Some Effects of the Intrinsic Magnetic Moment of the Electron

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Effects of the magnetic moment of the electron of the six-vector type are calculated. According to the hypothesis followed here, the electron's behavior in the field of a nucleus is described only approximately as that of a point charge. More accurately, it behaves as a particle having mainly the properties of a point charge which are corrected by the addition of a small "intrinsic" magnetic moment. In the present note the purely empirical point of view of an intrinsic magnetic moment is followed without any attempt at a justification from the point of view of quantum electrodynamics which has been given by Schwinger. Formulas are worked out for the effect on hyperfine structure, on the Landé g factor and the contribution to the Lamb-Retherford line shift. The arrangement of the calculations is such as to have the same system for the ordinary hyperfine structure and the other effects. The finite rather than zero size of the nuclear current system is explicitly included in the formulas.

I. INTRODUCTION

THE measurements of the hyperfine structure of hydrogen and deuterium in its ground state made by Nafe, Nelson, and Rabi[1] disagreed with theoretical prediction.[2] The observed energy difference of hyperfine structure levels $h\nu$ appeared to be too large for both isotopes, and the ratio of the energy difference of the lighter to that of the heavier isotope also appeared to be too large in comparison with theory. Part of the latter effect has disappeared as a result of refinements in experimental precision. The remainder of this effect has been explained by A. Bohr[3] as a result of the centering of the electron's wave function on the proton within the deuteron. Unpublished attempts by the writer to explain the factor $\sim$1.0024 by which both ν_H and ν_D exceed theoretical prediction were made during the summer of 1947 on the basis of the distortion of electronic wave functions by the magnetic field of nuclear moments. These were not successful and suggested that any explanation having to do with nuclear properties would be likely to be a forced one because it could not reproduce in a simple way the close equality of fractional discrepancy for H and D. It was natural to look for an explanation in an incomplete understanding of the nature of the electron. This point of view makes the discrepancy between theory and experiment automatically the same for H and D. Since the magnetic moment of the electron in Dirac's theory is the result of the Schroedinger vibratory motion of a point charge and since there are reasons for doubting the exact validity of ordinary space-time concepts, it appeared reasonable to question the exact relationship between the effective magnetic moment and the fundamental constants. It appeared, on the other hand, that it would be safest to make the modification in the theory so as to have a relativistically covariant answer. Pauli's investigation[4] of covariant forms pointed to the interaction energy having the form of Eq. (1) of the next section and it was proposed[5] to consider the possibility of the addition to the Hamiltonian of an interaction energy of the Pauli type. The order of magnitude of the intrinsic Pauli type moment μ_e required by experiment appeared to be a reasonable one because μ_e/μ_0 turned out to be of the order α where μ_0 is the Bohr magneton. In a reference system moving with a velocity close to the velocity of light c the Bohr magneton μ_0 appears as an electric moment equivalent to displacing a charge e through a distance $\hbar/2mc$. A limitation in the validity of space time concepts of the order e^2/mc^2 could conceivably result in a modification of the length $h/2mc$ by the amount e^2/mc^2 and introduce a relative uncer-

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[1] J. E. Nafe, E. B. Nelson, and I. I. Rabi, Phys. Rev. **71**, 914 (1947).

[2] E. Fermi, Zeits. f. Physik **60**, 320 (1930); G. Breit, Phys. Rev. **35**, 1447 (1930). The relativistic corrections of the latter paper are too small to account for the observed discrepancy.

[3] A. Bohr, Phys. Rev. **73**, 1109 (1948). The writer would like to thank to Dr. Bohr for letting him see the manuscript of this paper before publication.

[4] W. Pauli, *Handbuck der Physik* (Verlag Julius Springer, Berlin, 1933), Vol. **24/1**, p. 211.

[5] G. Breit, Phys. Rev. **72**, 984 (1947).

tainty of the order $(e^2/mc^2)/(\hbar/mc)=\alpha$ into the value of the moment.

The writer's first publication contained an error[6] in the evaluation of the expected contribution to the hyperfine splitting of s terms which exaggerated the other expected effects. Among the effects of the moment μ_e which were listed[5] there was mentioned the possibility of finding a characteristic deviation from the Landé g formula. It was noticed by Rabi that this possibility was especially suitable for experimental test and as a result the effect of μ_e was found by Kusch and Foley.[7] Soon after the writer became aware of the development by Schwinger of an improved quantum electrodynamics which predicts $\mu_e/\mu_0=-\alpha/2\pi$ and a more detailed publication appeared to be unwarranted at the time.

In reviewing the calculations which led to the first publication it appeared that there might be considerable work involved in reproducing them or in developing abridged methods. The present note has been, therefore, prepared even though the more vital physical questions involved have in the meanwhile been treated more deeply by others and especially by Schwinger. It contains formulas for the effects of μ_e and compares the effects with those of ordinary hyperfine structure allowing for some of the effects of finite nuclear dimensions. A brief comparison with experimental results is also included.

II. THE INTERACTION ENERGY

The Hamiltonian is taken to be

$$H=-eA_0-c\boldsymbol{\alpha}(\mathbf{p}+e\mathbf{A}/c)-\beta mc^2 +\mu_e[\rho_3(\mathcal{H}\boldsymbol{\sigma})-\rho_2(\mathcal{E}\boldsymbol{\sigma})] \quad (1)$$

in the original Dirac notation, with β standing for Dirac's α_4. The matrices σ_x, σ_y, σ_z are the four-row square matrices. The externally applied electric and magnetic fields are $\mathcal{E}$, $\mathcal{H}$. The Pauli part of the electron's moment is μ_e. The nucleus is considered to be fixed in position. Its magnetic field is taken to be produced by a current system distributed through a volume which is supposed to give rise to a magnetic field

$$\mathcal{H}=[\nabla(\boldsymbol{\mu}_N\nabla)-\boldsymbol{\mu}_N\Delta]u=\mathrm{curl}[\nabla u\times\boldsymbol{\mu}_N], \quad (1.1)$$

where

$$u\sim 1/r. \quad (1.2)$$

Here r is the distance from the center and the asymptotic form for u is supposed to apply for large r. The nuclear magnetic moment is μ_N. The vector potential due to the nucleus is

$$\mathbf{A}_N=[\nabla u\times\boldsymbol{\mu}_N], \quad (1.3)$$

as is seen from Eq. (1.1). According to Eqs. (1.2), (1.3), this vector potential approaches asymptotically that of a magnetic dipole μ_N for any function u which has asymptotically the same gradient as $1/r$. For such u

$$\mathbf{A}_N\sim[\boldsymbol{\mu}_N\times\mathbf{r}]/r^3, \quad (1.3')$$

which is the vector potential of a dipole μ_N. The same is clear from the first expression for $\mathcal{H}$ in Eq. (1.1) according to which

$$\mathcal{H}\sim\nabla(\boldsymbol{\mu}_N\nabla)(1/r), \quad (1.3'')$$

which is directly related to the picture of a dipole as two coincident poles. The validity of this representation does not depend on whether u is assumed to be spherically symmetric or not. It will be assumed that Δu is finite and differentiable as well as continuous everywhere. In this representation the nuclear current density is

$$\mathbf{J}=(c/4\pi)[\boldsymbol{\mu}_N\times\nabla(\Delta u)], \quad (1.3''')$$

and is finite everywhere. For this type of current density one has

$$\mathrm{div}\mathbf{J}=0.$$

If instead of ∇u one had an arbitrary vector in Eq. (1.3) the divergence of the current would not necessarily be zero.

In the special case of

$$u=-r^2/2a^3+3/2a \quad (r<a)$$
$$=1/r \quad (r>a)$$

the current distribution becomes a spherical current sheet on a sphere of radius a and the magnetic field inside the sphere is then $2\boldsymbol{\mu}_N/a^3$. In this case $-\boldsymbol{\mu}_N\Delta u=3\boldsymbol{\mu}_N/a^3$ for $r<a$ and 0 for $r>a$. It has a discontinuity at $r=a$. On the surface of the sphere the current density integrated through the thickness of the sphere is $(3c/4\pi)[\boldsymbol{\mu}_N\times\mathbf{r}]/a^4$. This corresponds to the change of $\Delta u=-3/a^3$ inside the sphere to $\Delta u=0$ outside which gives in the integration of Eq. (1.3'') through the sphere

[6] Julian Schwinger, Phys. Rev. **73**, 415 (1948); G. Breit, Phys. Rev. **73**, 1410, (1948).
[7] P. Kusch and H. M. Foley, Phys. Rev. **72**, 1256 (1947).

thickness the value $3\mathbf{r}/a^4$ for the second factor of the vector product. In this special case Δu has a discontinuity at $r=a$. This case can be considered as the limit of a continuous u and discontinuous $\mathcal{H}$. It is mentioned so as to add concreteness to the meaning of u. In particular, Δu, when integrated through the volume inside the sphere, gives -4π and Δu has a constant value inside the sphere. On account of Eq. (1.2) and Gauss' theorem the integral over-all space is -4π also in the general case.

The interaction energy arising from effects other than the electrostatic energy will be broken up into a sum of three parts.

$$H'=-e(\boldsymbol{\alpha}\mathbf{A}),\quad H''=\mu_e\rho_3(\mathcal{H}\boldsymbol{\sigma}),\quad H'''=-\mu_e\rho_2(\mathcal{E}\boldsymbol{\sigma}). \tag{1.4}$$

The energy H' enters standard discussions of hyperfine structure. For H'' one has

$$H''=\mu_e\rho_3[(\boldsymbol{\sigma}\boldsymbol{\nabla})(\boldsymbol{\mu}_N\boldsymbol{\nabla})-(\boldsymbol{\sigma}\boldsymbol{\mu}_N)\Delta]u. \tag{1.5}$$

Denoting the nuclear spin by $\mathbf{I}$, the nuclear spin quantum number by I and the absolute value of $\boldsymbol{\mu}_N$ by μ_N one has

$$\boldsymbol{\mu}_N=\mu_N\mathbf{I}/I \tag{1.51}$$

and

$$H''=(\mathbf{B}''\mathbf{I}), \tag{1.52}$$

where

$$\mathbf{B}''=(\mu_e\mu_N/I)\rho_3[(\boldsymbol{\sigma}\boldsymbol{\nabla})\boldsymbol{\nabla}-\boldsymbol{\sigma}\Delta]u. \tag{1.53}$$

For a spherically symmetric u

$$\mathbf{B}''=\frac{\mu_e\mu_N}{I}\left\{\left[-\boldsymbol{\sigma}+\frac{\mathbf{r}}{r}\left(\frac{\mathbf{r}}{r}\boldsymbol{\sigma}\right)\right]\Delta u+\left[\boldsymbol{\sigma}-3\frac{\mathbf{r}}{r}\left(\frac{\mathbf{r}}{r}\boldsymbol{\sigma}\right)\right]\frac{u'}{r}\right\}\rho_3. \tag{1.54}$$

The diagonal matrix elements of H'' for hyperfine structure levels of quantum number F are obtained as

$$\Delta E''=\tfrac{1}{2}[F(F+1)-J(J+1)-I(I+1)]\times(\mathbf{B}''\mathbf{J})_J/J(J+1), \tag{1.55}$$

where $\mathbf{J}$ is the operator representing the electronic angular momentum and J is the inner quantum number so that $J(J+1)$ are the characteristic values of $\mathbf{J}^2$. The quantity $(\mathbf{B}''\mathbf{J})_J$ is the diagonal element of $\mathbf{B}''\mathbf{J}$ in the J representation. In the evaluation of this matrix element one needs to know the matrix elements of products of $\mathbf{J}$ with $U\mathbf{r}(\mathbf{r}\boldsymbol{\sigma})r^{-2}\rho_3$ where U can be either Δu or u'/r. For its evaluation as well as that of other quantities it is convenient to use the radial functions f, g for which the radial equations are

$$\begin{cases}(p_0+mc)f-\hbar[dg/dr+(1+k)g/r]=0\\(p_0-mc)g+\hbar[df/dr+(1-k)f/r]=0.\end{cases} \tag{1.6}$$

The Dirac matrix

$$\epsilon=(\boldsymbol{\alpha}\mathbf{r})/r \tag{1.61}$$

corresponds in terms of these wave functions to the multiplication rule

$$\epsilon\begin{pmatrix}f\\g\end{pmatrix}=\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}f\\g\end{pmatrix}=\begin{pmatrix}-i&g\\i&f\end{pmatrix} \tag{1.62}$$

as has been shown by Dirac.[8] This rule means that $\epsilon\psi$ is obtained from ψ by changing the f into $-ig$ and the g into if. In the representation which corresponds to Eq. (1.6) the matrix ρ_3 is diagonal and has diagonal elements 1, 1, -1, -1 in ascending order of μ in ψ_μ. When operating on the two-row-column matrix having f, g as its elements the operator ρ_3 leaves f unchanged and changes g into $-g$. Its effect amounts to a change of sign of g^2 in the final expression. For the calculation of matrix elements of

$$U(\mathbf{J}\mathbf{r})(\mathbf{r}\boldsymbol{\sigma})r^{-2}\rho_3, \tag{1.63}$$

one has

$$(\mathbf{r}\boldsymbol{\sigma})/r=\rho_1\epsilon, \tag{1.64}$$

and hence

$$(\mathbf{J}\mathbf{r})(\mathbf{r}\boldsymbol{\sigma})r^{-2}=(\mathbf{L}\mathbf{r})(\mathbf{r}\boldsymbol{\sigma})r^{-2}+\tfrac{1}{2}(\mathbf{r}\boldsymbol{\sigma})^2r^{-2}=\tfrac{1}{2}(\rho_1\epsilon)^2=\tfrac{1}{2}. \tag{1.65}$$

It follows that

$$(\psi,\ U(\mathbf{J}\mathbf{r})(\mathbf{r}\boldsymbol{\sigma})r^{-2}\rho_3\psi)=\tfrac{1}{2}\int_0^\infty(f^2-g^2)Ur^2dr, \tag{1.66}$$

where the normalization is such that

$$\int_0^\infty(f^2+g^2)r^2dr=1. \tag{1.67}$$

In the result of substituting Eq. (1.54) into Eq. (1.55) there occur, besides, terms of the type

$$U(\boldsymbol{\sigma}\mathbf{J})\rho_3. \tag{1.7}$$

Their evaluation brings in Dirac's quantum num-

[8] P. A. M. Dirac, Proc. Roy. Soc. **A118**, 351 (1928). In accordance with later custom the symbol j of Dirac's article is called k in the present paper.

ber k which is an eigenvalue of the operator

$$k=\rho_3[(\mathbf{L}\boldsymbol{\sigma})+1]. \quad (1.71)$$

One has

$$(\boldsymbol{\sigma}\mathbf{J})=\boldsymbol{\sigma}\mathbf{L}+\tfrac{3}{2} \quad (1.72)$$

and hence according to Eq. (1.71)

$$(\boldsymbol{\sigma}\mathbf{J})\rho_3=\tfrac{3}{2}\rho_3+k-\rho_3=k+\tfrac{1}{2}\rho_3, \quad (1.73)$$

so that

$$(\psi,\ U(\boldsymbol{\sigma}\mathbf{J})\rho_3\psi) = \int_0^\infty [(k+\tfrac{1}{2})f^2+(k-\tfrac{1}{2})g^2]r^2dr. \quad (1.74)$$

Combining Eqs. (1.54), (1.66), (1.74) one obtains

$$(\mathbf{B}''\mathbf{J})_J=\frac{\mu_e\mu_N}{I}\int_0^\infty \left\{-k(f^2+g^2)\left(\Delta u-\frac{u'}{r}\right) + (g^2-f^2)\frac{u'}{r}\right\}r^2dr, \quad (1.75)$$

where $'$ denotes differentiation with respect to r. The terms in u' in this expression arose from corresponding terms in Eq. (1.54). For $u=1/r$ the part of H'' arising from these terms has the same form as the interaction energy between magnetic doublets $\boldsymbol{\mu}_N$, $\mu_e\boldsymbol{\sigma}$. There is present in addition a contribution in Δu. According to Eq. (1.3''') this quantity is confined to nuclear dimensions. For the current sheet considered right after Eq. (1.3'') the value of Δu was seen to be finite everywhere. There are clearly many types of volume distributions of current density for which Δu is finite. The example of the current sheet shows that at $r=0$ the quantity Δu is of the order $-3\langle a^{-3}\rangle$ where the average is taken over the nucleus and a is the radius of the elementary spherical current sheet into which the volume distribution is broken down. The value of Δu at $r=0$ is, therefore, not necessarily much larger than for the spherical current sheet. The functions f and g become infinite at $r=0$ for s terms. The integral of $(f^2+g^2)r^2dr$ converges. In view of the finiteness of Δu the integral of the term involving Δu in Eq. (1.75) converges also.

The results will be considered next for *light nuclei*, i.e., for the limit $Z=0$ where Z is the atomic number. For these nuclei the radial functions f, g differ from their non-relativistic approximations which are finite only in a narrow range of values close to $r=0$. In this case the term in Δu matters only for s terms. It contributes to the right side of Eq. (1.75) the amount

$$-(\mu_e\mu_N/I)g^2(0). \quad (1.8)$$

The function g is related to the non-relativistic Schroedinger function by

$$g^2=4\pi\psi_S^2, \quad (Z=0) \quad (1.81)$$

where subscript S stands for Schroedinger.

The quantity u'/r is $-1/r^3$ outside the nucleus. Inside the spherical current sheet it is $-1/a^3$ which is of the same order as Δu. The integral over the nuclear interior arising from $(u'/r)g^2$ is therefore comparable with that originating from $g^2\Delta u$. Neither of these is important except for s terms. For these, however, the combination $(1+k)g^2$ which occurs with u' vanishes because $k=-1$ for s terms. The integral of $-(1+k)f^2ru'dr$ does not vanish but since f/g is of order αZ it is much smaller than the corresponding contribution of Δu. The terms in u'/r are important, therefore, only for non-s terms. Neglecting the contribution of f^2 to these terms one obtains for their contribution

$$-(\mu_e\mu_N/I)(1+k)\langle r^{-3}\rangle. \quad (1.82)$$

Combining Eqs. (1.75), (1.8), (1.81), (1.82) one obtains the approximation

$$(\mathbf{B}''\mathbf{J})_J\cong(\mu_e\mu_N/I)\{-4\pi\psi_S^2(0)\delta_{L,0} -(1+k)\langle r^{-3}\rangle\delta'_{L,0}\}, \quad (1.83)$$

where $\delta_{L,0}$ is the Kronecker δ and

$$\delta'_{L,0}=1-\delta_{L,0}. \quad (1.84)$$

The term H''' of Eq. (1.4) brings in the interaction which is expected in classical anology on account of the motion of $\boldsymbol{\mu}_e$ in the electric field of the nucleus. This field is

$$\mathcal{E}=Ze\mathbf{r}/r^3, \quad (1.9)$$

and one can express H''' as

$$H'''=-Ze\mu_e\rho_2(\mathbf{r}\boldsymbol{\sigma})/r^3=i\mu_eZe\rho_3\epsilon/r^2. \quad (1.91)$$

In accordance with Eq. (1.62) ϵ operates on f, g and

$$i\rho_3\epsilon=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \quad (1.92)$$

One has thus

$$\langle H'''\rangle=(\psi,\ H'''\psi)=2\mu_eZe\int_0^\infty fgdr. \quad (1.93)$$

The interaction energy H' occurs in the ordinary theory of hyperfine structure. Since the current distribution is not ordinarily taken into account and since the ordinary way of deriving hyperfine structure formulas can be put into a neater form, a brief derivation of the formula for the expectation value of H' will now be sketched. According to Eqs. (1.3), (1.4) and similarly to Eq. (1.55), one has for the first-order perturbation effect of H'

$$E'=\tfrac{1}{2}[F(F+1)-J(J+1)-I(I+1)]\times(\mathbf{B}'\mathbf{J})_J/J(J+1), \quad (2)$$

where

$$\mathbf{B}'=(e\mu_N/I)u'[(\mathbf{r}/r)\times\boldsymbol{\alpha}], \quad (2.1)$$

and only the case of spherically symmetric u is under consideration. The quantity $(\mathbf{B}'\mathbf{J})_J$ is analogous to $(\mathbf{B}''\mathbf{J})_J$ of Eq. (1.55). Its evaluation can be made by noting that

$$\mathbf{J}[\mathbf{r}\times\boldsymbol{\sigma}]=\{(\mathbf{J}\boldsymbol{\sigma})(\boldsymbol{\sigma}\mathbf{r})-(\mathbf{J}\mathbf{r})\}/i, \quad (2.11)$$

as a consequence of multiplication rules for the components of $\boldsymbol{\sigma}$. Multiplying both sides of the above equation on the right by L_1/r one obtains

$$\mathbf{J}[(\mathbf{r}/r)\times\boldsymbol{\alpha}]=(\mathbf{J}\boldsymbol{\sigma})\epsilon/i-(\mathbf{J}\mathbf{r})/ir. \quad (2.12)$$

One also has the following identities

$$(\mathbf{J}\boldsymbol{\sigma})=\rho_3 k+\tfrac{1}{2}, \quad (2.13)$$

$$(\mathbf{J}\mathbf{r})/r=\tfrac{1}{2}\rho_1\epsilon. \quad (2.14)$$

Combining Eqs. (2.12), (2.13), (2.14) there follows

$$\mathbf{J}[(\mathbf{r}/r)\times\boldsymbol{\alpha}]=k\rho_3\epsilon/i. \quad (2.15)$$

The operator k commutes with both ϵ and ρ_3. For a state having a definite eigenvalue k the operator k amounts to multiplication by the c number k. Multiplication of such a state by the right side of Eq. (2.15) amounts, therefore, to multiplication by $\rho_3\epsilon/i$ followed by multiplication by the c number k. For operations on the f, g column matrix of Eq. (1.62)

$$\rho_3\epsilon/i=-\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad (2.16)$$

and hence, combining Eqs. (2.1), (2.12), (2.15), (2.16),

$$(\mathbf{B}'\mathbf{J})_J=2k(e\mu_N/I)\int_0^\infty fg(-u'r^2)dr. \quad (2.2)$$

For a nuclear current distribution concentrated at very small r the factor $(-u'r^2)$ is unity and one has the usual approximation[9]

$$(\mathbf{B}'\mathbf{J})_J\cong 2k(e\mu_N/I)\int_0^\infty fgdr; \quad (a=0). \quad (2.3)$$

In this limiting case the integral occurring in the expression for the hyperfine structure interval factor is the same as that for the effect of the nuclear electric field which occurs in Eq. (1.93). The latter should really be modified also because Eq. (1.9) does not take into account the finite—rather than zero—nuclear volume. This may be done in the approximation of a spherically symmetric charge distribution for which

$$\mathcal{E}=-Ze\mathbf{r}v'/r \quad (2.4)$$

corresponding to an electrostatic potential

$$Zev(r), \quad (2.5)$$

and a revised

$$\langle H'''\rangle=2\mu_e Ze\int_0^\infty fg(-v'r^2)dr. \quad (2.6)$$

For very concentrated charge distributions $-v'r^2=1$ and Eq. (1.93) is obtained again.

III. LIMIT OF SMALL Z AND SMALL NUCLEAR RADIUS

For a small nuclear radius and small Z the effect of μ_e on the hyperfine structure is given by Eqs. (1.55), (1.83). The effect of the interaction of $\mathcal{E}$ and μ_e is given by Eq. (1.93) which is the limiting form of Eq. (2.6) for small nuclear radius. The hyperfine structure arising from effects other than μ_e is obtained from Eqs. (2), (2.3). For Eqs. (1.93), (2.3) one must still make the approximations corresponding to small Z. These are obtained by means of the first Eq. (1.6) which gives

$$f\cong(\hbar/2mc)[dg/dr+(1+k)g/r]. \quad (3)$$

This approximation breaks down for small values of r if the Coulomb potential is supposed to be valid everywhere. The radial integrals involved are, however, familiar ones from the theory of hyperfine structure and it is known that they

[9] G. Breit, Phys. Rev. **38**, 463 (1931). The quantity referred to as **B**′ in the present paper is called **A** in reference 9.

may be approximated by neglecting the difference between p_0+mc and $2mc$ as is done in Eq. (3) and by identifying g with the Schroedinger radial function at small r where g is infinite. The reason why this procedure works is that the contribution to the radial integral of Eq. (2.3) which arises from the small range of values of r within which the approximations are poor is not significant and also because the two approximations have mutually compensating effects. The approximate value of the radial integral is

$$\int_0^\infty fgdr \cong (\hbar/2mc)\left[-2\pi\psi_S^2(0) + (1+k)\langle r^{-3}\rangle\delta'_{L,0}\right], \quad (3.1)$$

so that

$$(\mathbf{B}'\mathbf{J})_J=(2\mu_0\mu_N/I)[2\pi\psi_S^2(0) + k(1+k)\langle r^{-3}\rangle\delta'_{L,0}], \quad (3.2)$$

and

$$\langle H'''\rangle \cong 2Z\mu_0\mu_e[-2\pi\psi_S^2(0) + (1+k)\langle r^{-3}\rangle\delta'_{L,0}]. \quad (3.3)$$

Here the positive number

$$\mu_0=e\hbar/2mc \quad (3.4)$$

is the Bohr magneton. The ratio of the effect of μ_e on hyperfine structure to the normal effect is

$$(\mathbf{B}''\mathbf{J})_J/(\mathbf{B}'\mathbf{J})_J=-\mu_e/\mu_0 (L=0)$$
$$(\mathbf{B}''\mathbf{J})_J/(\mathbf{B}'\mathbf{J})_J=-(1/2k)(\mu_e/\mu_0)(L\neq 0). \quad (3.5)$$

The values of the ratio for different terms are as in Table I.

The effect of an external magnetic field is to give an addition to the energy

$$(\Delta E)_{|\mathcal{H}|}=-\mu_e(\mathcal{H}\boldsymbol{\sigma}), \quad (4)$$

because for small Z multiplication by ρ_3 amounts to multiplication by -1. This result may be compared with

$$\mu_0(\mathcal{H}\boldsymbol{\sigma}), \quad (4.1)$$

which is the contribution to the energy involving $\mathcal{H}$ and $\boldsymbol{\sigma}$ in the absence of μ_e. The effect of μ_e is, therefore, to increase the contribution of the electron spin to the Landé g value by the factor

$$1-\mu_e/\mu_0, \quad (4.2)$$

TABLE I. Values of $(\mathbf{B}''\mathbf{J})_J/(\mathbf{B}'\mathbf{J})_J$ for single electron spectra.

Term =	$s_{1/2}$	$p_{1/2}$	$p_{3/2}$	$d_{3/2}$	$d_{5/2}$
Ratio (μ_e/μ_0) =	-1	$-\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{6}$

so that the modified Landé g factor becomes

$$g'=1+[1-2(\mu_e/\mu_0)]\times\frac{J(J+1)+S(S+1)-L(L+1)}{2J(J+1)}$$
$$=g-2(\mu_e/\mu_0)(g-1), \quad (4.3)$$

which means that

$$g'-1=[1-2\mu_e/\mu_0](g-1). \quad (4.4)$$

Here g is the ordinary Landé g factor.

The experimental discrepancy[1] between the atomic and molecular beam values of the proton's and deuteron's magnetic moments indicated[5] the probable existence of μ_e. In order to obtain the value of μ_e from these experiments it is essential to note that the molecular beam measurements[10] have been made, making use of a calibration of the magnetic field by means of the deflection produced in virtue of the atomic magnetic moment of a single electron s term. According to Eq. (4.4) this moment is $\mu_0-\mu_e$. The molecular beam experiment thus measures

$$\mu_N/(\mu_0-\mu_e). \quad (4.5)$$

The atomic beam experiments on the hyperfine structure of the ground states of hydrogen are essentially independent of applied magnetic fields since they are concerned with absorption frequencies extrapolated to zero field. The theoretically expected value of hyperfine structure separation is proportional to

$$\mu_N(\mu_0-\mu_e), \quad (4.6)$$

in accordance with Eq. (3.5) and Table I. The theoretically expected ratio of the atomic to the molecular beam values is, therefore,

$$(\mu_0-\mu_e)^2/\mu_0^2 \cong 1-2\mu_e/\mu_0. \quad (4.7)$$

This ratio represents the theoretically expected

[10] S. Millman and P. Kusch, Phys. Rev. 60, 91 (1941). The writer is grateful to Professors I. I. Rabi and N. F. Ramsey for discussions concerning the way in which the Columbia group standardized their measurements by measuring different moments in the same magnetic field.

value of the "measured" to the "calculated" value of the hyperfine structure of the ground state where by "calculated" one means calculated neglecting μ_e. The experimental value of this ratio is believed to be[11] close to 1.00242, which indicates that

$$\mu_e/\mu_0 \cong -0.00121. \qquad (4.8)$$

No correction is being made here for the effect of centering of the electron's wave function on the proton within the deuteron which has been brought out by Bohr.[3] The latter effect is an order of magnitude smaller than that of Eq. (4.8). It is absent for the lighter isotope of hydrogen and need not be considered in connection with Eq. (4.8) if only measurements on the lighter isotope are used.

From Eq. (4.3) one obtains for the ratio of two values of g' for two spectroscopic terms I, II

$$\frac{g'_{\mathrm{I}}/g'_{\mathrm{II}}}{g_{\mathrm{I}}/g_{\mathrm{II}}} \cong 1+\frac{2\mu_e}{\mu_0}\left(\frac{1}{g_{\mathrm{I}}}-\frac{1}{g_{\mathrm{II}}}\right)$$

$$\cong 1+0.00242\left(\frac{1}{g_{\mathrm{II}}}-\frac{1}{g_{\mathrm{I}}}\right), \qquad (5)$$

which gives

$$g'(p_{3/2})/g'(p_{1/2}) \cong 2\times 1.00182, \qquad (5.1)$$

$$g'(s_{1/2})/g'(p_{1/2}) \cong 3\times 1.00242 \qquad (5.2)$$

as the expected ratios of measured g values. The first of these may be compared with the experimental value $2(1.00172\pm 0.00006)$ of Kusch and Foley.[7] The disagreement is almost within experimental error. The agreement of the results of Kusch and Foley with expectation is even better if instead of the experimental value -0.00121 of Eq. (4.8) one makes use of Schwinger's[6] value $-\alpha/2\pi=-0.001163$ for this number. The number taking place of 1.00182 on the right side of Eq. (5.1) is then 1.00174(4). The preliminary result of Foley and Kusch[12] for the ratio of the g' for $s_{1/2}$ of Na to the g' for $p_{1/2}$ of Ga is 3.00732 ± 0.00018 which according to Eq. (5) is $1-2\mu_e/\mu_0$ which gives $2\mu_e/\mu_0=-0.00242\pm 0.00006$ in good agreement with the value in Eq. (4.8). It is understood,[13] however, that there is other experimental evidence which indicates that Na is an exception and that Schwinger's $-\alpha/2\pi$ is favored by experiment.

The expectation value of H''' is expressed in terms of a radial integral by means of Eq. (1.93). Substitution of the Schroedinger function approximation for the radial integral gives Eq. (3.3). The electric field is supposed to be central but not necessarily Coulombian in this formula. For the Coulombian case well-known formulas[14] for $\psi_S{}^2(0)$ and $\langle r^{-3}\rangle$ give on substitution into Eq. (3.3)

$$\langle H'''\rangle_s=-2(\mu_e/\mu_0)\alpha^2 Z^4 n^{-3} Ry, \qquad (6)$$

where n is the principal quantum number and Ry is the value of the Rydberg in ergs. Also for the same n the ratio of $\langle H'''\rangle$ for a term with $L\neq 0$ to that for $L=0$ is

$$\langle H'''\rangle/\langle H'''\rangle_s=-1/[k(2L+1)]. \qquad (6.1)$$

In connection with the Lamb-Retherford-Bethe[15] line shift one has substituting numbers into Eq. (6)

$$\begin{aligned}\langle H'''\rangle_{ns}&=-11.64(\mu_e/\mu_0)(Z^4/n^3)\ \mathrm{cm}^{-1}\\ &=-3.49\times 10^5(\mu_e/\mu_0)(Z^4/n^3)\ \mathrm{mc/s}, \qquad (6.2)\end{aligned}$$

$$\begin{aligned}\langle H'''\rangle_{2s}&=-1.455(\mu_e/\mu_0)Z^4\ \mathrm{cm}^{-1}\\ &=-4.36\times 10^4(\mu_e/\mu_0)Z^4\ \mathrm{mc/s}. \qquad (6.3)\end{aligned}$$

For the value of μ_e/μ_0 corresponding to Eq. (4.8) one has for the displacement of $2s$ the value $0.00176\ \mathrm{cm}^{-1}=53$ mc/sec. and for $\mu_e/\mu_0=-\alpha/2\pi$ the displacement is $0.00169(2)\ \mathrm{cm}^{-1}=51$ mc/sec. These amounts are small in comparison with the total shift[15] of about 1000 mc/sec. According to Eq. (6.1) the shifts of the $2p_{1/2}$, $2p_{3/2}$ terms are, respectively, $-\frac{1}{3}$ and $+\frac{1}{6}$ of the shift of the $2s$ term. The expected separation between $2s$ and $2p_{1/2}$ produced by H''' is $0.00234\ \mathrm{cm}^{-1}=70$ mc/sec. for Eq. (4.8) and $0.00226\ \mathrm{cm}^{-1}=68$ mc/sec. for $\mu_e/\mu_0=-\alpha/2\pi$.

IV. CONCLUSION

The addition of a six-vector type of interaction between the electromagnetic field and the electron spin variables to the usual four-vector type is in good agreement with experiments on the

[11] I. I. Rabi, private communication.
[12] H. M. Foley and P. Kusch, Phys. Rev. **73**, 271 (1948).
[13] I. I. Rabi and P. Kusch, private communication.
[14] H. A. Bethe, *Handbuch der Physik* (Verlag Julius Springer, Berlin, 1933), Vol. **24/1**, p. 211.
[15] Willis E. Lamb and Robert C. Retherford, Phys. Rev. **72**, 241 (1947).

hyperfine structure of the hydrogen isotopes and on the Zeeman Effect of gallium and other elements.

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Note in proof. The results of the present paper which matter most for comparison with experiment are the ratios of contributions to the hyperfine structure interval factors and to the spin part of the Landé g factor contained in Eqs. (3.5), (4.2). It appears desirable to supply an explanation of these results making more use of physical pictures than has been done above. This may be done in the following way. The four-component Dirac equation can be reduced to an approximate two-component form which shows[16] the ordinary hyperfine structure in the absence of μ_e as arising from three effects: (a) an interaction of nuclear spin and electronic orbital angular momentum having an r^{-3} dependence and proportional to the scalar product of the two angular momenta; (b) an interaction of the nuclear spin with the electronic spin having the same mathematical form as the interaction energy between two magnetic doublets; (c) an additional interaction energy of nuclear and electronic spins which matters only for s terms, giving in this case the whole hyperfine structure effect.

[16] G. Breit, Phys. Rev. **37**, 51 (1931).

For $L\neq 0$ only effects (a) and (b) matter. When their sum is projected onto the total electronic angular momentum one finds that the answer depends only on L and J in agreement with the fact that the interval factor is proportional to $L(L+1)$. But it is clear that the scalar product of the interaction energy with $\mathbf{J}$ must contain $(\mathbf{L}+\boldsymbol{\sigma}/2)\mathbf{L}$ for the effect (a) and consequently $-(\boldsymbol{\sigma}\mathbf{L})/2$ for effect (b). The ratio of the contribution coming from interactions with electronic spin to the total is, therefore,

$$\begin{aligned}&-(\boldsymbol{\sigma}\mathbf{L})/2L(L+1)\\&=-[J(J+1)-L(L+1)-S(S+1)]/2L(L+1)\\&=1/2k.\end{aligned}\tag{7}$$

For $L=0$ on the other hand only effect (c) matters and the ratio of the electronic spin contribution to the total is unity. One has thus

$$\begin{aligned}\frac{\text{Spin contribution}}{\text{Total}}&=1/2k \qquad (L\neq 0)\\ \frac{\text{Spin contribution}}{\text{Total}}&=1 \qquad (L=0)\end{aligned}\tag{7.1}$$

which differs from Eq. (3.5) only through the absence of the factor $-\mu_e/\mu_0$. This is as it should be because the contribution of μ_e arises, in the approximation of small Z from a spin current which differs from the current that gives rise to ordinary hyperfine structure in the absence of μ_e the proportionality factor $-\mu_e/\mu_0$. This is seen to fit in with Eq. (4.2) above and in the limit of small Z the effects of μ_e just discussed appear as though the number μ_0 were changed into $\mu_0-\mu_e$ wherever the interaction of the magnetic moment of the electron spin is dealt with.