Theory of Scattering Processes*

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The formal infinite series for the probability amplitudes are transformed by (*i*) a regrouping procedure which separates out "repetitive" terms from all orders beyond the second and combines them to produce a common factor multiplying all orders, (*ii*) a procedure of summation to a closed form (essentially an analytical continuation) replacing the above common factor by a generalized energy denominator, and (*iii*) unlimited repetition of (*i*) and (*ii*). Procedure (*ii*) is based on the generalized energy quantity

$$\mathcal{E}_{gh\cdots np}=E_p+\sum_{q\neq gh\cdots np}\frac{V_{pq}V_{qp}}{(q)}+\sum_{qr\neq gh\cdots np}\frac{V_{pq}V_{qr}V_{rp}}{(q)(r)}+\cdots$$

and the formal identity

$$[E-\mathcal{E}_{gh\cdots np}]^{-1}=\frac{1}{(p)}\left[1+\sum_{q\neq gh\cdots n}\frac{V_{pq}V_{qp}}{(p)(q)}+\sum_{qr\neq gh\cdots n}\frac{V_{pq}V_{qr}V_{rp}}{(p)(q)(r)}+\cdots\right]$$

employing the notation $(x)=E-E_x$.

INTRODUCTION

THE present discussion of scattering theory resembles an earlier treatment of the energy eigenvalue problem.[1] Both utilize a formal transformation to modify the energy denominators and reduce simultaneously the number of intermediate states in all orders beyond the second. In the absence of interaction the scattering system and the incident particle are described by the Hamiltonian operator H and the complete ortho-normal set of eigenfunctions ϕ_x with eigenvalue E_x. These functions are obtained as a discrete set by the imposition of a periodic boundary condition in a large cube of side L. Transitions are produced by an interaction operator V. The calculations are facilitated by including the diagonal matrix elements of the interaction operator in H (as a diagonal matrix) and in E_x. What remains of the interaction operator is then denoted by V.

I seek a solution of the equation

$$(E_a-H-V)\psi=0, \tag{1}$$

subject to the asymptotic boundary condition

$$\psi\rightarrow\phi_a+\text{outgoing spherical waves.} \tag{2}$$

Equation (1) can be expressed in the form

$$\psi'=(E_a-H)^{-1}V\psi'+\phi_a, \tag{3}$$

provided that a suitable interpretation, consistent with the asymptotic boundary condition, is found for the inverse operator $(E_a-H)^{-1}$. It is well known that the analytical device of displacing the singular point from the real axis into the upper half of the complex energy plane produces a solution containing no incoming spherical waves.[2] With $E=E_a+i\gamma$. $(\gamma>0)$, Eq. (3) is replaced by

$$\psi'=(E-H)^{-1}V\psi'+\phi_a \tag{4}$$

and all transition probabilities and cross sections are evaluated in the limit $\gamma\rightarrow 0$. In view of the profound difference between Eqs. (1) and (4) it is not surprising that function ψ' does not always, in the limit $\gamma\rightarrow 0$, include the incident wave ϕ_a with unit amplitude.[3]

Equation (4) possesses the formal solution[4]

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[1] E. Feenberg, Phys. Rev. **74**, 206 (1948). Presented at the second Symposium on Applied Mathematics of the American Mathematical Society, July 29–31, 1948.

[2] A. Sommerfeld, *Wellenmechanik* (Frederick Ungar Publishing Company, New York), p. 442; P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford University Press, London, 1930), p. 184; J. Schwinger, *The General Theory of Scattering Processes* (unpublished seminar notes).

[3] An alternative equation $\psi=(E-H)^{-1}V\psi-\phi_a(\phi_a{}^*, (E-H)^{-1}V\psi)+\phi_a=(E-H)^{-1}V\psi-(1/i\gamma)\phi_a(\phi_a{}^*, V\psi)+\phi_a$ has the desirable property of automatically insuring that the incident wave ϕ_a appears with unit amplitude in the wave function.

[4] The notation $(x)\equiv E-E_x$ is used in Eqs. (6) and (7) and in the remainder of the paper.

$$\psi' = \sum_{v=0}^{\infty} \{(E-H)^{-1}V\}^v \phi_a \qquad (5)$$
$$= \sum_x \phi_x S'_{xa}$$

$$S'_{xa}(x\neq a) = \frac{V_{xa}}{(x)} + \sum_b \frac{V_{xb}V_{ba}}{(x)(b)} + \sum_{bc} \frac{V_{xb}V_{bc}V_{ca}}{(x)(b)(c)} + \cdots \qquad (6)$$

$$S'_{aa} = 1 + \sum_b \frac{V_{ab}V_{ba}}{(a)(b)} + \sum_{bc} \frac{V_{ab}V_{bc}V_{ca}}{(a)(b)(c)} + \cdots. \qquad (7)$$

The unsatisfactory character of this solution is evident in the apparent deviation of S'_{aa} from the initially assumed value of unity. Even more disturbing is the fact that

$$\operatorname*{Limit}_{\gamma\to 0} (E-E_a)S'_{aa}$$

appears not to vanish. However, one should not pass adverse judgment too hurriedly on the solution represented by Eq. (6). The more obvious blemishes are removed in the following section.

The explicit consideration of convergence difficulties may be deferred by introducing a suitable convergence factor in the interaction operator V. The matrix elements are then functions of a cutoff parameter λ. I suppose that V_{bc} approaches the value $V_{bc}{}^0$ of the unmodified theory as $\lambda\to 0$ and decreases rapidly with increasing λ for $E_b \neq E_c$. In this manner it is possible to insure the absolute convergence of Eqs. (6) and (7) excluding a finite region, centered about the origin, in the λ, γ plane. Obviously the extent of the excluded region can be reduced by transformations having the general character of analytical continuation. Such transformations may, however, make extremely difficult the direct verification that ψ' is a solution of Eq. (4).

FACTORIZATION OF THE AMPLITUDES

The transformation of Eqs. (6) and (7) required to overcome the difficulties mentioned in the introduction begins with the algebraic identity

$$\sum_{bc\cdots mn} \frac{V_{xb}V_{bc}\cdots V_{mn}V_{na}}{(b)(c)\cdots(m)(n)} \equiv \sum_{bc\cdots mn\neq a} \frac{V_{xb}V_{bc}\cdots V_{mn}V_{na}}{(b)(c)\cdots(m)(n)} + \frac{V_{xa}}{(a)} \sum_{cd\cdots mn} \frac{V_{ac}V_{cd}\cdots V_{mn}V_{na}}{(c)(d)\cdots(m)(n)}$$
$$\sum_{b\neq a} \frac{V_{xb}V_{ba}}{(a)(b)} \sum_{d\cdots mn} \frac{V_{ad}\cdots V_{mn}V_{na}}{(d)\cdots(m)(n)} + \cdots + \sum_{bc\cdots l\neq a} \frac{V_{xb}V_{bc}\cdots V_{la}}{(a)(b)(c)\cdots(l)} \sum_n \frac{V_{an}V_{na}}{(n)}. \qquad (8)$$

With the aid of Eq. (8) the infinite series of sums denoted by $S'_{xa}(x\neq a)$ can be rearranged in the factored form

$$S'_{xa}(x\neq a) = \frac{1}{(x)}\left[V_{xa} + \sum_{b\neq a} \frac{V_{xb}V_{ba}}{(b)} + \sum_{bc\neq a} \frac{V_{xb}V_{bc}V_{ca}}{(b)(c)} + \cdots\right] S'_{aa}. \qquad (9)$$

It is immediately clear that a failure of the condition $S'_{aa}=1$ does not in fact complicate the theory nor does it create difficulties for the physical interpretation. The reduced amplitudes

$$S_{xa} = S'_{xa}/S'_{aa} \qquad (10)$$

yield a solution in which the coefficient of the incident wave has precisely the value required by the asymptotic boundary condition stated in Eq. (2):

$$\psi = \sum_x \phi_x S_{xa},$$
$$S_{aa} = 1,$$
$$S_{xa}(x\neq a) = \frac{1}{(x)}\left[V_{xa} + \sum_{b\neq a} \frac{V_{xb}V_{ba}}{(b)} + \sum_{bc\neq a} \frac{V_{xb}V_{bc}V_{ca}}{(b)(c)} + \cdots\right]. \qquad (11)$$

The reduced amplitudes are characterized by one significant property: they are all linear functionals of the incident wave ϕ_a. *There are no repetitive terms involving the state a as an inter-*

mediate station on the road from a to x. Eq. (11) establishes the result that such repetitive terms are devoid of physical meaning.

A second algebraic identity is useful in factoring out repetitive terms involving the final state x as an intermediate station on the road from a to x:

$$\sum_{bc\cdots mn\neq a}\frac{V_{xb}V_{bc}\cdots V_{mn}V_{na}}{(b)(c)\cdots(m)(n)} \equiv \sum_{bc\cdots mn\neq ax}\frac{V_{xb}V_{bc}\cdots V_{mn}V_{na}}{(b)(c)\cdots(m)(n)} + \sum_{bc\cdots lm\neq a}\frac{V_{xb}V_{bc}\cdots V_{mx}}{(b)(c)\cdots(m)}\frac{V_{xa}}{(x)} + \sum_{bc\cdots l\neq a}\frac{V_{xb}V_{bc}\cdots V_{lx}}{(b)(c)\cdots(l)} \times\sum_{n\neq ax}\frac{V_{xn}V_{na}}{(x)(n)}+\cdots+\sum_{b\neq a}\frac{V_{xb}V_{bx}}{(b)} \times\sum_{de\cdots mn\neq ax}\frac{V_{xd}V_{de}\cdots V_{mn}V_{na}}{(x)(d)(e)\cdots(m)(n)}. \quad (12)$$

Proceeding as in the derivation of Eq. (11) one finds that the amplitude S_{xa} factors into

$$(x)S_{xa}=\left[V_{xa}+\sum_{b\neq ax}\frac{V_{xb}V_{ba}}{(b)}+\sum_{bc\neq ax}\frac{V_{xb}V_{bc}V_{ca}}{(b)(c)}+\cdots\right]\cdot\left[1+\sum_{b\neq a}\frac{V_{xb}V_{bx}}{(x)(b)}+\sum_{bc\neq a}\frac{V_{xb}V_{bc}V_{ca}}{(x)(b)(c)}+\cdots\right]. \quad (13)$$

Dirac and others have shown that the first order pole $1/(x)$ in the amplitude S_{xa} has the consequence that only states for which energy is conserved occur in the asymptotic wave function. What then can be the meaning of the additional powers of $1/(x)$ (of arbitrarily high order) present in the second square bracket? A partial and perhaps not altogether satisfactory answer is that this factor is meaningless as it stands, but can be given meaning by a process of analytical continuation. One may however consider the possibility that the theory requires modification in the sense proposed by Heitler, i.e., the more-or-less general omission of repetitive terms. Equation (13) suggests the simple modification

$$(x)S_{xa}=V_{xa}+\sum_{b\neq ax}\frac{V_{xb}V_{ba}}{(b)}+\sum_{bc\neq ax}\frac{V_{xb}V_{bc}V_{ca}}{(b)(c)}+\cdots, \quad (14)$$

in which neither a nor x occurs as an intermediate station on the road from a to x.

THE REGROUPING PROCEDURE

The set of functions

$$E_{gh\cdots np}=E_p+\sum_{q\neq gh\cdots np}\frac{V_{pq}V_{qp}}{(q)}+\sum_{qr\neq gh\cdots np}\frac{V_{pq}V_{qr}V_{rp}}{(q)(r)}+\cdots \quad (15)$$

represents a formal generalization of the complex energies (displaced energy levels and associated damping constants) occurring in the energy denominators of the Dirac resonance scattering theory. To establish the connection with energy denominators, observe that (for $p\neq gh\cdots n$)

$$[E-\mathcal{E}_{gh\cdots np}]^{-1}=\frac{1}{(p)}\sum_{v=0}^{\infty}\left[\frac{\mathcal{E}_{gh\cdots np}-E_p}{(p)}\right]^v =\frac{1}{(p)}\left[1+\sum_{q\neq gh\cdots n}\frac{V_{pq}V_{qp}}{(p)(q)}+\sum_{qr\neq gh\cdots n}\frac{V_{pq}V_{qr}V_{rp}}{(p)(q)(r)}+\cdots\right]. \quad (16)$$

To verify the equality, the right-hand member of the first line may be expanded as a sum of products in which the variable indices run over all values excluding $gh\cdots np$. The second line can be expressed similarly, again restricting the variable indices to values different from $gh\cdots np$. The following example serves to illustrate the procedure as regards the expansion of the sums in the second line of Eq. (16):

$$\sum_{qrs\neq gh\cdots n}\frac{V_{pq}V_{qr}V_{rs}V_{sp}}{(p)(q)(r)(s)}=\sum_{qrs\neq gh\cdots np}\frac{V_{pq}V_{qr}V_{rs}V_{sp}}{(p)(q)(r)(s)}+\frac{1}{(p)^2}\left[\sum_{q\neq gh\cdots np}\frac{V_{pq}V_{qp}}{(q)}\right]^2. \quad (17)$$

Now Eq. (13) reduces to

$$S_{xa}=\frac{1}{E-\mathcal{E}_{ax}}\left[V_{xa}+\sum_{b\neq xa}\frac{V_{xb}V_{ba}}{(b)}+\sum_{bc\neq xa}\frac{V_{xb}V_{bc}V_{ca}}{(b)(c)}+\cdots\right], \quad (18)$$

a form similar to Eq. (14), but differing from it in important respects. Continuing the development of the regrouping procedure, write

$$(E-\mathcal{E}_{ax})S_{xa}=V_{xa}+\sum_{b\neq xa}\frac{V_{xb}}{(b)}\times\left[V_{ba}+\sum_{c\neq xa}\frac{V_{bc}V_{ca}}{(c)}+\cdots\right], \quad (19)$$

and apply the procedure leading from Eq. (11) to Eq. (13) to the factor in square brackets. The result is

$$(E-\mathcal{E}_{ax})S_{xa}=V_{xa}+\sum_{b\neq xa}\frac{V_{xb}V_{ba}}{E-\mathcal{E}_{xab}}+\sum_{bc\neq xa}\frac{V_{xb}V_{bc}}{(E-\mathcal{E}_{xab})(c)}\times\left[V_{ca}+\sum_{d\neq xa}\frac{V_{cd}V_{da}}{(d)}+\cdots\right]. \quad (20)$$

Equation (20) is designed to suggest unlimited repetition of the regrouping procedure. The limiting formula for S_{xa}, obtained by induction, is

$$(E-\mathcal{E}_{ax})S_{xa}=V_{ax}+\sum_{b\neq xa}\frac{V_{xb}V_{ba}}{E-\mathcal{E}_{xab}}+\sum_{\substack{b\neq xa\\ c\neq abx}}\frac{V_{xb}V_{bc}V_{ca}}{(E-\mathcal{E}_{xab})(E-\mathcal{E}_{xabc})}+\cdots. \quad (21)$$

In Eq. (21) *all repetitive terms in which a given station on the road from x to a is traversed more than once have been removed from the explicit numerator products* and relegated to relatively harmless positions in the generalized energy denominators.

An immediate consequence of Eq. (15) is

$$S'_{aa}=\frac{E-E_a}{E-\mathcal{E}_a}=\frac{i\gamma}{E_a-\mathcal{E}_a+i\gamma}. \quad (22)$$

Equation (22) contains two important special cases. In the first $E_a-\mathcal{E}_a$ vanishes in the limit as the fundamental volume is allowed to become infinite. Then $S'_{aa}=1$ for all values of γ and $\psi'=\psi$. The second case occurs in connection with the Dirac frequency shift and line breadth.[5] If $E_a-\mathcal{E}_a$ does not vanish Eq. (22) implies

$$\operatorname{Limit}_{\gamma\to 0} S'_{aa}=0, \quad (E_a\neq\mathcal{E}_a). \quad (23)$$

Casual inspection of Eq. (7) suggests that the limiting value is infinity rather than zero. This observation supplies a measure of the profound modification produced in the convergence properties of the wave amplitudes by repeated applications of the regrouping transformation.

The function $\mathcal{E}_{gh\cdots np}$ is itself a proper subject for the regrouping transformation. Regrouping from left to right as in Eq. (21) the result is

$$\mathcal{E}_{gh\cdots np}=E_p+\sum_{q\neq gh\cdots np}\frac{V_{pq}V_{qp}}{E-\mathcal{E}_{gh\cdots npq}}+\sum_{\substack{q\neq gh\cdots np\\ r\neq gh\cdots npq}}\frac{V_{pq}V_{qr}V_{rp}}{(E-\mathcal{E}_{gh\cdots npq})(E-\mathcal{E}_{gh\cdots npqr})}+\cdots. \quad (24)$$

The reverse order yields

$$\mathcal{E}_{gh\cdots np}=E_p+\sum_{q\neq gh\cdots np}\frac{V_{pq}V_{qp}}{E-\mathcal{E}_{gh\cdots npq}}+\sum_{\substack{q\neq gh\cdots np\\ r\neq gh\cdots npq}}\frac{V_{pr}V_{rq}V_{qp}}{(E-\mathcal{E}_{gh\cdots npqr})(E-\mathcal{E}_{gh\cdots npq})}+\cdots. \quad (25)$$

[5] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford University Press, London, 1930).

REFORMULATION

An additional set of functions

$$U_{bgh\cdots mnx;xb}=V_{xb}+\sum_{q\neq bgh\cdots mnx}\frac{V_{xq}V_{qb}}{(q)}+\sum_{qr\neq bgh\cdots mnx}\frac{V_{xq}V_{qr}V_{rb}}{(q)(r)}+\cdots, \quad (26)$$

proves useful in providing a convenient formulation of the preceding results. The subscripts xb represent the indices of a matrix element in the usual sense while $bgh\cdots mnx$ stand for states excluded from the summations involved in the definition of $U_{bgh\cdots mnx;xb}$. Equation (26) is equivalent to an implicit (integral) equation; in particular

$$U_{ax;xa}=V_{xa}+\sum_{b\neq ax}\frac{V_{xb}U_{ax;ba}}{(b)} \quad (27)$$

an equation of interest because the systematic omission of states for which $E_b\neq E_a$ reduces it to the Heitler integral equation.[6]

An immediate consequence of Eqs. (15) and (26) is

$$\mathcal{E}_{gh\cdots mnp}=E_p+U_{gh\cdots mnp;pp}. \quad (28)$$

The regrouping procedure applied to Eq. (26) yields

$$U_{bgh\cdots mnx;xb}=V_{xb}+\sum_{p\neq xbg\cdots mn}\frac{V_{xp}V_{pb}}{E-\mathcal{E}_{xbg\cdots mnp}}+\sum_{\substack{p\neq xbg\cdots mn\\ q\neq xbg\cdots mnp}}\frac{V_{xp}V_{pq}V_{qb}}{(E-\mathcal{E}_{xbg\cdots mnp})(E-\mathcal{E}_{xbg\cdots mnpq})}+\cdots$$

$$=V_{xb}+\sum_{p\neq bg\cdots mn}\frac{V_{xp}U_{xbgh\cdots mnp;pb}}{E-E_p-U_{xbg\cdots mnp;pp}}. \quad (29)$$

At the end of the infinite chain of relations one has

$$U_{ax;xa}=V_{xa}+\sum_{b\neq ax}\frac{V_{xb}U_{xab;ba}}{E-E_b-U_{xab;bb}}. \quad (30)$$

DISCUSSION

The scattering of a particle by a static potential provides a trivial, but illuminating, illustration of the regrouping transformation. With plane waves for the first order eigenfunctions the matrix elements all vary inversely as L^3. Since the density of states in momentum space is proportional to this volume it is apparent from Eq. (15) that

$$\mathcal{E}_{gh\cdots np}=E_p+0(L^{-3}),\quad \gamma\neq 0.$$

Consequently the transformed energy denominators revert back to the original form if the fundamental volume is increased indefinitely while γ is held fixed at a non-zero value. At the same time the sums over intermediate states in Eqs. (11) and (21) approximate to integrals in momentum space. Both Eqs. (11) and (21) yield the same limiting amplitude for a fixed value of γ. However, using Eq. (11), the convergence to the integral form is obviously nonuniform in γ because of the second and higher order poles (in γ) produced by the coincidence of two or more variable indices in the multiple summations. Such duplications and the attendant nonuniform convergence to the limiting integral form are absent in Eq. (21).

Similar remarks apply to the problem of elastic and inelastic scattering of a particle by a composite system. In this case

$$\psi=\sum_{\alpha p}\phi_{\alpha p}S_{\alpha p\alpha_0 p_0} \quad (31)$$

with

$${}_{\alpha p}=\phi_\alpha(\text{scatterer})L^{-\frac{3}{2}}e^{ip\cdot r/\hbar}. \quad (32)$$

The method of Dirac and Sommerfeld yields

$$\psi\longrightarrow-\frac{2\pi L^{\frac{3}{2}}}{h^2c^2}\sum_{\alpha}\phi_\alpha U_{\alpha p\alpha_0 p_0}\frac{e^{ipr/\hbar}}{r} \quad (33)$$

for the asymptotic behavior of ψ. In Eq. (33) ψ is an outgoing spherical wave and $E_{\alpha_0}+E_{p_0}=E_\alpha+E_p$. The differential cross section for scattering into the element of solid angle $d\Omega$ is

$$\sigma_{\alpha p,\,\alpha_0 p_0}d\Omega=\frac{4\pi^2h^2}{c^4}\frac{E_pE_{p_0}p}{p_0}\left|\frac{L^3}{h^3}U_{\alpha p\alpha_0 p_0}\right|^2 d\Omega, \quad (34)$$

using relativistic expressions for the total energy of the incident and scattered particles. To simplify the formulae only the matrix subscripts appear explicitly on the amplitudes U (the full notation is $U_{\alpha_0 p_0\alpha p;\alpha p\alpha_0 p_0}$).

[6] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, 1944), p. 240.

In the limit of infinitely large fundamental volume Eq. (27) assumes the limiting form

$$L^3 U_{\alpha p \alpha_0 p_0} = L^3 V_{\alpha p \alpha_0 p_0} + \frac{1}{c^2 h^3}$$
$$\times \sum_{\alpha'} \int \frac{L^3 V_{\alpha p \alpha' p'} L^3 U_{\alpha' p' \alpha_0 p_0} p' dE p' d\Omega'}{E_{\alpha_0} + E_{p_0} - E_{\alpha'} - E_{p'} + i\gamma}$$
$$= L^3 V_{\alpha p \alpha_0 p_0} + \frac{1}{c^2 h^3}$$
$$\times \sum_{\alpha'} P \left[\int \frac{L^3 V_{\alpha p \alpha' p'} L^3 U_{\alpha' p' \alpha_0 p_0} E_{p'} dE_{p'} d\Omega'}{E_{\alpha_0} + E_{p_0} - E_{\alpha'} - E_{p'}} \right]$$
$$- \frac{i\pi}{c^2 h^3} \sum_{\alpha'} \int L^3 V_{\alpha p \alpha' p'} L^3 U_{\alpha' p' \alpha_0 p_0} p' d\Omega', \quad (35)$$

the symbol P denoting, as usual, the principal value of the following integral. The omission of the principal value term reduces Eq. (35) to the form of Heitler's integral equation. If only $L^3 V_{\alpha p \alpha_0 p_0}$ is retained one has the first order Born approximation.

The derivation of the Thomson scattering cross section, including the classical damping denominator, provides another simple application of the regrouping formalism. However, the calculation is identical with that given by Heitler,[7] since the energy denominators all reduce to the usual difference of unperturbed energies in the limit $L \to \infty$.

From the point of view of the full utilization of the advantages inherent in the regrouping formalism the foregoing applications are trivial. Non-trivial applications first appear in problems involving the interaction of quantized fields with matter. In such problems the energy displacements $U_{gh \cdots np; pp}$ do not vanish in the limit $L \to \infty$; moreover these displacements are complex numbers with negative imaginary components (in the limit $\gamma \to 0$). Thus the factor $(E - \mathcal{E}_{ax})^{-1}$ in Eq. (21) for S_{xa} is non-singular along the real energy axis (certainly an improvement over the singularities of arbitrarily high order at $E_x = E_a$ in S_{xa} defined by Eq. (13)). However, the derivation of Eqs. (33) and (34) clearly reveals that Eq. (21) cannot yield conservation of energy in the classical sense. Classically the total energy of the system is given by the sum of the energies of the several parts (scatterer and scattered particle or particles) when they are well separated in space. In quantized field theories the existence of the interaction reveals itself in the failure of additivity for the energy of field and matter. Even if the real part of the energy shift produced by the interaction is absorbed into a correction to the mass of the scattered particle, there remains the imaginary part of $\mathcal{E}_{ax}$ to cause failure of conservation. Perhaps this result is not unsatisfactory; if, however, it is rejected the modification of the theory by the substitution of (x) for $E - \mathcal{E}_{ax}$ in Eq. (21) offers a workable alternative.

A further important point is that the multiple sums in Eqs. (21), (25) and (29) have precisely the form required for a straight forward transition to multiple integrals in the limit $L \to \infty$. The nature of the complication which is here avoided appears clearly in Eq. (17), the left-hand sum having the limiting form of a double integral plus the square of a single integral.

In the light of these remarks it seems likely that the regrouping formalism will prove useful in the study of field theories, particularly where the true physical content is obscured at present by inadequate perturbation methods.

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[7] Reference 6, p. 249.