity of the angular distribution of the outgoing particles is limited by the following theorem: *If only incoming waves of orbital angular momentum L contribute appreciably to the reaction, the angular distribution of the outgoing particles in the center of mass system is an even polynomial of $\cos\theta$ with maximum exponent not higher than $2L$.* Here θ is the angle between the incoming and the outgoing particles in the center-of-mass system of reference.

To prove this let us consider the collision between two particles A and P with definite ($=a$ and p) components of spin along the z axis, and definite total and z component relative orbital angular momenta L and m. (We use the center-of-mass system throughout.) The incoming wave

[1] C. L. Critchfield and E. Teller, Phys. Rev. **60**, 10 (1941).
[2] E. Eisner and R. G. Sachs, Phys. Rev. **72**, 680 (1947); L. Wolfenstein and R. G. Sachs, Phys. Rev. **73**, 528 (1948).

function is, at large distances r_{AP} between A and P:

$$\frac{1}{r_{AP}}\sin(k_{AP}r_{AP}-\tfrac{1}{2}L\pi)\psi_A{}^a\psi_P{}^p Y_{Lm}(\theta_P, \phi_P), \qquad (2)$$

where $\psi_A{}^a$, $\psi_P{}^p$ are normalized internal wave functions of particles A and P; θ_P, ϕ_P describe the direction of motion of the particle P; and $Y_{Lm}(\theta, \phi)$ is the normalized spherical harmonics of order Lm.

The asymptotic behavior of the wave function at large values of r_{BQ} is of the form

$$\frac{1}{r_{BQ}}\exp(ik_{BQ}r_{BQ})\sum_{b,q}\psi_B{}^b\psi_Q{}^q f_{bq}{}^{apm}(\theta_Q, \phi_Q). \qquad (3)$$

In reaction (1), if we choose as the z axis the direction of motion of particle P, it is clear that when the incoming wave is expanded into partial waves with definite total and z component orbital angular momenta L and m, only terms with $m=0$ occur. Under the assumption stated in the theorem we can neglect all terms except the spherical harmonic Y_{L0}. The differential cross section of reaction (1) is, therefore,

$$d\sigma = (\text{constant})d\Omega_Q \sum_{b,q}|f_{bq}{}^{ap0}(\theta_Q, \phi_Q)|^2. \qquad (4)$$

For unpolarized incoming particles we get

$$d\sigma = (\text{constant})d\Omega_Q \sum_{apbq}|f_{bq}{}^{ap0}(\theta, \phi)|^2. \qquad (5)$$

The requirement of invariance under rotation will now be introduced. Consider a new coordinate system (primed system) obtained from the old by a rotation of the coordinate axis. Let $(m'/m)^{(L)}$ be the matrix element[3] of the irreducible representation $\mathfrak{D}^L$ of the three-dimensional rotation group. We have

$$Y_{Lm'}(\theta', \phi') = \sum_m(m'|m)^{(L)}Y_{Lm}(\theta, \phi),$$
$$\psi_A{}'^{a'} = \sum_a(a'|a)^{(S_A)}\psi_A{}^a, \qquad (6)$$

where S_A = spin of particle A, which may be an integer or a half-odd integer. By $\psi'_A{}^{a'}$ is meant the function $\psi_A{}^{a'}$ of the *primed* internal coordinates. The proof of our theorem consists in showing that

[3] E. Wigner, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren* (Braunschweig, 1931), p. 180.

(a) the superposition principle requires that f be transformed according to $\mathfrak{D}^{S_A} \times \mathfrak{D}^{S_P} \times \mathfrak{D}^L \times \mathfrak{D}^{S_{B^*}} \times \mathfrak{D}^{S_{Q^*}}$ and (b) the expression

$$I^{mm'}(\theta, \phi) = \sum_{apbq} [f_{bq}{}^{apm}(\theta, \phi)]^* f_{bq}{}^{apm'}(\theta, \phi) \qquad (7)$$

transforms according to

$$\mathfrak{D}^{L^*} \times \mathfrak{D}^L = \mathfrak{D}^{2L} + \mathfrak{D}^{2L-2} + \cdots.$$

The linear combinations of (7) that transform according to $\mathfrak{D}^{2L-1}$, $\mathfrak{D}^{2L-3}$, etc., vanish identically.

(a) Consider the following incoming wave

$$\frac{1}{r_{AP}} \sin(k_{AP} r_{AP} - \tfrac{1}{2} L \pi) \psi_A{}'^{a'} \psi_P{}'^{p'} Y_{Lm'}(\theta_P', \phi_P'). \quad (2')$$

To an observer in the primed coordinate system this has exactly the same form as (2). Hence the outgoing wave must be

$$\frac{1}{r_{BQ}} \exp(ik_{BQ} r_{BQ}) \sum_{b'q'} \psi_B{}'^{b'} \psi_Q{}'^{q'} f_{b'q'}{}^{a'p'm'}(\theta_Q', \phi_Q'). \qquad (3')$$

Notice that we use the *same f* instead of an f', because there is *no physically observable distinction* between the two coordinate systems. Using (6) one can. express (2') as a superposition of waves (2)

$$\sum_{apm} (a'|a)(p'|p)(m'|m) \left[\frac{1}{r_{AP}} \sin(k_{AP} r_{AP} - \tfrac{1}{2} L \pi) \right. $$
$$\left. \times \psi_A{}^a \psi_P{}^p Y_{Lm}(\theta_{P'} \phi_P) \right].$$

Here we have omitted the superscripts S_A, S_P, L from $(a'|a)^{(S_A)}$, $(p'|p)^{(S_P)}$, $(m'|m)^{(L)}$ for simplicity. The outgoing wave must therefore be a corresponding superposition of waves (3) with the same coefficients:

$$\sum_{apm} (a'|a)(p'|p)(m'|m) \left[\frac{1}{r_{BQ}} \exp(ik_{BQ} r_{BQ}) \right.$$
$$\left. \times \sum_{bq} \psi_B{}^b \psi_Q{}^q f_{bq}{}^{apm}(\theta_{Q'} \phi_Q) \right].$$

Equating this to (3') and using (6) to express $\psi_B{}'^{b'}$, $\psi_Q{}'^{q'}$ in terms of $\psi_B{}^b$, $\psi_Q{}^q$, we get finally, by identifying the coefficient of $\psi_B{}^b \psi_Q{}^q$,

$$\sum_{apm} (a'|a)(p'|p)(m'|m) f_{bq}{}^{apm}(\theta, \phi)$$
$$= \sum_{b'q'} (b'|b)(q'|q) f_{b'q'}{}^{a'p'm'}(\theta', \phi').$$

This reduces to the following form

$$f_{b'q'}{}^{a'p'm'}(\theta', \phi') = \sum_{apmbq} (a'|a)(p'|p)(m'|m)$$
$$\times (b'|b)^*(q'|q)^* f_{bq}{}^{apm}(\theta, \phi) \qquad (8)$$

through the orthogonality relations

$$\sum_m (m'|m)(m''|m)^* = \delta_{m'm''}. \qquad (9)$$

Equation (8) expresses the transformation property of f.

(b) To obtain the transformation property of expression (4) we investigate the behavior of expression (7) under rotation. By (8) and (9)

$$I^{m''m'}(\theta', \phi') = \sum_{abpq} \sum_{mm'''} (m''|m''')^*(m'|m)$$
$$\times [f_{bq}{}^{apm'''}(\theta, \phi)]^* f_{bq}{}^{apm}(\theta, \phi) \qquad (10)$$
$$= \sum_{mm'''} (m''|m''')^*(m'|m) I^{m'''m}(\theta, \phi).$$

Now the differential cross section is proportional to I^{00}. If we put in (10) $m'' = m' = 0$ and take the rotation from the unprimed to the primed coordinate system to be a rotation around the z axis by an angle ξ we have $\theta' = \theta$ and $\varphi' = \varphi + \xi$. Since then $(0|m) = \delta_{0m}$ it is evident that

$$I^{00}(\theta, \varphi + \xi) = I^{00}(\theta, \varphi),$$

showing that I^{00} is independent of φ. To study its dependence on θ we put in (10) $m' = m'' = 0$. It is well known[4] that if $\theta = \varphi = 0$,

$$(0|m) = Y_{L, -m}(\theta', \varphi').$$

Hence (10) becomes

$$I^{00}(\theta', \varphi') = \sum_{mm'''} Y_{L, -m'''}{}^*(\theta', \phi')$$
$$\times Y_{L, -m}(\theta', \varphi') I^{m'''m}(0, 0).$$

On application of the reduction theorem[4] of products of spherical harmonics this leads directly to our theorem.

If instead of a rotation we had chosen an inversion of the coordinates, it is evident that (8)

[4] H. Bethe, *Handbuch der Physik*, (Springer, 1933) Vol. 24/1, Chapter 3, Section 65.

would become

$$f_{bq}{}^{apm}(\pi-\theta, \pi+\phi)$$

$$= P_A P_P P_B P_Q(-1)^L f_{bq}{}^{apm}(\theta, \varphi), \quad (11)$$

where P_A, P_P, etc., are the intrinsic parities of the nuclei. This shows that

$$|f(\pi-\theta, \pi+\varphi)|^2 = |f(\theta, \varphi)|^2,$$

and it follows that the angular dependence must be an even function of $\cos\theta$, a fact that is already established by (10). Equation (11) further shows that any odd power of $\cos\theta$ in the angular dependence must come from an interference term between orbital wave functions of opposite parity.

The symmetry requirements of the wave function under interchanges of the nucleons do not, in general, lead to any new conclusions about the properties of f.[5] However, in the special case in which the two incoming particles or the two outgoing particles are identical, more detailed consideration is necessary. An example of such a case is the reaction

$$Li^7 + H^1 \rightarrow He^4 + He^4.$$
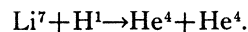
Since the outgoing particles are spinless and satisfy Bose-Einstein statistics and since Li^7 has an odd parity, the value of L must be odd in order to have a balance of parity. This means that $f=0$ unless L is odd. At low energies, therefore, the effective orbital angular momentum is 1.

Another example is the $D^2 + D^2$ reaction:

$$D^2 + D^2 \rightarrow n' + He^3,$$
$$D^2 + D^2 \rightarrow H^1 + H^3.$$
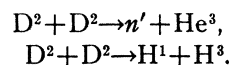
This reaction has recently been considered theoretically by Konopinski and Teller.[6] Because of the symmetry nature of the deuterons it is no longer convenient to specify the spin of the two incoming particles separately. Instead we should group the nine possible incoming states into a quintet, a triplet, and a singlet. The space-wave functions for the quintet and the singlet states are symmetrical with respect to the exchange of the two deuterons, and those for the triplet states are antisymmetrical. Strictly speaking, the proof of our theorem does not apply to such a case where the space-wave function depends on the orientation of the spins of the particles. But since all the states in the same multiplet have the same *a priori* probability, it is evident that the difference of the space-wave function for the different multiplets does not affect the validity of our theorem.

The Coulomb field affects the waves of different orbital angular momenta in such a way as to favor those with higher angular momenta at low energies.[7] This accounts for the reason why at bombarding energies as low as 20 kev the angular distribution in the $D+D$ reaction is not spherically symmetrical.[6,8] We shall not go into this point in any further detail here.

We conclude this section by stating a variation of the theorem proved above: *When contributions from incoming waves with orbital angular momenta $>L$ are neglected, the angular distribution in reaction (1) in the center-of-mass system is a polynomial of $\cos\theta$ with maximum exponent not higher than $2L$. This holds even if the contributing compound nuclear states have angular momenta $>L$.*

It will be noticed that when both even and odd values of the orbital angular momenta in the incoming beam are effective in producing the reaction, the angular distribution contains odd powers of $\cos\theta$. This, however, will not happen when either (a) the reaction goes through a *single* compound nuclear state (e.g. near a strong resonance level); or (b) symmetry requirements exclude even (or odd) L values as in the $Li^7 + H^1 \rightarrow He^4 + He^4$ reaction discussed above.

RELATIVISTIC CASE

We shall in this section generalize the result of the last section to the case when the particle P is an electron and has relativistic velocities. (The nuclei A, B, and Q are still supposed to be non-relativistic.) No such process has been experimentally realized. We shall, however,

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[5] To understand this it is best to introduce the idea of *channels* in the configuration space which was first discussed by G. Breit, Phys. Rev. **58**, 1068 (1940); J. A. Wheeler, Phys. Rev. **52**, 1107 (1937). An interchange of the nucleons in general results in an interchange of the channels, except for the case when either A and P or B and Q are identical.

[6] E. J. Konopinski and E. Teller, Phys. Rev. **73**, 822 (1948).

[7] H. Bethe and E. J. Konopinski, Phys. Rev. **54**, 130 (1938).

[8] E. Bretscher, A. P. French, and F. G. P. Seidl, Phys. Rev. **73**, 815 (1948).

discuss it to illustrate our method. It will be proved that *if only partial waves of orbital angular momentum L in the electron wave function contribute to the reaction, the angular distribution is a polynomial of cosθ with maximum exponent not higher than 2L+1.*

Instead of the stationary picture used in the last section, we shall here use a non-stationary description of the process. The electron wave function at time $t=0$ is a product of a spin wave function with four components and a space-wave function e^{ikz}. The spin of the electron along the z axis is a constant of motion and is denoted by $p(=\pm\frac{1}{2})$. If we expand the space-wave function into partial waves of definite orbital angular momenta L, the first term $(L=0)$ would give rise to allowed transitions, the second term $(L=1)$ first forbidden transitions, etc. To study the angular distribution arising from the contribution of the partial wave of orbital angular momentum L we need to decompose it again into normalized waves ψ_{LJPm} of definite L, J (total angular momentum of the electron), P (parity), and m (z component of J).* The advantage of using these ψ's is that they have simple transformation properties under rotation. The possible values of J are $L\pm\frac{1}{2}$. Under the assumption that we are considering only the contribution from a definite L value, the wave function at $t=0$ can be replaced by

$$\sum_{P=\pm1}\sum_{J=L\pm\frac{1}{2}}\alpha_{LJPp}\psi_{LJPp}. \qquad (12)$$

We have put $m=p$ because the z component of the orbital angular momentum is zero.

Let us now first study the reaction arising from the electron wave ψ_{LJPm}. Starting at $t=0$ with ψ_{LJPm} and nucleus A with a definite value a for the z component of spin, we shall denote by $f_{bq}{}^{LJPam}(\theta_Q, \varphi_Q)$ the *probability amplitude* at any later time $t>0$ for that outgoing state in which the z component of spin of the particles B and Q are b and q, and in which the momentum of Q is in the direction θ_Q, φ_Q. The absolute value of the outgoing momentum (which is not fixed because the energy is not necessarily conserved when t is small) should also enter the function f as an

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* The parity can be either 1 or −1 for any given L, J, and m. However, for slow electrons the amplitude of waves with $P=-(-1)^L$ is very small. Cf. end of this section.

independent variable, but has been omitted for the sake of simplicity in writing.

Now the probability amplitudes are additive when we superpose states. Since under a rotation the different waves ψ_{LJPm} with the same LJP values combine linearly, the argument which led to (8) in the last section would now lead to

$$f_{b'q'}{}^{LJPa'm'}(\theta', \varphi') = \sum_{ambq}(a'|a)(b'|b)^*$$

$$\times (m'|m)(q'|q)^*f_{bq}{}^{LJPam}(\theta, \varphi). \qquad (13)$$

Returning now to the wave (12) at $t=0$, we see that the differential cross section is proportional to

$$d\Omega_Q|\sum_{PJ}\alpha_{LJPp}f_{bq}{}^{LJPap}(\theta, \varphi)|^2.$$

This will have to be summed over a, b, p, and q. Since the coefficients in (12') are independent of a, b and q, the final expression is

$$d\Omega_Q\sum_{JPJ'P'}\sum_p\alpha_{LJPp}\alpha^*_{LJ'P'p}$$

$$\times[\sum_{a,b,q}\{f_{bq}{}^{LJ'P'ap}(\theta, \phi)\}^*f_{bq}{}^{LJPap}(\theta, \phi)]. \qquad (14)$$

By (13) the individual terms under the summation sign $\sum_{abq}$ transform under a rotation according to $\mathfrak{D}^{SA}\times\mathfrak{D}^{SA*}\times\mathfrak{D}^{SB}\times\mathfrak{D}^{SB*}\times\mathfrak{D}^{SQ}\times\mathfrak{D}^{SQ*}\times\mathfrak{D}^{J}\times\mathfrak{D}^{J'*}$. But after the summation over a, b, and q is carried out, the sum transforms more simply according to

$$\mathfrak{D}^{J}\times\mathfrak{D}^{J'*}=\mathfrak{D}^{J+J'}+\mathfrak{D}^{J+J'-1}+\cdots.$$

This means that the expression in the square bracket in (14) is a sum of spherical harmonics of order $\mathfrak{L}\mathfrak{M}$ with $\mathfrak{L}\leq J+J'$. But both J and J' are $\leq L+\frac{1}{2}$. The theorem stated at the beginning of this section follows immediately.

If we introduce the requirement of invariance under inversion, Eq. (14) shows that those terms with $P'P=+1$ give rise to angular correlation functions that are even under the transformation $\theta\rightarrow\pi-\theta$, and those with $P'P=-1$ give rise to odd angular correlation functions. A consequence of this is the following. If the velocity v of the electron is small compared to the velocity of light c, and if the spin wave function of the electron is expanded in powers of v/c, the first term, i.e., the term that does not

vanish as $v \to 0$, is invariant under an inversion. This term would therefore give rise to terms with $P = (-1)^L$. The opposite parity first appears in the next term of the expansion and is proportional to v/c. Hence those terms in (14) with $PP' = -1$ contain a factor v/c. Thus *the odd powers of $\cos\theta$ in the angular correlation have coefficients smaller than the even powers by a factor of v/c.*

β-NEUTRINO CORRELATION

In β-decay we have the particularly simple situation in which both the electron and the neutrino have wave-lengths long compared to the dimension of the nucleus. The argument of the last section can now be applied to both these particles and we can prove that *the angular correlation between the electron and the neutrino emitted in a β-decay is a polynomial of $\cos\theta$ up to a maximum exponent $K+1$, where*[9] $K=0$ *for allowed transitions, $K=1$ for first forbidden transitions, etc.*

The idea of the proof is that for first forbidden transitions one has either $L=1$ for the electron and $L_1=0$ for the neutrino or $L=0$ for the electron and $L_1=1$ for the neutrino. The waves $L=1$ and $L_1=1$ occur together only in second forbidden processes. Now the intensity produced by the $L=0$, $L_1=1$ waves has an angular correlation function that goes up to $\cos\theta$ to the first power, according to the theorem of the last section. Similarly, fixing our attention on the neutrino wave function we can draw the same conclusion about the $L=1$, $L_1=0$ waves. The interference term of the $L=1$, $L_1=0$ waves with the $L=0$, $L_1=1$ waves, however, gives an angular distribution that contains $\cos^2\theta$, which is the highest power of $\cos\theta$ possible for this case.

The proof is as follows. Consider the β-decay

$$A \to B + e + \nu.$$

Let a and b be the z components of the spin of the nuclei A and B, θ_e, ϕ_e and θ_ν, ϕ_ν the directions of motion of the electron and the neutrino, and s and s_1 the spin components of the electron and the neutrino in their respective directions of motion. Starting with the nucleus A at $t=0$,

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[9] Notice that when the interaction involves derivatives of the wave function, as in the Konopinski-Uhlenbeck type of interaction, we always expand the wave function *before* taking the derivatives.

the probability amplitude at any later time t of the β-decay for given θ_e, ϕ_e, θ_ν, ϕ_ν, s, s_1, a, and b will be denoted by

$$f_{bss_1}{}^a(\theta_e, \varphi_e, \theta_\nu, \phi_\nu). \qquad (15)$$

Now let the electron wave function be expanded into waves Φ_{LJP_s}, as done before in (12), with the only difference that here Φ_{LJP_s} represents a wave function with total angular momentum along the direction θ_e, ϕ_e (instead of along the z axis), equal to s. The coefficients α in (12) remain unchanged. Now Φ_{LJP_s} can be further expanded into waves ψ_{LJPm} with definite total angular momentum along the z axis. The final result is

$$\sum_{LJPm} \alpha_{LJP_s}(s \,|\, m)_e \psi_{LJPm}, \qquad (16)$$

where e represents a rotation of the coordinate axes so that the z axis changes from the direction of motion of the electron (i.e., the direction specified by θ_e, ϕ_e) into the laboratory z axis. It is evident that the choice of the x and y axes perpendicular to the direction θ_e, ϕ_e affects only the phase of Φ_{LJP_s} and would not in any way influence our final result. In (16) $(s \,|\, m)_e$ is the only factor that depends on θ_e, ϕ_e. A similar expansion of the neutrino wave will now be made

$$\sum_{L_1J_1P_1n} \beta_{L_1J_1P_1s_1}(s_1 \,|\, n)_\nu \psi_{\nu L_1J_1P_1n}. \qquad (17)$$

The wave amplitude (15) is evidently given by

$$f_{bss_1}{}^a(\theta_e, \phi_e, \theta_\nu, \phi_\nu) = \sum_{LJPL_1J_1P_1} \sum_{mn}$$

$$\times \alpha_{\lambda s}{}^*\beta_{\lambda_1 s_1}{}^*(s \,|\, m)_e{}^*(s_1 \,|\, n)_\nu{}^* F_{\lambda\lambda_1 bmn}^a, \qquad (18)$$

where λ and λ_1 are abbreviations for LJP and $L_1J_1P_1$. We have taken the complex conjugates of the waves (16) and (17) because they represent final states. In (18) F represents the probability amplitude of the final state specified by b, $\psi_{\lambda m}$, and $\psi_{\nu\lambda_1 n}$, the initial state being specified by a.

The probability of the β-decay is proportional to

$$\sum_{abss_1} |f_{bss_1}{}^a(\theta_e, \varphi_e, \theta_\nu, \varphi_\nu)|^2. \qquad (19)$$

Writing

$$\sum_{a,b} F_{\lambda\lambda_1 bmn}^a (F_{\lambda\lambda_1 b\overline{m}\overline{n}}^a)^* = G_{\lambda m\overline{m}n\overline{n}}, \qquad (20)$$

and

$$\alpha\lambda S^* \beta\lambda_1 s_1{}^* \alpha\bar\lambda S \beta\bar\lambda_1 s_1 = \Gamma_{\Lambda s s_1}, \qquad (21)$$

where Λ is an abbreviation for $\lambda, .\lambda_1, \bar\lambda, \bar\lambda_1,$ expression (19) becomes

$$\sum_{\Lambda s s_1} \Gamma_{\Lambda s s_1} \sum_{m\bar m n\bar n} G_{\Lambda m \bar m n}{}_-(s\,|\,m)_e{}^*$$
$$\times (s\,|\,\bar m)_e (s_1\,|\,n)_\nu{}^*(s_1\,|\,\bar n)_\nu. \qquad (22)$$

We shall show later that

$$\sum_{m\bar m n\bar n} G_{\Lambda m \bar m n \bar n}(s\,|\,m)_e{}^*(s\,|\,\bar m)_e$$
$$\times (s_1\,|\,n)_\nu{}^*(s_1\,|\,\bar n)_\nu \qquad (23)$$

is a polynomial of $\cos\theta$ with maximum exponent $\leqq$ both $J+\bar J$ and $J_1+\bar J_1$, θ being the angle between the directions of motion of the electron and the neutrino. But $J=L\pm\frac12$, $J_1=L_1\pm\frac12$. Hence expression (23), which represents the (cross) term in the probability of the β-decay between waves LL_1 and $\bar L\bar L_1$, is a polynomial of $\cos\theta$ with maximum exponent $\leqq$ both $L+\bar L+1$ and $L_1+\bar L_1+1$.

The classification of β-decays into allowed, first forbidden, etc., processes consists of an expansion in powers of $r/\lambda_e \sim r/\lambda_\nu$ $(\sim \frac{1}{10})$, λ_e, λ_ν being the wave-lengths of the electron and the neutrino, and r the dimension of the nucleus. In an allowed transition only the waves $L=0, L_1=0$ are effective for the process. Contributions from other waves are negligible because with increasing values of L the amplitude of the wave ψ_{LJPm} inside the nucleus decreases as $(r/\lambda_e)^L$. In a first forbidden process $F^a{}_{\lambda\lambda_1 bmn}\,|\,_{L=L_1=0}$ vanishes because of selection rules and the contributing waves are the following two:[10] $L=1, L_1=0$ and $L=0, L_1=1$, In general, for a Kth forbidden transition only waves with $L+L'\leqq K$ are important.[9] This means that in the summation over Λ in (22) only $L+L_1\leqq K, \bar L+\bar L_1\leqq K$ terms need be retained. Hence $L_1+\bar L_1\leqq 2K-(L+\bar L)$. Thus the maximum exponent of $\cos\theta$ is $\leqq$ both $L+\bar L+1$ and $2K-(L+\bar L)+1$; hence it is $\leqq K+1$, which proves our theorem.

It remains to be proved that the above statement about (23) is true. This we do by noticing first that F represents the probability amplitude

of the final state b, $\psi_{\lambda m}$, $\psi_{\nu\lambda_1 n}$ if the initial state is represented by a. If R is any rotation of coordinates, $\sum_a (a'/a)_R F^a{}_{\lambda\lambda_1 bmn}$ would give the probability amplitude of these same final states resulting from an initial state obtained by rotating nucleus A in state a' by R^{-1}. Thus

$$\sum_a (a'/a)_R F^a{}_{\lambda\lambda_1 bmn} = \sum_{b'MN} F^{a'}{}_{\lambda\lambda_1 b'MN}$$
$$\times (b'\,|\,b)_R (M\,|\,m)_R (N\,|\,n)_R, \qquad (24)$$

which means that $F_{\lambda\lambda_1}$ is invariant under $\mathfrak{D}^{S_A} \times \mathfrak{D}^{S_B{}^*} \times \mathfrak{D}^{J^*} \times \mathfrak{D}^{J_1{}^*}$. The definition (20) therefore shows that G_Λ is invariant under $\mathfrak{D}^{J^*} \times \mathfrak{D}^{J_1{}^*} \times \mathfrak{D}^J \times \mathfrak{D}^{J_1}$. That is,

$$G_{\Lambda m\bar m n\bar n} = \sum_{M\bar M N\bar N} G_{\Lambda M\bar M N\bar N}(M\,|\,m)_R$$
$$\times (\bar M\,|\,\bar m)_R{}^*(N\,|\,n)_R(\bar N\,|\,\bar n)_R{}^*.$$

Hence

$$\sum_{m\bar m} (M\,|\,m)_R{}^*(\bar M\,|\,\bar m)_R G_{\Lambda m\bar m n\bar n}$$
$$= \sum_{N\bar N} G_{\Lambda M\bar M N\bar N}(N\,|\,n)_R(\bar N\,|\,\bar n)_R{}^*.$$

Putting $R=e$, $M=\bar M=S$, we see that (23) can be written

$$\sum_{N\bar N n\bar n} G_{\Lambda s s N\bar N}(N\,|\,n)_e(\bar N\,|\,\bar n)_e{}^*(s_1\,|\,n)_\nu{}^*(s_1/\bar n)_\nu$$

$$= \sum_{N\bar N n\bar n} G_{\Lambda s s N\bar N}(n\,|\,N)_{e^{-1}}{}^*(\bar n\,|\,\bar N)_{e^{-1}}$$
$$\times (s_1\,|\,n)_\nu{}^*(s_1\,|\,\bar n)_\nu$$

$$= \sum_{N\bar N} G_{\Lambda s s N\bar N}(s_1\,|\,N)_{\nu e^{-1}}{}^*(s_1\,|\,\bar N)_{\nu e^{-1}}. \qquad (23')$$

This is evidently independent of the choice of the laboratory coordinate system. If these be so chosen that $\theta_e=\phi_e=0$, the rotation represented by e becomes the identity and (23') shows that (23) is a polynomial of $\cos\theta$ with maximum exponent $\leqq J_1+\bar J_1$. A similar argument shows that it is also $\leqq J+\bar J$. This completes the proof.

If we fix our attention on one end of the spectrum where the electron momentum p is $\ll$ the neutrino momentum q, the waves that contribute most in a Kth order forbidden transition are those with $L=0, L_1\leq K$. By the theorem proved in the last section we see that *the maximum exponent of $\cos\theta$ in the angular coorelation is 1.* This evidently applies also when $q\ll p$.

If $p\ll mc$ the spin function of the electron can be separated from the space-wave function.

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[10] It may happen that $F^a{}_{\lambda\lambda_1 bmn}\,|\,L=L_1=0$ is not zero but is $\sim F^a{}_{\lambda\lambda_1 bmn}\,|\,L=1, L_1=0$. This happens in the usual interactions because of the presence of terms $\sim$nucleon velocity. In such cases we should include the $L=0, L_1=0$ wave. The conclusions are, however, unchanged as far as they concern only the complexity of the angular correlation.

Hence, after summation over the spin directions of the electron, the maximum exponent is both $\leq L+\bar{L}$ and $J_1+\bar{J}_1 \leq 2K-(L+\bar{L})+1$. We have $L+\bar{L}$ instead of $J+\bar{J}$, as in all non-relativistic cases. Thus *the maximum exponent is K.*

In case $p \ll q$ and $p \ll mc$, only $L=0$ wave is effective and the angular correlation is spherically symmetrical for transitions of any order. Thus when $p \rightarrow 0$ the angular correlation becomes spherically symmetrical. On the other hand, when $q \rightarrow 0$ the angular correlation becomes $1 + \alpha \cos\theta$ or 1 according as the mass of the neutrino is zero or otherwise.

Actual calculations of the angular correlation between the electron and the neutrino emitted in β-decays of different orders have been carried out by Hamilton,[11] using all the five usual types of interactions. The results, of course, conform with the theorems discussed above. Experimentally,[12] information about the angular correlation has been obtained by measuring the energy spectrum of the recoil nuclei or by coincidence measurements of the electrons and the recoil nuclei. Because of the indirect nature of these experiments, the results are not as yet very quantitative.

β–γ AND γ–γ CORRELATIONS

The method used in the last three sections evidently applies also to γ-rays. The rectangular components A_x, A_y, and A_z of the vector potential of the electromagnetic field is expanded into spherical harmonics. As is well known, the term $L=0$ leads to electric dipole processes, the term $L=1$ to magnetic dipole and electric quadrupole processes, etc. For each direction of propagation of the light quantum there are two possible waves with $L=0$, corresponding to the two different polarizations. Changing the direction of propagation we obtain other waves. But altogether there are only three linearly independent waves with $L=0$, and they transform among themselves under a rotation like a vector. Hence *the angular correlation between the γ-ray and any other particle in a nuclear process is of the form $1 + \alpha \cos^2\theta$ if the γ-ray process is of the electric dipole type.*

[11] D. R. Hamilton, Phys. Rev. **71**, 456 (1947).

[12] J. S. Allen, Phys. Rev. **61**, 692 (1942); J. C. Jacobsen, and Kofoed-Hansen, Kgl. Danske Vid. Sels. Math.-Fys. Medd **23**, No. 12 (1945); J. S. Allen, H. R. Paneth, and A. H. Morrish, Bull. Am. Phys. Soc. **23**, No. 3 (1948); C. N. Sherwin, Phys. Rev. **73**, 216 (1948).

TABLE I.

		Nuclear particle	Electron or neutrino	Photon
Name for different approximations	$L=0$	S wave	Allowed	El. dipole
	$L=1$	P wave	First forbidden	Mag. dipole and el. quadrupole
	$L=2$	D wave	Second forbidden	Mag. quadrupole and el. octapole
Power of $\cos\theta$		Even	Even and odd	Even
Max. exponent of $\cos\theta$		$2L$	$2L+1$	$2L+2$

The odd power of $\cos\theta$ does not appear because the photon wave has a definite parity. This conclusion can be immediately generalized into magnetic dipole and electric quadrupole processes where the angular correlation is $1 + \alpha \cos^2\theta + \beta \cos^4\theta$. This holds even when *both* the magnetic dipole and the electric quadrupole transitions are present. Similar theorems obtain in higher multipole processes.

In general, we can study a process with any number of incoming and outgoing particles. We assume that the incoming particles are unpolarized. If one of the particles (whether incoming or outgoing), say P, has a wave-length long compared to the dimension of the space-region in which it interacts with the other particles, the process can be classified according to to the effective orbital angular momentum L of P. The angular correlation between P and any other particle Q in the process would then be a polynomial of $\cos\theta$ with a maximum exponent determined by L, θ being the angle between the directions of propagation of P and Q. The presence of other particles in the process does not affect the result because a summation over the directions of motion and over the spin of these "redundant" particles must always be carried out. We may say that these particles do not produce any preferential direction in space. The general results when P is a nucleon, an electron, or a photon are summarized in Table I.

The application to the angular correlation between successive γ-rays emitted by a nucleus is straightforward. Actual calculations of this correlation for dipole-dipole, dipole-quadrupole, and quadrupole-quadrupole transitions (all electric poles) have been published.[13] They have the

[13] D. R. Hamilton, Phys. Rev. **58**, 122 (1940); experimental evidence has been reported by L. Brady and M. Deutsch, Phys. Rev. **72**, 870 (1947).

form

$1+\alpha \cos^2\theta$ (dipole-dipole, dipole-

quadrupole), (25)

$1+\alpha \cos^2\theta+\beta \cos^4\theta$ (quadrupole-quadrupole),

agreeing with our results. In these calculations the line width of the second γ-ray process is assumed to be large compared to the hyperfine splitting of the atom, so that the lifetime of the intermediate nucleus is small compared to the time required for the nuclear spin to precess appreciably. Also the assumption is made that there is no magnetic dipole transition mixed with the electric quadrupole. It is evident that neither of these assumptions is necessary for the validity of our theorems, and that the angular correlation is quite generally of the form (25). It should be remarked, of course, that in case either of these two assumptions is violated the coefficients α and β in (25) may not have the values tabulated by Hamilton.

Another application is found in the problem of the angular correlation between the electron and the γ-ray emitted by a nucleus in succession. Since one of the particles is a photon, only even powers of $\cos\theta$ can occur in the correlation function. Using Table I, taking the electron to be P, we conclude that *for all allowed β-transitions the correlation is spherically symmetrical.* This appears at first sight very strange because, e.g., for the Gamow-Teller type of interaction the matrix element involves the spin of the nucleus and one would expect that the emission of an electron in a definite direction would result in a preferential distribution of the spin orientation of the intermediate nucleus and hence would affect the angular distribution of the γ-rays. *For first forbidden β-transitions the correlation is $1+\alpha \cos^2\theta$.* Falkoff and Uhlenbeck have made actual calculations for the first forbidden electric dipole process, using various types of β-interactions.[14] As in the $\gamma-\gamma$ case discussed above, we remark here that our conclusions hold independently of any assumption about the lifetime of the intermediate nucleus, and independently of the multipole nature of the γ-radiation. Also it is not necessary to neglect the term in the β-interaction that is proportional to the nucleon velocity.

REMARKS ABOUT OTHER PARTICLES

Table I can be extended to include mesons of spin 0 and 1. The treatment is very similar to the treatment of the electron if we use Kemmer's[15] representation of the meson wave functions. In this representation a scalar meson has a five-component and a vector meson a ten-component wave function. We shall assume that the rest mass is not zero. Let us take a plane wave

$$\psi = \phi \exp((i/\hbar)(\mathbf{p}\cdot\mathbf{x}-Et)) \qquad (26)$$

and expand it into waves with definite orbital angular momentum L. Under a rotation the spin function ϕ is transformed by a matrix S. The total angular momentum can go as high as $L+1$. Notice that this is true for scalar mesons as well as vector mesons.[15] Thus if only orbital wave L contribute to the reaction the angular correlation between a meson and any other particle is a polynomial of $\cos\theta$ with maximum exponent $\leqq 2L+2$.

If further the meson has non-relativistic velocities v, as must actually be the case in order that the wave-length of the meson may be long compared to nuclear dimensions, we can expand ϕ into a power series in v/c.

$$\phi = \phi_0 + \frac{|v|}{c}\phi_1 + \cdots. \qquad (27)$$

It can be readily proved that the following points are true:

(a) ϕ_0 has a definite parity and can be made independent of the direction of the velocity. The theorem proved in the section about nucleons can therefore be applied here and we see that to the order $(v/c)^0$ the angular correlation is an even polynomial of $\cos\theta$ with maximum exponent $\leqq 2L$.

(b) ϕ_1 has a definite parity which is the opposite of that of ϕ_0. Thus the interference term between ϕ_0 and ϕ_1 gives rise to odd powers of $\cos\theta$ only and we have the result that the terms in the angular correlation to the first order of v/c is an odd polynomial of $\cos\theta$.

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[14] D. L. Falkoff and G. E. Uhlenbeck, Bull. Am. Phys. Soc. 22, No. 5 (1947).

[15] N. Kemmer, Proc. Roy. Soc. A173, 91 (1939).