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Phenomenological Theory of Exchange Currents in Nuclei

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On the assumption that the interaction between nuclear particles involves a space exchange operator, it is shown that an addition must be made to the conventional current density for the nucleons in order to establish the equation of continuity within the nucleus. A general expression is found for this exchange current and the corresponding exchange magnetic moment. This phenomenological theory has application to the calculation of magnetic moments of nuclei and to the calculation of transition probabilities for the absorption and emission of radiation by nuclei. In this paper, application is limited to the exchange moments of H^3 and He^3. It is found that the exchange moments are six or seven times too small to account for the observed moments. In view of results obtained by Villars, it is concluded that the important contributions to the magnetic moment are directly related to the properties of the field (meson field) which describes the nuclear interaction, so the exchange moment may be of use for obtaining direct information concerning the nature of this field.

I. INTRODUCTION

As a result of recent measurements[1] of the magnetic moments of the nuclear three-body systems H^3 and He^3, consideration has been given to the possibility that there exist exchange currents in nuclei.[2] The term exchange current is used here to denote the net flow of charge between nucleons which may be considered to be a consequence of the charge exchange nature of their interaction potentials. A current of this type is expected to introduce a contribution to the magnetic moment with the result that the magnetic moments of nuclear systems could not, in general, be obtained by simply adding the spin and orbital contributions of the nucleons.

It was first suggested by Siegert[3,4] that the existence of exchange forces in nuclei implies the existence of an exchange current. He showed that under certain circumstances, one would expect the exchange current to be proportional to the exchange potential. The arguments presented by Siegert were based on a field theoretical description of the nuclear interaction, although his final result is independent of the properties of the nuclear field other than the interaction potential it produces. This suggests strongly that Siegert's exchange current is a property associated with the exchange potential which is quite independent of the nature of the field producing the potential. It will be shown below that this is indeed the case.

It is not to be assumed that this is the only contribution to the exchange current. For example, the anomalous contributions to the magnetic moments of the neutron and proton may be due to a charge bearing field, and the corresponding currents may be to some extent independent of the neutron-proton interaction potential. In general, then, one expects the exchange current to contain two terms, one depending in some way on the details of the field describing the interactions between nucleons.

Villars[2] has shown that for a specific type of nuclear field (pseudoscalar symmetric) one can account for the observed exchange moment of H^3 and He^3. It is to be expected that other types of field could also account for the observations.[5] In spite of the apparent agreement, the situation cannot be considered to be completely satisfactory because the field theories lead to unsatisfactory interaction potentials. An explanation of the exchange moments based on the "field independent" exchange current described above would not labor under this difficulty since it can be directly expressed in terms of the phenomenological potentials. Therefore, the contribution of this exchange current to the magnetic moments of H^3 and He^3 is estimated below. It is found that the result is not in agreement with observa-

[1] H. L. Anderson and A. Novick, Phys. Rev. **71**, 372 (1947); *ibid.* **73**, 919 (1948); F. Bloch, A. C. Graves, M. Packard, and R. W. Spence, *ibid.* **71**, 373, 551 (1947).
[2] F. Villars, Phys. Rev. **72**, 256 (1947); Helv. Phys. Acta **20**, 476 (1947).
[3] A. F. Siegert, Phys. Rev. **52**, 787 (1937).
[4] W. E. Lamb and L. I. Schiff, Phys. Rev. **53**, 651 (1938).
[5] A. Thellung and F. Villars, Phys. Rev. **73**, 924 (1948), have considered vector fields and a Møller-Rosenfeld mixture with negative results.

tions, so one may conclude that the exchange currents do depend in a detailed way on the nature of the nuclear fields. This would seem to indicate that the exchange moments may be a means for determining experimentally the transformation properties of the nuclear fields. However, a satisfactory comparison of this nature does not seem feasible until there is developed a field theory that leads to reasonable interaction potentials which may then be used to fix the constants in the theory.

In order to establish the relationship between exchange current and exchange potential, use will be made of Wheeler's velocity dependent formulation of the exchange potential.[6] The resulting theory of the interaction between the nuclear system and the electromagnetic field is applicable to radiation problems as well as to the magnetic moment problem. The consequences of the theory with respect to the emission and absorption of radiation by nuclei have not been investigated but it is hoped that that investigation will be carried out in time.

II. FORMULATION OF THE PROBLEM

We consider only the Majorana type of exchange interaction since it can be shown that a spin exchange operator in the interaction does not introduce any additional contributions to the exchange current. The exchange operator will be denoted by P_{jk}; this operator exchanges the position coordinates of the jth and kth particles.

If the usual expression of the equation of continuity which arises from the Schroedinger equation is considered, it becomes immediately obvious that an exchange potential requires the addition of a term in the exchange current. In the usual way we find

$$\frac{\partial}{\partial t}|\psi|^2+(i\hbar/2M)\sum_j \operatorname{div}_j(\psi\,\operatorname{grad}_j\psi^*-\psi^*\,\operatorname{grad}_j\psi)=(i/\hbar)(\psi V\psi^*-\psi^* V\psi). \quad (1)$$

For ordinary potentials the expression on the right vanishes, but for exchange potentials of the form $V=\sum J_{jk}P_{jk}$, with $J_{jk}=J(\mathbf{r}_j-\mathbf{r}_k)$, it will not vanish except for very special types of wave function. However, in the special case of the deuteron, it will always vanish since the wave functions are necessarily either symmetric or antisymmetric for interchange of the two nucleons. The wave functions of other nuclei will usually contain both symmetric and antisymmetric terms.

Although Eq. (1) shows clearly that the exchange current can be expected to depend directly on the exchange potential, it is not the most convenient form for establishing the exact relationship. For that purpose we look into the question of the gauge invariance of the Schroedinger equation involving exchange potentials.

It is well known that gauge invariance usually implies charge conservation. Stated more precisely, if it is possible to obtain the field equations (Schroedinger equation) from a gauge invariant Lagrangian, then a "natural" expression for the current density can usually be obtained by applying an infinitesimal gauge transformation to the Lagrangian and identifying the equation resulting from invariance under this transformation with the equation of continuity for charge and current.

If, then, the Schroedinger equation involving an exchange potential can be obtained from a Lagrangian formalism, the requirement of gauge invariance will lead directly to an expression for the exchange current. The problem is, therefore, reduced to that of obtaining the Lagrangian which will properly describe the exchange interactions between nucleons in the presence of an external electromagnetic field.

III. THE LAGRANGIAN DENSITY

In the absence of external fields, the Lagrangian density, L, may be taken to be

$$L=-i\hbar(\psi,\partial\psi/\partial t)+\sum_k(\hbar^2/2m_k)(\operatorname{grad}_k\psi,\cdot\operatorname{grad}_k\psi)+\sum_k\sum_{j<k}(\psi, J_{jk}P_{jk}\psi), \quad (2)$$

where the labels j and k denote particular nucleons and J_{jk} is the potential function which depends on the distance between the particles j and k and may be an operator involving the spins of those particles. The scalar product notation includes only the sum over spin coordinates; thus $(\psi,\partial\psi/\partial t)=\sum_\sigma\psi_\sigma^*\partial\psi_\sigma/\partial t$, etc. The condition

$$\delta\int L dt d\tau_1 d\tau_2\cdots d\tau_N=0, \quad (3)$$

[6] J. A. Wheeler, Phys. Rev. **50**, 643 (1936).

in which ψ^* is varied but ψ is kept constant, clearly leads to the Schroedinger equation

$$i\hbar\partial\psi/\partial t = -\sum_k(\hbar^2/2m_k)\nabla_k{}^2\psi + \sum_k \sum_{j<k} J_{jk}P_{jk}\psi, \quad (4)$$

which is the required form. The conjugate complex of Eq. (4) is obtained if ψ is varied and ψ^* is held fixed in Eq. (3). To show this it is necessary to note that

$$\int(\psi, J_{jk}P_{jk}\delta\psi)d\tau_j d\tau_k = \int(J_{jk}P_{jk}\psi, \delta\psi)d\tau_k d\tau_j \quad (5)$$

in virtue of the symmetry of the integration with respect to the variables $\mathbf{r}_j$ and $\mathbf{r}_k$. It is assumed, of course, that J_{jk} is symmetric in the coordinates of the two particles and Hermitian.

In the presence of an external electromagnetic field described by a vector potential $\mathbf{A}$ and a scalar potential φ, the first two terms in the Lagrangian density given by Eq. (2) are to be modified by the substitution

$$\begin{aligned} \mathrm{grad}_k\psi &\rightarrow \mathrm{grad}_k\psi - (ie_k/\hbar c)\mathbf{A}_k\psi, \\ \frac{\partial\psi}{\partial t} &\rightarrow \frac{\partial\psi}{\partial t} + (i/\hbar)\sum_k e_k\varphi_k\psi, \end{aligned} \quad (6)$$

where e_k is the charge of the kth particle (e for protons, 0 for neutrons) and $\mathbf{A}_k = \mathbf{A}(\mathbf{r}_k)$, $\varphi_k = \varphi(\mathbf{r}_k)$. Then these two terms are invariant under the gauge transformation

$$\begin{aligned} \mathbf{A}\rightarrow\mathbf{A}+\mathrm{grad}G, \quad \varphi\rightarrow\varphi - \partial G/\partial(ct), \\ \psi\rightarrow\psi e\{i\sum_k e_k G_k/\hbar c\}, \end{aligned} \quad (7)$$

where $G_k = G(\mathbf{r}_k)$ and $e\{x\}$ is used to denote e^x.

The third term in the Lagrangian Eq. (2) is certainly not gauge invariant as it stands since

$$P_{jk}\psi e\{i\sum_m e_m G_m/\hbar c\} = e\{(i/\hbar c)(\sum_{m\neq j,k} e_m G_m + e_j G_k + e_k G_j)\}P_{jk}\psi.$$

Only those terms for which $e_j = e_k$ (i.e., involving exchange of two protons or two neutrons) behave in such a way as to leave $(\psi, J_{jk}P_{jk}\psi)$ unchanged. In order to obtain the appropriate gauge-invariant modification of this term it is convenient to express the exchange operator as a differential operator as proposed by Wheeler.[6] Wheeler called attention to the fact that the function $P_{jk}\psi(\mathbf{r}_j, \mathbf{r}_k)$ may be expanded in Taylor series about the point $(\mathbf{r}_j, \mathbf{r}_k)$ with the result:

$$\begin{aligned} P_{jk}\psi = \sum_{(n)}(-1)^{n_4+n_5+n_6} &\times \frac{(x_k-x_j)^{n_1+n_4}(y_k-y_j)^{n_2+n_5}(z_k-z_j)^{n_3+n_6}}{n_1!n_2!n_3!_4!n_5!n_6!} \\ &\times\left(\frac{\partial}{\partial x_j}\right)^{n_1}\left(\frac{\partial}{\partial y_j}\right)^{n_2}\left(\frac{\partial}{\partial z_j}\right)^{n_3} \\ &\times\left(\frac{\partial}{\partial x_k}\right)^{n_4}\left(\frac{\partial}{\partial y_k}\right)^{n_5}\left(\frac{\partial}{\partial z_k}\right)^{n_6}\psi. \end{aligned}$$

It is to be noted that this equation may be put into simpler form by treating $\mathbf{r}_{kj}$ and $\mathbf{r}_{jk}$ as constant vectors in the equation

$$P_{jk}\psi = \sum_n \frac{1}{n!}(\mathbf{r}_{kj}\cdot\mathrm{grad}_j + \mathbf{r}_{jk}\cdot\mathrm{grad}_k)^n\psi, \quad (8)$$

and setting

$$\mathbf{r}_{kj} = \mathbf{r}_k - \mathbf{r}_j, \quad \mathbf{r}_{jk} = \mathbf{r}_j - \mathbf{r}_k = -\mathbf{r}_{kj}, \quad (9)$$

after each term in the series has been expanded. If we now denote by D_{kj} the differential operator $(\mathbf{r}_{kj}\cdot\mathrm{grad}_j)$, Eq. (8) takes the simple operational form

$$P_{jk}\psi = e\{D_{kj} + D_{jk}\}\psi. \quad (10)$$

Using this operator, the last term in the Lagrangian is given by

$$L_P = \sum_k \sum_{j<k}(\psi, J_{jk}e\{D_{kj}+D_{jk}\}\psi). \quad (11)$$

The gauge-invariant modification which takes into account the effect of external electromagnetic fields is now obvious. According to Eq. (6), the required substitution is

$$D_{kj}\rightarrow D_{kj} - (ie_j/\hbar c)(\mathbf{r}_{kj}\cdot\mathbf{A}_j). \quad (12)$$

Setting

$$f_{kj} = -(e_j/\hbar c)(\mathbf{r}_{kj}\cdot\mathbf{A}_j), \quad (13)$$

the potential term in the Lagrangian becomes in place of Eq. (11)

$$L_P(f) = \sum_k \sum_{j<k}(\psi, J_{jk} \times e\{D_{kj}+D_{jk}+if_{kj}+if_{jk}\}\psi). \quad (14)$$

This expression is clearly gauge invariant. Thus the complete gauge-invariant Lagrangian density is

$$L = -i\hbar(\psi, \partial\psi/\partial t) + (\psi, \sum_k e_k\varphi_k\psi) + \sum_k (1/2m_kc^2)([\hbar c\,\mathrm{grad}_k\psi - ie_k\mathbf{A}_k\psi], \cdot[\hbar c\,\mathrm{grad}_k\psi - ie_k\mathbf{A}_k\psi]) + \sum_k\sum_{j<k}(\psi, J_{jk} \times e\{D_{kj}+D_{jk}+if_{kj}+if_{jk}\}\psi). \quad (15)$$

IV. PROPERTIES OF THE EXCHANGE OPERATOR

In order that the Schroedinger equations for ψ and ψ^* which are produced by the Lagrangian Eq. (15) be consistent with each other, it is necessary that the operator $e\{D_{kj}+D_{jk}+if_{kj}+if_{jk}\}$ be Hermitian. To establish this property it will be convenient to consider some of the properties of the operator $\exp(D+g)$, where

$$D = (\boldsymbol{\alpha}\cdot\mathrm{grad}), \quad (16)$$

with $\boldsymbol{\alpha}$ a constant vector, and g a function of the point $\mathbf{r}$.

We first note that if ds is the element of length along a path parallel to $\boldsymbol{\alpha}$, then

$$D = \alpha(d/ds). \quad (17)$$

If we define a function $\Gamma(\mathbf{r})$ to be the line integral

$$\Gamma = \frac{1}{\alpha}\int^r g ds \quad (18)$$

along a straight line path parallel to $\boldsymbol{\alpha}$ and passing through the point $\mathbf{r}$, we find that

$$D\Gamma = g. \quad (19)$$

Now any function F which defines a differentia operator $F(D)$ through a power series will have the property

$$F(D+g)\psi = e^{-\Gamma}F(D)e^{\Gamma}\psi. \quad (20)$$

Proof: From Eq. (19) it follows that

$$De^{\Gamma}\psi = \psi(De^{\Gamma}) + e^{\Gamma}(D\psi) = e^{\Gamma}(D+g)\psi.$$

Iterating this result, we find

$$D^n e^{\Gamma}\psi = e^{\Gamma}(D+g)^n\psi,$$

or

$$e^{-\Gamma}D^n e^{\Gamma}\psi = (D+g)^n\psi.$$

The validity of this equation for each term in he series $F(D)$ leads immediately to the result Etq. (20).

A further result which will prove to be useful is that for any two functions φ and χ

$$e^D(\varphi\chi) = e^D\varphi e^D\chi. \quad (21)$$

Proof: e^D is a displacement operator such that $e^D\varphi(\mathbf{r}) = e^D\varphi(\mathbf{r}+\boldsymbol{\alpha})$ if $\mathbf{r}$ is a point on the path along which the operator D is defined. Thus

$$e^D\varphi\chi = \varphi(\mathbf{r}+\boldsymbol{\alpha})\chi(\mathbf{r}+\boldsymbol{\alpha}) = e^D\varphi e^D\chi.$$

If we now apply Eq. (21) to Eq. (20) with $F(D)$ replaced by the special function e^D, we find

$$e^{D+g}\psi = e^{-\Gamma}e^D(e^{\Gamma}\psi) = (e^{-\Gamma}e^De^{\Gamma})e^D\psi. \quad (22)$$

It is to be noted that the factor $(e^{-\Gamma}e^De^{\Gamma})$ is just a multiplicative factor consisting of a series of derivatives of g.

Now to return to the physical problem. The operator appearing in Eq. (14) can clearly be written as

$$e\{D_{kj}+D_{jk}+if_{kj}+if_{jk}\} = e\{D_{kj}+if_{kj}\}e\{D_{jk}+if_{jk}\}$$

since the two differential operators affect different coordinates. Then, from Eq. (22) it is found that

$$e\{D_{kj}+D_{jk}+if_{kj}+if_{jk}\}\psi = [e\{-i\Phi_{kj}-i\Phi_{jk}\}e\{D_{kj}+D_{jk}\} \times e\{i\Phi_{kj}+i\Phi_{jk}\}]e\{D_{kj}+D_{jk}\}\psi, \quad (23)$$

or, according to Eq. (10),

$$= [e\{-i\Phi_{kj}-i\Phi_{jk}\}P_{jk}e\{i\Phi_{kj}+i\Phi_{jk}\}]P_{jk}\psi,$$

where

$$\Phi_{kj} = \frac{1}{r_{kj}}\int^{r_j} f_{kj}ds_j = -(e_j/\hbar c)\left(\frac{\mathbf{r}_{kj}}{r_{kj}}\cdot\int^{r_j}\mathbf{A}_j ds_j\right),$$

according to the definition of f_{kj}. Thus

$$\Phi_{kj} = -(e_j/\hbar c)\int^{r_j} A_j^s ds_j, \quad (24)$$

where A^s is the component of $\mathbf{A}$ along the path of integration, which is parallel to $\mathbf{r}_{kj}$. Similarly,

$$\Phi_{jk} = -(e_k/\hbar c)\int^{r_k} A_k^s ds_k.$$

Since ds_k is parallel to $\mathbf{r}_{jk} = -\mathbf{r}_{kj}$ it is antiparallel to ds_j and both integrals can be expressed in terms of the same path which we choose to be parallel to ds_j. It is to be noted that the sign of the component A^s of the vector is changed by this transformation of paths so

$$\Phi_{kj}+\Phi_{jk}=-\left[(e_j/\hbar c)\int^{r_j} A^s ds + (e_k/\hbar c)\int^{r_k} A^s ds\right]. \quad (25)$$

Now

$$P_{jk}(\Phi_{kj}+\Phi_{jk})=-\left[(e_j/\hbar c)\int^{r_k} A^s ds + (e_k/\hbar c)\int^{r_j} A^s ds\right].$$

So Eq. (23) becomes

$$e\{D_{kj}+D_{jk}+if_{kj}+if_{jk}\}\psi = e\left\{[i(e_k-e_j)/\hbar c]\int_{r_j}^{r_k} A^s ds\right\}P_{jk}\psi, \quad (26)$$

since

$$\int^{r_k} A^s ds - \int^{r_j} A^s ds = \int_{r_j}^{r_k} A^s ds. \quad (27)$$

To establish the hermiticity of the operator we observe that the quadratic form $(\psi_a, e\{D_{kj}+D_{jk}+if_{kj}+if_{jk}\}\psi_b)$ involves integrals of the type

$$I_{ab}=\int\int \psi_a{}^* e\left\{[i(e_k-e_j)/\hbar c]\int_{r_j}^{r_k} A^s ds\right\} \times P_{jk}\psi_b d\tau_j d\tau_k.$$

In the integration, the variables $\mathbf{r}_j$ and $\mathbf{r}_k$ may be interchanged without altering the value of the integral. Thus

$$I_{ab}=\int\int P_{jk}\left[\psi_a{}^* e\left\{[i(e_k-e_j)/\hbar c] \times \int_{r_j}^{r_k} A^s ds\right\}\right]\psi_b d\tau_j d\tau_k.$$

Since

$$P_{jk}\int_{r_j}^{r_k} A^s ds = -\int_{r_j}^{r_k} A^s ds,$$

as can be seen from Eq. (27),

$$I_{ab}=\int\int \psi_b e\left\{-[i(e_k-e_j)/\hbar c]\int_{r_j}^{r_k} A^s ds\right\} \times P_{jk}\psi_a{}^* d\tau_j d\tau_k = I_{ba}{}^*. \quad (28)$$

This establishes the desired result since integration over the variables referring to other particles will not alter the essential relationship.

V. THE EQUATIONS OF MOTION AND THE EXCHANGE CURRENT

Making use of Eq. (26), the Lagrangian density Eq. (15) may now be expressed in the form

$$L=-i\hbar(\psi, \partial\psi/\partial t)+(\psi, \textstyle\sum_k e_k\varphi_k\psi) + \sum_k(1/2m_kc^2)([\hbar c\,\mathrm{grad}_k\psi - ie_k\mathbf{A}_k\psi], \cdot[\hbar c\,\mathrm{grad}_k\psi - ie_k\mathbf{A}_k\psi]) + \sum_k \sum_{j<k}\left(\psi, J_{jk}e\left\{[i(e_k-e_j)/\hbar c] \times\int_{r_j}^{r_k} A^s ds\right\}P_{jk}\psi\right). \quad (29)$$

The corresponding Euler-Lagrange differential equation is

$$i\hbar\partial\psi/\partial t = -\textstyle\sum_k(1/2m_kc^2)(\hbar c\,\mathrm{grad}_k - ie_k\mathbf{A}_k)^2\psi + \sum_k e_k\varphi_k\psi + \sum_k\sum_{j<k} e\left\{[i(e_k-e_j)/\hbar c] \times\int_{r_j}^{r_k} A^s ds\right\}P_{jk}\psi. \quad (30)$$

This is the required, gauge-invariant modification of the Schroedinger equation. From Eq. (28) it follows that the equation for ψ^* may be obtained by taking the conjugate complex of Eq. (30).

The complete expression for the current density may now be obtained in the usual manner from consideration of an infinitesimal gauge transformation. Since L given by Eq. (29) is invariant under the gauge transformation Eq. (4), it will in particular be invariant if G is an infinitesimal. That is, for the variation

$$\begin{aligned} \delta\mathbf{A} &= -\mathrm{grad}G, \\ \delta\varphi &= \partial G/\partial(ct), \\ \delta\psi &= -(i/\hbar c)\textstyle\sum_j e_jG_j\psi, \\ \delta\psi^* &= (i/\hbar c)\textstyle\sum_j e_jG_j\psi^*, \end{aligned} \quad (31)$$

we have

$$\delta\int Ldtd\tau_1d\tau_2\cdots d\tau_N=0.$$

The variation in the integral produced by $\delta\psi$ and $\delta\psi^*$ vanishes in virtue of the equations of motion, Eq. (30) or Eq. (3). Thus we are left with

$$\int\left(\frac{\delta L}{\delta\mathbf{A}}\cdot\delta\mathbf{A}+\frac{\delta L}{\delta\varphi}\delta\varphi\right)dtd\tau_1\cdots d\tau_N=0. \quad (32)$$

Now, from Eq. (29)

$$(\delta L/\delta\varphi)\delta\varphi=(\psi,\ \textstyle\sum_k e_k\delta\varphi_k\psi), \quad (33)$$

and

$$(\delta L/\delta\mathbf{A})\cdot\delta\mathbf{A}=-\textstyle\sum_k(\hbar e_k/2m_kc)[(i\delta\mathbf{A}_k\psi,\cdot\text{grad}_k\psi)$$
$$+(\text{grad}_k\psi,\cdot i\delta\mathbf{A}_k\psi)-(2e_k/\hbar c)\mathbf{A}_k\cdot\delta\mathbf{A}_k(\psi,\psi)]$$
$$+\textstyle\sum_k\sum_{j<k}[i(e_k-e_j)/\hbar c]\displaystyle\int_{r_j}^{r_k}\delta A^s ds$$
$$\left(\psi,\ J_{jk}e\left\{[i(e_k-e_j)/\hbar c]\int_{r_j}^{r_k}A^s ds\right\}P_{jk}\psi\right). \quad (34)$$

We consider the limit of these expressions as $\mathbf{A}\to 0$, $\varphi\to 0$ since we are not interested in the small currents produced by the external fields. Then

$$(\delta L/\delta\mathbf{A})\cdot\delta\mathbf{A}=\textstyle\sum_k(i\hbar e_k/2m_kc)\delta\mathbf{A}_k$$
$$\cdot[(\psi,\text{grad}_k\psi)-(\text{grad}_k\psi,\psi)]+(i/\hbar c)\textstyle\sum_k$$
$$\times\textstyle\sum_{j<k}(e_k-e_j)\displaystyle\int_{r_j}^{r_k}\delta A^s ds(\psi,\ J_{jk}P_{jk}\psi). \quad (35)$$

Inserting the expressions for $\delta\mathbf{A}$ and $\delta\varphi$ from Eq. (31), we find

$$(\delta L/\delta\varphi)\delta\varphi=(\psi,\ \textstyle\sum_k e_k\partial G_k/\partial(ct)), \quad (36)$$

and

$$(\delta L/\delta\mathbf{A})\cdot\delta\mathbf{A}=-\textstyle\sum_k(i\hbar e_k/2m_kc)$$
$$\times[(\psi,\text{grad}_k\psi)-(\text{grad}_k\psi,\psi)]\cdot\text{grad}_kG_k$$
$$-(i/\hbar c)\textstyle\sum_k\sum_{j<k}(e_k-e_j)$$
$$\times(G_k-G_j)(\psi,\ J_{jk}P_{jk}\psi), \quad (37)$$

where $G_k=G(\mathbf{r}_k)$. The last term can be put into a more useful form by noting that

$$G_j=P_{jk}G_k=e\{D_{jk}\}G_k=\sum_{n=0}^{n=\infty}\frac{1}{n!}(\mathbf{r}_{jk}\cdot\text{grad}_k)^nG_k. \quad (38)$$

Introducing Eqs. (36)–(38) into Eq. (32) and integrating by parts, we find

$$\textstyle\sum_k\displaystyle\int dtd\tau_1\cdots d\tau_NG_k\left\{-e_k\frac{\partial}{\partial t}(\psi,\psi)\right.$$
$$+(ie_k\hbar/2m_k)\,\text{div}_k[(\psi,\text{grad}_k\psi)-(\text{grad}_k\psi,\psi)]$$
$$+(i/\hbar)\textstyle\sum_{j<k}(e_k-e_j)\displaystyle\sum_{n=1}^{n=\infty}\frac{(-1)^n}{n!}(\text{div}_k\mathbf{r}_{jk})^n$$
$$\left.\times(\psi,\ J_{jk}P_{jk}\psi)\right\}=0, \quad (39)$$

where the operator

$$(\text{div}_k\mathbf{r}_{jk})=-3+\mathbf{r}_{jk}\cdot\text{grad}_k.$$

Now by making use of the fact that G_k has an arbitrary functional form, Eq. (39) may be written as the conventional equation of continuity

$$\partial\rho/\partial t+\text{div}(\mathbf{S}_0+\mathbf{S}_x)=0, \quad (40)$$

with

$$\rho(\mathbf{r})=\textstyle\sum_k e_k\displaystyle\int(\psi,\psi)_{r_k=r}$$
$$\times d\tau_1\cdots d\tau_{k-1}d\tau_{k+1}\cdots d\tau_N, \quad (41)$$

$$\mathbf{S}_0(\mathbf{r})=\textstyle\sum_k(e_k\hbar/2im_k)\displaystyle\int[(\psi,\text{grad}_k\psi)$$
$$-(\text{grad}_k\psi,\psi)]_{r_k=r}d\tau_1\cdots d\tau_{k-1}d\tau_{k+1}\cdots d\tau_N, \quad (42)$$

and

$$\mathbf{S}_x(\mathbf{r})=-(i/\hbar)\textstyle\sum_k\sum_{j<k}(e_k-e_j)$$
$$\times\int\left[\mathbf{r}_{jk}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\right.$$
$$\left.\times(\text{div}_k\mathbf{r}_{jk})^{n-1}(\psi,\ J_{jk}P_{jk}\psi)\right]_{r_k=r}$$
$$\times d\tau_1\cdots d\tau_{k-1}d\tau_{k+1}\cdots d\tau_N. \quad (43)$$

The expressions for ρ and $\mathbf{S}_0$ are the conventional charge and current densities and $\mathbf{S}_x$ may clearly be interpreted as the exchange current density.

In order to complete the description of the exchange current, it will be shown that the energy of interaction between a weak external field and the exchange current has the conventional form

$$U=-(1/c)\int \mathbf{A}\cdot\mathbf{S}_x d\tau. \qquad (44)$$

From the Schroedinger equation, Eq. (30), the average value of the exchange energy is found to be

$$\sum_k \sum_{j<k}\int (\psi,\, J_{jk}P_{jk}\psi)e\left\{(i/\hbar c)(e_k-e_j)\times\int_{r_j}^{r_k} A^s ds\right\}d\tau_1\cdots d\tau_N.$$

For weak external fields the exponential may be expanded. The linear term then yields the interaction U in the desired approximation. Thus

$$U=\sum_k \sum_{j<k}(i/\hbar c)\int\left[(\psi,\, J_{jk}P_{jk}\psi)(e_k-e_j)\times\int_{r_j}^{r_k} A^s ds\right]d\tau_1\cdots d\tau_N. \qquad (45)$$

The line integral in Eq. (45) can be re-expressed by making use of the expansion of $A^s(\mathbf{r})$ about the point $\mathbf{r}_k$:

$$A^s(\mathbf{r})=\sum_{n=0}^{\infty}\frac{1}{n!}(\Delta\mathbf{r}_k\cdot\mathrm{grad}_k)^n A^s(\mathbf{r}_k) =\sum_{n=0}^{\infty}\frac{1}{n!}|\Delta\mathbf{r}_k|^n\left(\frac{d^n}{ds^n}A^s\right)_{r_k=r},$$

where $\Delta\mathbf{r}_k=\mathbf{r}-\mathbf{r}_k$ so $|\Delta\mathbf{r}_k|=s$, the length of the path measured from the point $\mathbf{r}_k$. Thus

$$\int_{r_j}^{r_k} A^s ds=-\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{d^n}{ds^n}A^s\right)_{r_j}\int_{r_k}^{r_j} s^n ds \qquad (46)$$
$$=-\sum_{n=0}^{\infty}\frac{1}{(n+1)!}(\mathbf{r}_{jk}\cdot\mathrm{grad}_k)^n(\mathbf{r}_{jk}\cdot\mathbf{A}_k).$$

Inserting this expression in Eq. (45) and integrating by parts n times, we obtain

$$U=-(i/\hbar c)\sum_k \sum_{j<k}(e_k-e_j)\times\int\left[(\mathbf{A}_k\cdot\mathbf{r}_{jk})\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}(\mathrm{div}_k\mathbf{r}_{jk})^n\times(\psi,\, J_{jk}P_{jk})\right]d\tau_1\cdots d\tau_N.$$

The introduction of $\mathbf{S}_x$ from Eq. (43) leads immediately to the desired result, Eq. (44).

This result indicates that variation with respect to the potentials of the total Lagrangian describing both the particles and the electromagnetic field will lead to Maxwell's equations involving the total current $\mathbf{S}_0+\mathbf{S}_x$ in the usual way.

VI. THE EXCHANGE MOMENT

In a uniform magnetic field the interaction energy with the exchange current can also be expressed as

$$U=-(\mathbf{M}_x\cdot\mathbf{H}), \qquad (47)$$

where $\mathbf{M}$ is the exchange magnetic moment. Since the vector potential for such a field is $\mathbf{A}=-(1/2)[\mathbf{r}\times\mathbf{H}]$, we find from Eq. (44) the usual expression

$$\mathbf{M}_x=(1/2c)\int[\mathbf{r}\times\mathbf{S}_x]d\tau. \qquad (48)$$

However, because of the complicated form of $\mathbf{S}_x$, it is somewhat simpler to deal directly with the expression, Eq. (45). Inserting the value of $\mathbf{A}$ in Eq. (45) we find that the line integral can be immediately evaluated since $[\mathbf{r}\times\mathbf{H}]^s$ is constant along the path of integration:

$$\int_{r_j}^{r_k} A^s ds=-\tfrac{1}{2}([\mathbf{r}_k\times\mathbf{r}_j]\cdot\mathbf{H}). \qquad (49)$$

Thus

$$U=-(i/2\hbar c)\sum_k \sum_{j<k}(e_k-e_j)\times\int(\psi,\, J_{jk}P_{jk}\psi)([\mathbf{r}_k\times\mathbf{r}_j]\cdot\mathbf{H})d\tau_1\cdots d\tau_N,$$

or, comparing Eq. (47)

$$\mathbf{M}_x=(i/2\hbar c)\sum_k \sum_{j<k}(e_k-e_j)\times\int(\psi,\, J_{jk}P_{jk}\psi)[\mathbf{r}_k\times\mathbf{r}_j]d\tau_1\cdots d\tau_N. \qquad (50)$$

It is to be noted that in the double sum defining $\mathbf{M}_x$ those terms vanish for which both j and k refer to neutrons or those for which both j and k refer to protons, since in such cases $e_k - e_j \doteq 0$.

Equation (50) may therefore be replaced by

$$\mathbf{M}_x = (ie/2\hbar c)\sum_\nu \sum_\pi \int [\mathbf{r}_\pi \times \mathbf{r}_\nu] \times (\psi, J_{\pi\nu} P_{\pi\nu} \psi) d\tau_1 \cdots d\tau_N, \quad (51)$$

where the indices π refer only to protons and the indices ν to neutrons. This expression is antisymmetric for interchange of neutrons and protons, therefore *the exchange moments of conjugate pairs of nuclei* (i.e., those that can be obtained from one another by interchange of neutrons and protons) *are equal in magnitude and opposite in sign.* It also follows that *the exchange moments of self-conjugate nuclei vanish.* Thus one cannot expect to explain the anomalies[7] in the moments of H^2, Li^6, B^{10}, N^{14} by the introduction of an exchange current.

VII. APPLICATION TO H^3 AND He^3

The exchange moments of the nuclei H^3 and He^3 may now be obtained from Eq. (51). As an immediate consequence of the theorem stated at the end of the foregoing paragraph we find that the exchange moments of H^3 and He^3 are equal in magnitude and opposite in sign. Villars[2] obtained the same result in his field theoretical treatment of the moments. One can conclude that the sum of the moments of the two nuclei does not involve the exchange moments so the validity of the general theorem[8] concerning the sum of the moments is not affected:

$$\mu(H^3) + \mu(He^3) = \mu_p + \mu_n - 2(\mu_p + \mu_n - \tfrac{1}{2})(3D^2 - {}^4P^2 + 2\,{}^2P^2)/3. \quad (52)$$

Thus, for a given nuclear wave function it is necessary to calculate the exchange moment for only one of the two nuclei. The calculation will be carried through for H^3.

In order to evaluate the integral in Eq. (51) it will be necessary to make some assumption concerning the nature of the potential, $J_{\pi\nu}$, and the form of the wave functions. It will be assumed that the potential is the sum of a spin-independent and a tensor interaction. The functional form of the potential will be taken to be Gaussian with the constants determined from the properties of the deuteron.[9] The wave function which has been found[10] using this potential will be used here. No serious difference would be expected if the Gerjuoy-Schwinger[11] functions were used. This wave function contains four percent ^4D function and no ^2P or ^4P function. In the calculation which produced this function the tensor interaction was treated as an ordinary potential rather than an exchange potential, but either of these potentials will give about the same result for the D state probability. The present considerations are based on the assumption that the tensor interaction involves a Majorana exchange operator.

If the positions of the neutrons in H^3 are $\mathbf{r}_1$ and $\mathbf{r}_2$, and the position vector of the proton is $\mathbf{r}_3$, Eq. (51) gives for the exchange moment

$$\mathbf{M}_x = (ie/2\hbar c)\int \{[\mathbf{r}_3 \times \mathbf{r}_1](\psi, J_{13}P_{13}\psi) + [\mathbf{r}_3 \times \mathbf{r}_2](\psi, J_{23}P_{23}\psi)\} d\tau_1 d\tau_2 d\tau_3. \quad (53)$$

Now the wave function ψ is the sum of an S function and a D function. The integrand will involve a product of two S functions, a product of an S and D function, and the product of two D functions. Since $(\mathbf{r}_3 \times \mathbf{r}_1)$ and $(\mathbf{r}_3 \times \mathbf{r}_2)$ have the transformation properties of a P function under space rotations, the only contribution to the integral from the spin independent term in J arises from the product of the two D functions. The tensor interaction term in J has the transformation properties of a D function so this term will make a contribution for the product of S and D functions as well as the product of the two D functions. If we denote by S and D the amplitudes of the S and D wave functions, then $S^2 = 0.96$, $D^2 = 0.04$. A direct calculation of the average exchange movement based on Eq. (53), using the above-mentioned wave functions,[10]

[7] R. G. Sachs, Phys. Rev. **69**, 611 (1946).
[8] R. G. Sachs and J. Schwinger, Phys. Rev. **70**, 41 (1946).
[9] S. Moszkowski and R. G. Sachs, Phys. Rev. **73**, 184 (1948).
[10] M. Goeppert-Mayer and R. G. Sachs, Phys. Rev. **73**, 185 (1948).
[11] E. Gerjuoy and J. Schwinger, Phys. Rev. **61**, 138 (1942).

leads to the result

$$M_x = [0.080\gamma SD + 0.038D^2 - 0.073\gamma D^2] \times (J_0 a_0^2 M/\hbar^2) \quad (54)$$

if the magnetic moment is measured in units of nuclear magnetons. Here, J_0 is the strength of the Gaussian neutron-proton interaction, a_0 is the range of the interaction, M is the mass of the proton, and γ is the strength of the tensor interaction in the notation of Rarita and Schwinger.[12] The numerical coefficients depend explicitly on the H^3 wave functions, but in such an insensitive manner that Eq. (54) should yield a fair estimate of the exchange moment for values of J_0 and γ different from those used to determine the wave functions. It seems likely that Eq. (54) can also be used to estimate the moment for a square well potential. For the Gauss potential we take[9] $\gamma = 0.53$, $J_0 a_0^2 M/\hbar^2 = 4.29$ and find

$$M_x = 0.035 \text{ n.m.} \quad (55)$$

If we assume that Eq. (54) is adequate to represent the exchange moment for a square well potential, the Rarita-Schwinger[12] values $\gamma = 0.775$, $J_0 a_0^2 M/\hbar^2 = 2.80$ lead to the estimate

$$M_x \approx 0.041 \text{ n.m.} \quad (56)$$

VIII. CONCLUSION

The exchange moments given by Eq. (55) or Eq. (56) are much too small to account for the difference of 0.27 n.m. between the observed moment and that to be expected,[8] neglecting exchange, on the basis of a four percent D state probability. This result is to be contrasted to that obtained by Villars,[2] using the pseudoscalar symmetric meson theory. The principal difference between the two theories arises from the fact that Villars' exchange current has a non-vanishing average value in the S state, while the phenomenological theory leads only to contributions which are proportional to the small amplitude of the D function. This is a consequence of the dependence of one term in the Villars exchange moment operator on the spins of the nucleons. Villars also finds an orbital contribution to the exchange moment which is very similar in form to that obtained here. He ignores the latter contribution, since he treats the wave function of the nucleus as an S function.

This situation is analogous to that which obtains in the theory of the magnetic moment of the electron. If one bases the theory of the interaction of the electron with the electromagnetic field on the non-relativistic Schroedinger equation, no information is obtained concerning the intrinsic magnetic moment of the electron which is associated with the spin. On the other hand, the Dirac theory in the non-relativistic limit leads directly to an expression for the intrinsic moment. Similarly, a relativistic formulation of the theory of interacting nucleons is required to give, even in the non-relativistic limit, an expression for the spin dependent part of the exchange moment. The meson field theories provide a possible formulation of the theory. Unfortunately, the pseudoscalar symmetric theory which accounts for the observed exchange moment does not give a reasonable account of nuclear forces since it leads to the undesirable r^{-3} potential. On the other hand, work which has been carried out by Villars and Thellung,[5] indicates that modifications of the meson theories which eliminate the undesirable potential do not lead to the correct exchange moment.

It can be concluded that the orbital exchange moment which is obtained here on phenomenological grounds is well founded, but not adequate to account for the observations. The principal contribution to the exchange moment would appear to depend in a detailed way on the nature of the field describing the interaction between nucleons. Therefore, the magnetic moments of H^3 and He^3 may prove to be a means for obtaining direct information concerning the nature of the field.

This investigation was carried out as a direct consequence of a very interesting discussion with Professor Fermi.

[12] W. Rarita and J. Schwinger, Phys. Rev. 59, 436 (1941).