On the Classical Equations of Motion of Point Charges

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Recent attempts to obtain the force of radiative reaction in the classical equations of motion of point charges have proceeded from two different viewpoints. Each of these has to introduce one basic assumption in addition to Maxwell's equations, namely the conservation law for the electromagnetic energy-momentum tensor in field theory and the relation between the Lorentz force and the momentum of the particle in action-at-a-distance theory. In previous field-theoretical derivations the Lorentz-Dirac equations including radiation damping are obtained only if one takes the field produced by the particle to be the retarded field; but equations of motion without the damping term are obtained if one uses half the sum of retarded and advanced fields. On the other hand, the theory of action at a distance as developed by Wheeler and Feynman was able to obtain the radiation damping using fields symmetric in time. It is noted that the need for the exclusive use of retarded fields arose only in the field-theoretical derivations for the one-particle problem. The considerations of Wheeler and Feynman on the total field due to all particles of the universe are, however, applicable to field theory in the symmetric form as well as to action-at-a-distance theory. The acceptance of their condition of "complete absorption" again leads to the radiation damping term in the equations of motion of the many-body problem. Some implications of this result are discussed.

I. INTRODUCTION

ONE of the major problems of classical electrodynamics has been to account for the force experienced by a charge as a result of its motion. The first attempt at a solution due to Lorentz[1] was based on a model of an extended charge and attributed the force of radiative reaction to the action of one part of the particle on another. It appeared as the first term in a series in powers of the radius of the particle, and all higher terms depended upon the charge distribution assumed. This, and the fact that it appears to be very difficult to fit finite-sized elementary particles into the schemes of relativity and of quantum mechanics makes it desirable to treat these particles as mathematical points.[2] In Lorentz' theory, however, infinities appear in the equations if the radius goes to zero.

Recent attempts to obtain the force of radiative reaction have proceeded from two different viewpoints. One is that of field theory, which considers the *total* field at all points in space to be the fundamental physical quantity and the point charges as singularities of the field. The other is that of action at a distance, which considers only the forces exerted on a charge by *other* charges to be physically meaningful.

The field theoretical point of view was first applied successfully to this problem by Dirac,[3] who showed that the equations of motion are suggested by the conservation law for the electromagnetic energy-momentum tensor. He clearly stated it as follows: "The usual derivation of the stress-tensor is valid only for continuous charge distributions and we are here using it for point charges. This involves adopting as a fundamental assumption the point of view that energy and momentum are localized in the field in accordance with Maxwell's and Poynting's ideas." Using this conservation law and Maxwell's equations he obtained Lorentz' equations of motion, but whereas these equations were considered by Lorentz to be only approximate, Dirac concluded that "there is good reason for believing them exact, within the limits of the classical theory."

Another field theoretical derivation of these equations was given by Infeld and Wallace,[4] who succeeded in linking it with the general method of obtaining equations of motion in general relativity of Einstein, Infeld and Hoffmann.[5] The main interest of their paper for the present purpose is that they showed explicitly that, for the case of a single particle, the Lorentz-Dirac equations including radiation damping are ob-

[1] H. A. Lorentz, *Collected Papers* (M. Nijhoff, The Hague, 1936), Vol. II, pp. 281 and 343. Also *The Theory of Electrons* (Teubner, Leipzig, 1909), pp. 49 and 253.

[2] Cf. J. Frenkel, Zeits. f. Physik **32**, 518 (1925).

[3] P. A. M. Dirac, Proc. Roy. Soc. **A167**, 148 (1938). In the following quoted as D.

[4] L. Infeld and P. H. Wallace, Phys. Rev. **57**, 797 (1940). In the following quoted as IW.

[5] A. Einstein, L. Infeld and B. Hoffmann, Ann. Math. **39**, 65 (1938).

tained only if one takes the field produced by the particle to be the *retarded* field; but equations of motion without the damping term are obtained if one uses half the sum of retarded and advanced fields.

The first derivation of the force of radiative reaction on the basis of action at a distance is due to Wheeler and Feynman.[6] As it is necessary in any action-at-a-distance theory to introduce an assumption on the basic equations of motion, they had to assume that the Lorentz force acting on a particle equals its rate of change of momentum. On the other hand, they were able to obtain the Lorentz-Dirac equations taking the forces on the charges as determined by half the sum of the retarded and the advanced field.

It was of considerable importance to show that the fundamental law of force is symmetric with respect to past and future, and to settle the question first raised in 1909, in which "Ritz considers the limitation to retarded potentials as one of the foundations of the second law of thermodynamics, while Einstein believes that the irreversibility of radiation depends exclusively on considerations of probability."[7] It appeared that the theory of action at a distance was preferable to the point of view of field theory, which seemed incapable of explaining the radiation reaction using fields symmetric in time.

However, we shall show that this is actually not the case. The need for the exclusive use of retarded fields for the explanation of the radiative reaction arose only in the field-theoretical derivations for the *one-particle* problem. The considerations of WF on the total field due to *all* particles are, however, applicable to field theory as well as to the theory of action at a distance. The acceptance of their condition of "complete absorption" does not yield any new results in field theory if the retarded field alone is used, but in the case of half-advanced, half-retarded fields it does provide the radiation-damping term in the equations of motion. Therefore, subject to this condition, *one can obtain the Lorentz-Dirac equations in both theories starting with fields symmetric in time, in spite of the fundamentally different underlying physical ideas.*[8]

[6] J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **17**, 157 (1945). In the following quoted as WF.

[7] W. Ritz and A. Einstein, Phys. Zeits. **10**, 323 (1909).

II. FIELD THEORY: THE FORMULATION OF INFELD AND WALLACE

We shall first outline the field theoretical derivation of the equations of motion due to Infeld and Wallace.[4] We write down the conservation laws for the electromagnetic energy-momentum tensor:

$$T_{mn,\,n}=T_{mo,\,o} \qquad \text{(IW 2.1)}$$

$$T_{on,\,n}=T_{oo,\,o} \qquad \text{(IW 2.2)}$$

where

$$T_{mn}=\text{Maxwell stress tensor}=F_{ms}F_{ns}-F_{mo}F_{no}-1/4\delta_{mn}F_{rs}F_{rs}+1/2\delta_{mn}F_{so}F_{so} \qquad \text{(IW 2.3)}$$

$$T_{on}=\text{Poynting vector}=F_{so}F_{sn} \qquad \text{(IW 2.4)}$$

$$T_{oo}=\text{Energy density}=1/2F_{so}F_{so}+1/2F_{rs}F_{rs} \qquad \text{(IW 2.5)}$$

$F_{so}=E_s$ and $F_{mn}=\epsilon_{mns}H_s$ where E_s and H_s are the electric and magnetic field respectively, δ_{mn} is the Kronecker symbol, and ϵ_{mns} is the permutation symbol (Levi-Civita tensor density). Latin letters run from 1 to 3 and repetition of an index implies summation over this range. ",n" and ",o" denote partial derivatives with respect to the coordinates x^n and time, respectively. The velocity of light is taken as unity.

The conservation laws break down only at points occupied by a singularity. We shall choose a Lorentz frame of reference in which the point charge is instantaneously at rest at the origin at some moment t. Then if we take any surface enclosing the singularity it can be shown that the four surface integrals

$$\int(T_{mn}+\psi_{m,\,n})\lambda^n dS \qquad \text{(IW 2.10)}$$

$$\int(T_{on}+\psi_{o,\,n})\lambda^n dS \qquad \text{(IW 2.11)}$$

[8] In talking of the "equivalence" of the theory of action at a distance and field theory, WF refer only to the formal equivalence of the final equations of motion obtained in the two theories. However, the derivation of the Lorentz-Dirac equations requires, as noted above, in addition to Maxwell's equations common to both theories, assumptions which are fundamentally different for these theories, and which differentiate them by more than just "language". By treating Dirac's results as a "prescription", WF do not enter into an examination of these assumptions at all. This is the object of the present paper, which will show that the equivalence extends to the use of fields symmetric in time.

are independent of the shape and size of the surface chosen, and hence can depend only on quantities characterizing the singularity, in particular the coordinates of the singularity and their time derivatives. Here λ^n are the direction cosines of the normal to the surface of integration,

$$\psi_m = \Psi_m - \bar{\psi}_m \qquad \text{(IW 2.14)}$$

$$\psi_o = \Psi_o - \bar{\psi}_o \qquad \text{(IW 2.15)}$$

where $\bar{\psi}_m$, $\bar{\psi}_o$ are arbitrary harmonic functions and Ψ_m, Ψ_o are solutions of the Poisson equation which do not contain in their development with respect to r an harmonic function of the type $1/r$.

The motion of the singularity is determined by the assignment of consistent values to the four surface integrals. We can assume that they are equal to zero and shift the task of determining the motion to the choice of the arbitrary harmonic functions. Calling the coordinates of the singularity $\eta^r(t)$, and indicating derivatives with respect to time by dots, we have as the simplest non-trivial choice[9]

$$\bar{\psi}_m = m_o(\ddot{\eta}^m/r)$$

and obtain then for the equations of motion

$$m_0\ddot{\eta}^m = -1/4\pi \int (T_{mn} + \Psi_{m,n})\lambda^n d\xi. \qquad \text{(IW 2.16)}$$

We note that up to this stage it did not matter if retarded or advanced fields were used. We are only concerned with this problem when we evaluate the integral over T_{mn}.

The integrals over $\Psi_{m,n}$ can be shown to be zero. For the fields determining T_{mn} we have to take the total fields

$$F_{mn} = \gamma_{m,n} - \gamma_{n,m} + {}_{\text{ext}}F_{mn} \qquad \text{(IW 1.1)}$$

$$F_{mo} = \gamma_{m,o} - \gamma_{o,m} + {}_{\text{ext}}F_{mo} \qquad \text{(IW 1.2)}$$

where γ_o is the electromagnetic scalar potential, and γ_m the vector potential of the field of the point charge itself, and ${}_{\text{ext}}F_{\mu\nu}$ is the external field (including the fields of other point charges).

If we take for γ_μ half the sum of the retarded and advanced potentials, we obtain for the equations of motion

$$m_o\ddot{\eta}^m = e\,{}_{\text{ext}}E_m \qquad \text{(IW 2.17)}$$

where ${}_{\text{ext}}E_m$ is the external electric field (the fields of the other point charges also being taken as 1/2 (retarded+advanced) field) evaluated at the position of the singularity at the moment t. These are just the usual equations of motion without radiation reaction, *except for the special form of the fields of the other charges*.

If we choose for γ_μ retarded potentials only, we obtain

$$m_o\ddot{\eta}^m = e\,{}_{\text{ext}}E_m + 2/3e^2\dddot{\eta}^m \qquad \text{(IW 2.18)}$$

where we have to take in ${}_{\text{ext}}E_m$ the contribution due to other charges as retarded fields only. These equations do contain the radiation reaction.

The above equations held in a special coordinate system only. We consider a four-space with coordinates x^μ and signature $+ - - -$, Greek letters taking the values 0, 1, 2, 3 where x^0 is the time coordinate and x^1, x^2, x^3 are the space coordinates. The vector v^μ is defined as the four-dimensional velocity vector of the point charge. Accents indicate differentiation with respect to arc length in space-time. Then it may be shown that in the first case (from now on referred to as the symmetric case) the equations of motion become

$$m_o v'^\mu = e\,{}_{\text{ext}}F_\nu{}^\mu v^\nu \qquad \text{(IW 2.19)}$$

and in the second case (from now on referred to as the retarded case)

$$m_o v'^\mu = e\,{}_{\text{ext}}F_\nu{}^\mu v^\nu + 2/3e^2 v''^\mu + 2/3e^2 v'^2 v^\mu \qquad \text{(IW 2.20)}$$

where ${}_{\text{ext}}F_\nu{}^\mu$ is evaluated at the world point of the singularity and contains symmetric contributions from the other charges in the symmetric case and retarded contributions only in the retarded case.[10] Equations (IW 2.20) are just the Lorentz-Dirac equations.

[9] It has been shown in IW that this choice corresponds to the choice of an energy-momentum tensor for matter and that the arbitrariness can be removed by an appeal to the general relativity theory. The same argument holds also for the choice necessary at the corresponding stage of the derivation in *D*. Equations of motion based on the derivation of *D*, but using a different choice of the "arbitrary" functions, have been suggested by C. J. Eliezer, Proc. Camb. Phil. Soc. **42**, 278 (1946). It appears from the above that while these equations are perfectly consistent with the special theory of relativity, they are in conflict with the general theory.

[10] This distinction in the meaning of ${}_{\text{ext}}F_\nu{}^\mu$ in the two sets of equations has not been made explicit in IW, as that paper was not concerned with any application of those equations, but it is obvious from their derivation.

For future reference we shall make the distinction between the meaning of ${}_{\text{ext}}F_\nu{}^\mu$ in Eqs. (IW 2.19) and (IW 2.20) explicit by writing it as

$$ {}_{\text{ext}}F_\nu{}^\mu = \sum_{k\neq a} 1/2({}_{\text{ret}}F_\nu{}^{\mu(k)} + {}_{\text{adv}}F_\nu{}^{\mu(k)}) + {}_eF_\nu{}^\mu \quad (1)$$

in the symmetric case and

$$ {}_{\text{ext}}F_\nu{}^\mu = \sum_{k\neq a} {}_{\text{ret}}F_\nu{}^{\mu(k)} + {}_eF_\nu{}^\mu \quad (2)$$

in the retarded case. Here ${}_{\text{ret}}F_\nu{}^{\mu(k)}$ and ${}_{\text{adv}}F_\nu{}^{\mu(k)}$ are the retarded and advanced fields of the k-th charge, and ${}_eF_\nu{}^\mu$ is a solution of Maxwell's equations for empty space, which of course is not necessarily the same in the retarded and in the symmetric case.

We obtain, therefore, for the equations of motion

$$ m_a v_a{}'^\mu = e_a \sum_{k\neq a} 1/2({}_{\text{ret}}F_\nu{}^{\mu(k)} + {}_{\text{adv}}F_\nu{}^{\mu(k)})v_a{}^\nu + e_a\, {}_eF_\nu{}^\mu v_a{}^\nu \quad (3)$$

for the symmetric case, and

$$ m_a v_a{}'^\mu = e_a \sum_{k\neq a} {}_{\text{ret}}F_\nu{}^{\mu(k)} v_a{}^\nu + e_a\, {}_eF_\nu{}^\mu v_a{}^\nu + 2/3 e_a^2 v_a{}''^\mu + 2/3 e_a^2 v_a{}'^2 v_a{}^\mu \quad (4)$$

for the retarded case. All F's are evaluated at the world point of the a-th singularity, whose rest mass we have denoted by m_a now instead of m_o, its velocity vector by $v_a{}^\mu$, and its charge by e_a.

Dirac's results are entirely equivalent to those of Infeld and Wallace. As Dirac states, however, of Eq. (4) that it has been "obtained in a theory which is fundamentally symmetrical between retarded and advanced potentials", we shall show in the Appendix that actually he did not use any symmetry relations in his derivation of the above equations, but retarded fields only.

III. ACTION AT A DISTANCE: THE WHEELER-FEYNMAN THEORY

In the theory of action at a distance as developed by Fokker[11] and Wheeler and Feynman[6] the equations of motion of the a-th charge are assumed to be[12]

$$ m_a v_a{}'^\mu = e_a F_\nu{}^\mu v_a{}^\nu \quad (5)$$

where the right-hand side is the Lorentz force acting on the particle, and F is the field

$$ \sum_{k\neq a} 1/2({}_{\text{ret}}F^{(k)} + {}_{\text{adv}}F^{(k)}). \quad \text{(WF 38)}$$

Therefore, we have

$$ m_a v_a{}'^\mu = e_a \sum_{k\neq a} 1/2({}_{\text{ret}}F_\nu{}^{\mu(k)} + {}_{\text{adv}}F_\nu{}^{\mu(k)})v_a{}^\nu. \quad (6)$$

The expression (WF 38) can be broken down into three parts

$$ \sum_{k\neq a} {}_{\text{ret}}F^{(k)} + (1/2\ {}_{\text{ret}}F^{(a)} - 1/2\ {}_{\text{adv}}F^{(a)}) - \sum_{\text{all } k} 1/2({}_{\text{ret}}F^{(k)} - {}_{\text{adv}}F^{(k)}). \quad \text{(WF 39)}$$

The second term of this will contribute to the force an expression

$$ e_a(1/2\,{}_{\text{ret}}F_\nu{}^{\mu(a)} - 1/2\,{}_{\text{adv}}F_\nu{}^{\mu(a)})v_a{}^\nu.$$

This reduces, according to Dirac, to the form

$$ 2/3 e_a^2(v_a{}^\mu v_{a\nu}{}'' - v_a{}''^\mu v_{a\nu})v_a{}^\nu \quad \text{(WF 41)}$$

which can also be written

$$ 2/3 e_a^2(v_a{}''^\mu + v_a{}'^2 v_a{}^\mu).$$

The third term has no singularities anywhere and is, therefore, a solution of Maxwell's equations for empty space, which we shall call ${}_fF$ to distinguish it from the empty-space solutions introduced in (1) and (2), or

$$ {}_fF = \sum_{\text{all } k} 1/2({}_{\text{ret}}F^{(k)} - {}_{\text{adv}}F^{(k)}). \quad (7)$$

Then we obtain for the final equations of motion

$$ m_a v_a{}'^\mu = e_a \sum_{k\neq a} {}_{\text{ret}}F_\nu{}^{\mu(k)} v_a{}^\nu + 2/3 e_a^2(v_a{}''^\mu + v_a{}'^2 v_a{}^\mu) + e_{a\,f}F_\nu{}^\mu v_a{}^\nu \quad \text{(WF 44)}$$

for the case which WF call "incomplete absorption". The case which they call "complete absorption" is characterized by

$$ \sum_{\text{all } k} ({}_{\text{ret}}F^{(k)} - {}_{\text{adv}}F^{(k)}) = 0 \text{ (everywhere).} \quad \text{(WF 37)}$$

Using this relation,[13] we obtain for the equations

[11] A. D. Fokker, Zeits. f. Physik **58**, 386 (1929). We are not concerned here with other formulations of action at a distance using retarded interactions only.

[12] Here and in the following we shall change the notation of WF slightly to conform to the one employed above.

[13] While (WF 37) was obtained in WF from the relation

$$ \sum_{\text{all } k} (1/2\ {}_{\text{ret}}F^{(k)} + 1/2\ {}_{\text{adv}}F^{(k)}) = 0 \text{ (outside the absorber)} \quad \text{(WF 33)},$$

it is only (WF 37) which is used in the equations of motion. As (WF 33) necessitates a division of the universe into a part "inside" the absorber and a part "outside" it, it appears irreconcilable with any current cosmological of motion

$$m_a v_a'^{\mu} = e_a \sum_{k\neq a} {}_{\text{ret}}F_{\nu}{}^{\mu(k)} v_a{}^{\nu} + 2/3 e_a^2 (v_a''^{\mu} + v_a'^2 v_a^{\mu}). \quad \text{(WF 42)}$$

IV. FIELD THEORY AND THE CONSIDERATIONS OF WHEELER AND FEYNMAN

Following Infeld and Wallace we obtained two different sets of equations of motion for the retarded and the symmetric case (Eqs. (3) and (4)) respectively. If we considered the equations significant for the case of a single particle, we would obtain a force of radiative reaction in the retarded case only. However, it is clear that we can only compare those equations with experiment which take account of the existence of a large number of particles in the universe (which may or may not lead to the same conclusions as the simpler equations for the one-particle case). We shall show that we can take over the considerations of Wheeler and Feynman on the field of all particles in the universe into field theory and we shall then obtain the radiative reaction *also in the symmetric case* just as in the theory of action at a distance.

Except for its last term, Eq. (3) of field theory is of the same form as Eq. (6), the starting point of the theory of action at a distance. Its first term on the right-hand side is just the expression (WF 38) and we can break it down into three different fields exactly as WF have done, for none of their arguments (once Eq. WF 38 is accepted) involves any distinction between field theory and action at a distance. Therefore, we we obtain finally, corresponding to (WF 44),

$$m_a v_a'^{\mu} = e_a \sum_{k\neq a} {}_{\text{ret}}F_{\nu}{}^{\mu(k)} v_a{}^{\nu} + 2/3 e_a^2 (v_a''^{\mu} + v_a'^2 v_a^{\mu}) + e_a ({}_eF_{\nu}{}^{\mu} + {}_fF_{\nu}{}^{\mu}) v_a{}^{\nu} \quad (8)$$

where the sum in the last term is due to two fields each of which is a solution of Maxwell's equations for empty space and, therefore, still a solution for empty space.

If we accept (WF 37), we obtain corresponding to (WF 42)

$$m_a v_a'^{\mu} = e_a \sum_{k\neq a} {}_{\text{ret}}F_{\nu}{}^{\mu(k)} v_a{}^{\nu} + 2/3 e_a^2 (v_a''^{\mu} + v_a'^2 v_a^{\mu}) + e_a\, {}_eF_{\nu}{}^{\mu} v_a{}^{\nu}. \quad (9)$$

Equation (4) (the retarded case) is already of the form of Eq. (8) (except that the term involving ${}_fF$ if absent), and nothing new is obtained if one takes into account all particles in the universe.

We have considered in outline the derivation of the equations of motion from the point of view of field theory first in order to show that, starting from the *total* field, one may obtain the Eqs. (3) or (4) which apparently do not involve the total field, but only the "external" one, and are therefore of the same form as in action at a distance. This, then, enabled us to show that the application of the WF considerations to the symmetric case yields the radiation damping term which appears in the retarded case already without these considerations.

However, we could have seen without any calculation that under the so-called "complete absorption" conditions *any* results of field theory must be the same in the symmetric and in the retarded (and also the advanced) case. For the condition (WF 37) can also be written

$$\sum_{\text{all } k} {}_{\text{ret}}F^{(k)} = \sum_{\text{all } k} {}_{\text{adv}}F^{(k)} = \sum_{\text{all } k} (1/2\ {}_{\text{ret}}F^{(k)} + 1/2\ {}_{\text{adv}}F^{(k)}), \quad (10)$$

and these are the fields (due to sources) which have to be inserted into the energy-momentum tensor at the start of any field-theoretical calculation in the retarded, advanced and symmetric cases, respectively. As these are equal, it is obvious that the equations of motion must also be the same in all cases.[14]

While this demonstrates the equality of the equations of motion arrived at in the various cases, the explicit form of the equations must be obtained by a calculation such as those of Dirac or Infeld and Wallace.

It should be noted that in the theory of action at a distance, contrary to field theory, Eqs. (10) or (WF 37) do *not* lead to the same equations of

theory, while (WF 37) holds everywhere and is therefore consistent with, and might even be a consequence of cosmological considerations. Also (WF 33) does not have to hold for (WF 37) to be true. This makes it more plausible at present simply to take (WF 37) as an additional assumption of the WF-theory.

[14] The relation (10) was suggested by A. Einstein, Phys. Zeits. 10, 185 (1909). Cf. also reference 7.

TABLE I.

	Field theory			Action at a distance	
	Maxwell's equations				
Basic assumptions	Conservation laws for the electromagnetic energy-momentum tensor (IW 2.1, 2.2)			Lorentz Eq. (5)	
Fields used	Retarded	Symmetric		Symmetric	
Equations obtained	(4)	(3)		(6)	
Additional assumption	None	None	(WF 37)	None	(WF 37)
Final equations	(4)	(8)	(9)	(WF 44)	(WF 42)
Terms common to all final equations	$m_a v_a'^{\mu} = \sum_{k \neq a} {}_{\text{ret}}F_{\nu}{}^{\mu(k)} v_a{}^{\nu} + 2/3\, e_a{}^2 (v_a''^{\mu} + v_a'^2 v_a{}^{\mu})$				
Additional terms (to be multiplied by $e_a v_a{}^{\nu}$)	${}_eF_{\nu}{}^{\mu}$	${}_eF_{\nu}{}^{\mu} + {}_fF_{\nu}{}^{\mu}$	${}_eF_{\nu}{}^{\mu}$	${}_fF_{\nu}{}^{\mu}$	None

motion in all cases. As it is not the total field which enters the starting equations (5), the final equations (WF 42) can be obtained in the symmetric case only.

If one does not assume condition (WF 37), we have from (7)

$$\sum_{\text{all } k} {}_{\text{ret}}F^{(k)} = \sum_{\text{all } k} (1/2\ {}_{\text{ret}}F^{(k)} + 1/2\ {}_{\text{adv}}F^{(k)}) + {}_fF$$

$$= \sum_{\text{all } k} {}_{\text{adv}}F^{(k)} + 2\,{}_fF. \quad (11)$$

Therefore, the total fields in the three cases differ only by solutions of Maxwell's equations for empty space, and the equations of motion will only differ in the terms involving such solutions (cf. Eqs. 4 and 8).

In short, the mathematical reason for the similarity of (4) and (8) is simply that two solutions of the inhomogeneous wave equation can at most differ by a solution of the homogeneous one.

V. DISCUSSION

We shall now inquire into the relationship between the five different sets of equations of motion: Eqs. (4), (8) and (9) from field theory and (WF 42) and (WF 44) from action at a distance.

To facilitate the comparison we summarize the assumptions and results in Table I. The last row shows that these equations differ mathematically only in fields which represent empty-space solutions. However, physically the fields ${}_eF$ and ${}_fF$ do not have the same significance. ${}_eF$ is a solution of the homogeneous wave equation, which arises only in field theory and may be due wholly, partly, or not at all to sources; but at present field theory is unable to analyze it any further. ${}_fF$ on the other hand is uniquely determined by the sources according to relation (7), which states that it is half the difference of the retarded and advanced fields of all particles. As shown in section III, the term $e_a v_a{}^{\nu}\,{}_fF_{\nu}{}^{\mu}$ contains a part which just cancels the radiation damping term $2/3 e_a{}^2(v_a''^{\mu} + v_a'^2 v_a{}^{\mu})$. Therefore, the similarity of Eqs. (8) and (WF 44) to (4), (9) and (WF 42) is purely formal and (8) and (WF 44) actually do *not* describe radiating particles (cf. the discussion of "incomplete absorption" in WF).

If we adopt the assumption frequently used in field theory that ${}_eF = 0$, Eqs. (4) and (9) reduce to (WF 42). However, it should be kept in mind that while the solutions of the equations of the symmetric case (9) and (WF 42) are subject to the restriction (WF 37), those of the retarded case (4) are not. Therefore, we (and also WF) have *not* shown the complete equivalence of the retarded and symmetric cases.

Keeping the assumption ${}_eF = 0$, we have, on the other hand, complete equivalence of Eq. (9) and (WF 42) of field theory and action-at-a-distance theory, respectively, describing the motion of radiating charges. These identical equations have been obtained from two basically different starting points, while using in common Maxwell's equations, fields symmetric in time, and the condition (WF 37).

In both theories we have to introduce one additional assumption, namely, either the conservation law for the electromagnetic energy-momentum tensor, or the relation between Lorentz force and momentum. The advantage of action at a distance is the plausibility of the physical idea of reducing everything to the interaction of particles. But this is balanced by the difficulties connected with the conservation[11] and transmission of energy and momentum. Therefore, none of the viewpoints appears to be preferable to the other from considerations of simplicity.

Clearly a *direct* verification of the fundamental assumptions of the two viewpoints is impossible, as one cannot observe a field without a test charge, the effect of which, however, would have to be included in the total field. The only experimentally verifiable conclusions are precisely the equations of motion.

Comparing these equations, we see that there is ***no*** effect which would *require* the point of view of action at a distance. On the other hand, the demonstration of a non-vanishing ${}_eF$ would show an effect which can only be explained by field theory.

Such a demonstration, while possible in principle, appears to be impossible in practice, as it would amount to finding whether the observed field is "only" due to the retarded fields of all the charges in the universe, or whether there is still another part.

Therefore, as far as the symmetric case is concerned, there do not seem to be any compelling reasons at present to prefer either of the two points of view. It appears possible, however, that the application of these viewpoints to general relativity or to quantum mechanics will provide such reasons.

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APPENDIX

We want to show that Dirac's formulation of the field theory for the one-particle problem leads to the same conclusions as that of Infeld and Wallace, namely that one does obtain the term of radiative reaction only in the retarded, but not in the symmetric case. Dirac takes the actual field as[15]

$$ {}_{\text{act}}F = {}_{\text{ret}}F + {}_{\text{in}}F \qquad \text{(D 8)}$$

where ${}_{\text{in}}F$ is the incident field, a solution of the homogeneous wave equation, as is ${}_{\text{out}}F$, which is defined by

$$ {}_{\text{act}}F = {}_{\text{adv}}F + {}_{\text{out}}F. \qquad \text{(D 9)}$$

The difference

$$ {}_{\text{rad}}F = {}_{\text{out}}F - {}_{\text{in}}F \qquad \text{(D10)}$$

or

$$ {}_{\text{rad}}F = {}_{\text{ret}}F - {}_{\text{adv}}F \qquad \text{(D 11)}$$

is called the field of radiation in D. The equations of motion are obtained by substituting the actual field (D 8) into the energy-momentum tensor. After a rather long calculation, which, however, involves only the use of Maxwell's equations, his stress-tensor and (D 8), Dirac obtains the result

$$ mv'^{\mu} = ev^{\nu} f_{\nu}{}^{\mu} \qquad \text{(D 22)}$$

where

$$ f = {}_{\text{act}}F - 1/2({}_{\text{ret}}F + {}_{\text{adv}}F), \qquad \text{(D 13)}$$

or

$$ f = 1/2({}_{\text{ret}}F - {}_{\text{adv}}F) + {}_{\text{in}}F$$

from which we get (see (WF 41) above)

$$ mv'^{\mu} - 2/3e^2(v''^{\mu} + v'^2 v^{\mu}) = ev^{\nu}\, {}_{\text{in}}F_{\nu}{}^{\mu}. \qquad \text{(D 24)}$$

If we had used the field of what we called above symmetric case, we would have had

$$ {}_{\text{act}}F = 1/2({}_{\text{ret}}F + {}_{\text{adv}}F) + {}_{\text{in}}F.$$

It can be easily seen, following Dirac's calculations, that the introduction of this expression would still have led to (D 22) with the definition (D 13) or with

$$ f = {}_{\text{in}}F$$

and, therefore, we would have obtained as our equations of motion

$$ mv'^{\mu} = ev^{\nu}\, {}_{\text{in}}F_{\nu}{}^{\mu},$$

which do not include the term of radiative reaction.[16]

[15] Here, and in the following, we shall change the notation of D slightly to conform to the one employed above. Also, as we shall only have to follow the argument of D without any detailed calculations, we shall omit sub- and superscripts for convenience wherever there is no danger of confusion.

[16] This result has also been obtained by C. J. Eliezer, Rev. Mod. Phys. 19, 147 (1947) (case $k = -\frac{1}{2}$), in a different connection.

For the case of several particles, analogous considerations would lead to our Eqs. (4) and (3) respectively.

Therefore Dirac's derivation is in accord with the one of Infeld and Wallace outlined previously. However, we noted that Dirac himself calls his theory fundamentally symmetrical between retarded and advanced potentials. The contradiction seems to come from the fact that Dirac considers his theory symmetrical due to the apparent symmetry of the use of retarded and advanced fields in definitions (D 8) and (D 9) (and the analogous definitions (D 38) and (D 39) for the many-body problem).

However, it should be noted that in the derivation by Dirac outlined above, he is only using (D 8) (respectively D 38). The definition (D 9) never enters into any of his calculations which lead to his equations of motion. It is only used in a purely formal manner to introduce the notion of ${}_{\text{rad}}F$ (see (D 10)) into some of the equations;[17] but the results themselves are independent of it.

[17] The purely formal character of any definition of the radiation field can be seen also from the fact that in WF the term "radiation field" is used for just 1/2 ${}_{\text{rad}}F$ of D, without in any way leading to contradictions, as the only physically significant equations are the equations of motion of the point charges.

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The Photo-Voltaic Effect

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The *Schottky-Mott* theory of the barrier layer rectification is extended with respect to the action of light absorbed in the barrier layer. The essential physical assumptions to be used are as follows: (a) The barrier layer is a boundary layer of a semiconductor with a reduced density of mobile charges (either electrons or "holes"); (b) both positive *and* negative mobile charges are released by light; (c) the recombination within the barrier layer is negligible; and, (d) the electrons and "holes" have the same properties whether released by light or by thermal agitation. Thus an "equation of state" connecting photo-voltage, photo-current, light intensity, wave-length, external resistance, etc., is derived. Among others the regularities of short circuit current, open circuit voltage, photo-characteristic, dark characteristic (barrier layer rectification), power output, and spectral distribution of the quantum yield are involved.

I. INTRODUCTION

THE most successful theory in the explanation of the barrier rectification is the theory of W. Schottky[1] and N. F. Mott.[2] In his discussion of the action of light in a barrier layer of the nature described by the above theory, N. F. Mott has already succeeded in explaining the sign of the photo-voltage.[3] In this paper we generalize the fundamental assumptions of W. Schottky and N. F. Mott and derive a comprehensive formula for the barrier layer photo-effect.[4] The establishment of one formula for the manifold of barrier layer photo-cells varying in peculiarities is possible since the peculiarities of minor importance enter only into the parameters of the formula, which remain open to a further discussion in special cases.

We shall treat explicitly the photo-effect in semiconductors only for the case where the mobile carriers of charge are electrons (n-type semiconductors) and shall state the corresponding results for semiconductors with "hole" conductance (p-type semiconductors).[5]

II. BASIC CONCEPTS ABOUT THE BARRIER LAYER

In principle both barrier layer rectifiers and barrier layer photo-cells consists of a semicon-

[1] W. Schottky, Zeits. f. Physik **113**, 367 (1939); Zeits. f. Physik **118**, 539 (1942), and other publications.
[2] N. F. Mott, Proc. Roy. Soc. London **A171**, 27 (1939).
[3] N. F. Mott, Proc. Roy. Soc. London **A171**, 281 (1939).
[4] This formula has been communicated in a previous note by the author in Optik **1**, issue 3 (1946).

[5] K. Lehovec, Zeits. f. Naturforsch. **2**, 398 (1947).