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The Influence of Relaxation on the Two Velocity Field Model of Helium II

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In a previous paper (see reference 1) we have introduced the concept of a time of relaxation in discussing disturbances of thermal equilibrium between the populations of the "superfluid" state and the normal states in helium II, in order to relate second sound with heat conductivity.

For a complete discussion of the unrelaxed, the fully relaxed, and the transition region, we need at least one more independent variable than the two which specify the thermodynamic state of a system of one component in internal equilibrium. In a Taylor expansion of the energy as a function of such a set of three variables the coefficients at partial or zero relaxation will differ from the ordinary values calculated from equilibrium thermodynamics. The wave equations ordinarily deduced are therefore modified.

Expressions are obtained for the velocities of propagation of both sounds at infinite frequency (zero relaxation) and at zero frequency (full relaxation). At zero frequency, second sound reduces to thermal diffusion. Normal sound undergoes a sharp Einstein dispersion-and-attenuation at the lambda temperature where the relaxation time passes abruptly through the critical value for all generally used frequencies. The theory is used to compute x, the fraction of normal particles, from thermodynamic data and second sound velocity. The energy per normal particle is obtained as a function of temperature from x and the thermodynamic data on total heat content.

IN a previous paper[1] we have introduced the concept of a time of relaxation in discussing disturbances of thermal equilibrium between the populations of the "superfluid" state and the normal states in helium II, in order to relate second sound with the heat conductivity. For a complete discussion of the unrelaxed state (infinite frequency), the fully relaxed state (thermal equilibrium at zero frequency) and the transition region (partially relaxed states), we need at least one more independent variable than the two which specify the thermodynamical state of a system of one component when it is in internal equilibrium. Appropriate variables seem to be ρ, the density of the whole liquid, x, the fraction of the liquid in the normal states

$$x=\rho_x/\rho, \tag{1}$$

where ρ_x is the density of the normal component, and σ the entropy per unit mass of normal component:

$$\sigma=S/x, \tag{2}$$

where S is the entropy per unit mass of the liquid, contributed by the normal component. It will be assumed with Tisza[2] and Landau[3] that S is the whole entropy of the mixture. In this case the above three independent variables, ρ, x, σ, are sufficient to define the state of the system when it is not in a state of internal (thermal) equilibrium. In the case of full internal equilibrium ρ and x become equivalent to V and T, respectively, while σ ceases to be independent because in (2) both x and S are now definite functions of T. At the other extreme, in the entire absence of relaxation (infinite frequency) the entropy per normal particle remains constant as there is no time for interaction:

$$\Delta\sigma=0. \tag{3}$$

In a Taylor expansion of the variation of any function of the three variables, ρ, x and σ, only two variables will again appear explicitly; however, the coefficients at zero relaxation will differ from the ordinary equilibrium coefficients that apply to full relaxation. For a fully relaxed system, the variation of the energy may be written:

$$\Delta_rE=(\partial E/\partial\rho)_x\Delta\rho+(\partial E/\partial x)_\rho\Delta x+\text{second-order terms}. \tag{4}$$

As long as we consider only variations around an energy minimum, i.e., a state of equilibrium, the first-order terms vanish identically, and only second-order derivatives remain. For the completely unrelaxed case where (3) applies the variations of energy become, therefore,

$$\Delta_uE=\tfrac{1}{2}(\partial^2E/\partial\rho^2)_{x,\sigma}\Delta\rho^2+(\partial^2E/\partial x\partial\rho)_\sigma\Delta x\Delta\rho+\tfrac{1}{2}(\partial^2E/\partial x^2)_{\rho\sigma}\Delta x^2. \tag{5}$$

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[1] W. Band and L. Meyer, Phys. Rev. 73, 226 (1948).
[2] L. Tisza, Phys. Rev. 72, 838 (1947).
[3] L. Landau, J. Phys. U.S.S.R. 5, 711 (1941).

The second derivatives here have two suffixes to show that they are not the ordinary equilibrium partial derivatives for which σ would not be constant. These unrelaxed derivatives are evaluated in Appendix A.

To study the partially relaxed system in the intermediate region between the above two extremes we introduce a generalized temperature, T_x, defined by

$$T_x = (\partial E/\partial S)_{x\rho} = (1/x)(\partial E/\partial \sigma)_{x\rho} \tag{6}$$

(in any degree of relaxation). In Appendix A, it is shown that the first variation of energy can be expressed in the form

$$\Delta E = T_x \Delta S - P_x \Delta V + \mu_x \Delta x, \tag{7}$$

where

$$P_x \equiv \rho^2 (\partial E/\partial \rho)_{x\sigma} = P + (T_x - T)(\partial P/\partial T)_v \tag{8}$$

and

$$\mu_x = (C/x')(1 - T_x/T). \tag{9}$$

Here P_x is a generalized pressure, μ_x plays the part of a chemical potential for the mixture, C is the specific heat, and x' is $\partial x/\partial T$. In the fully relaxed limit $T_x \rightarrow T$, $P_x \rightarrow P$, and $\mu_x \rightarrow 0$.

The quantities $(\partial T_x/\partial T)_{x\sigma}$ and $(\partial T_x/\partial T)_{\rho\sigma}$ are related to each other. As already assumed, the heat content per gram, Q, of helium II is due to the normal particles only, so

$$Q = x\epsilon, \tag{10}$$

where ϵ is the mean energy per gram of normal particles. The specific heat per gram is then

$$C = dQ/dT = (dx/dT)\epsilon + (d\epsilon/dT)\cdot x. \quad (11)^*$$

In the completely unrelaxed case, variations of x at constant ρ and σ are not capable of altering ϵ because of absence of interaction with the superfluid. This means that, comparing equal variations in energy, relaxed and unrelaxed:

$$\Delta_u E = C\Delta T = x'\epsilon \cdot \Delta T_x)_{\rho\sigma}. \tag{12}$$

On the other hand, in an unrelaxed variation of ρ, keeping x constant, only ϵ can vary, so that

$$\Delta_u E = \epsilon' \cdot x \cdot \Delta T_x)_{x\sigma}. \quad (13)^{**}$$

Comparing (12) and (13) we find, writing a suffix "u" again to indicate the completely unrelaxed values:

$${}_u(\partial T_x/\partial T)_{\rho\sigma} = C/x'\epsilon, \quad {}_u(\partial T_x/\partial T)_{x\sigma} = C/\epsilon' x, \tag{14}$$

or by (10)

$${}_u(\partial T_x/\partial T)_{\rho\sigma} = xC/x'Q, \tag{15}$$

and

$${}_u(\partial T_x/\partial T)_{x\sigma} = 1/[1 - (x'Q/xC)]. \tag{16}$$

Consider two local adiabatic compressions, one ideal with full relaxation, the other actual with incomplete relaxation; they are accompanied by temperature increase T and T_x, respectively. The quantity

$$\chi = T_x - T \tag{17}$$

is a measure of the lack of internal equilibrium in the incompletely relaxed compression. Following Frenkel,[4] the tendency of the system to return to internal equilibrium will be assumed proportional to χ and we write

$$d\chi/dt = -\chi/\tau_2 + (d\chi/dt)_\infty \tag{18}$$

which, with Frenkel's operator

$$A_2 \equiv 1 + \tau_2 d/dt, \tag{19}$$

leads after some algebraic work (given in Appendix B) to the following result: The effective value $(\partial T_x/\partial T)^*$ under partial relaxation is given by

$$(\partial T_x/\partial T)^* - 1 = [(\partial T_x/\partial T)_u - 1](1 - A_2^{-1}), \tag{20}$$

where the expression $(\partial T_x/\partial T)_u$ appearing on the right side represents its value under complete absence of relaxation (infinite frequency).

This relation permits us to express the elastic coefficients of (5) for the partially relaxed case in terms of the values they have in the completely unrelaxed limit.*** The thermodynamic part of the problem is then solved. The general results, derived in Appendix B, are expressible in terms

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* We neglect the small difference between C_p and C_v in helium II.

** In writing ϵ' here, it is assumed that the increase of ϵ resulting from the work performed on the system during increase in density is distributed over the different excited states of the normal component with sufficient rapidity: the term "unrelaxed" refers only to the equilibrium between the superfluid and the excited states as a whole. This assumption will have to be confirmed by measurements on the dispersion of ordinary sound below the lambda-point.

[4] J. Frenkel, *Kinetic Theory of Liquids* (Oxford University Press, Cambridge, 1946).

*** These coefficients, which are physically real quantities, appear in this operator-type relaxation theory as complex quantities essentially because we have chosen to ignore the fact that additional terms in $\Delta\sigma^2$ would appear in (5) for the partially relaxed case.

of directly measurable (i.e., fully relaxed or thermal equilibrium) quantities, as follows:

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\begin{aligned}
(\partial^2E/\partial x^2)_{\rho\sigma}{}^* &= S/xx' + (C/Tx'^2 - S/xx') \\
&\quad \times (1 - xC/x'Q)(1 - A_2{}^{-1}), \\
(\partial^2E/\partial x\partial\rho)_\sigma{}^* &= (\partial P/\partial T)_v[ST/xC + (1/x') \\
&\quad \times (1 - xC/x'Q)(1 - A_2{}^{-1})], \qquad (21) \\
\rho^2(\partial^2E/\partial\rho^2)_{x\sigma}{}^* &= \rho^2(\partial^2E/\partial\rho^2)_s + (\partial P/\partial T)_v \\
&\quad \times (\partial T/\partial\rho)_s(xC/x'Q - 1)^{-1}(1 - A_2{}^{-1}).
\end{aligned}
$$

The Lagrangian method, first used in this connection by Tisza,[2] lends itself most naturally to the derivation of wave propagation in terms of these "elastic" coefficients. The details of the argument are given in Appendix C.

Instead of assuming that the x particles obey a mass continuity law which would be the case only in the completely unrelaxed limit, the following relation is obtained:

$$\Delta x/x = (x'Q/xC)\nabla\cdot\bar{u}_x(\partial T_x/\partial T)_{\rho\sigma}{}^*, \qquad (22)$$

which reduces, by (20) to the mass continuity at zero relaxation, but becomes the conservation of heat at full relaxation.

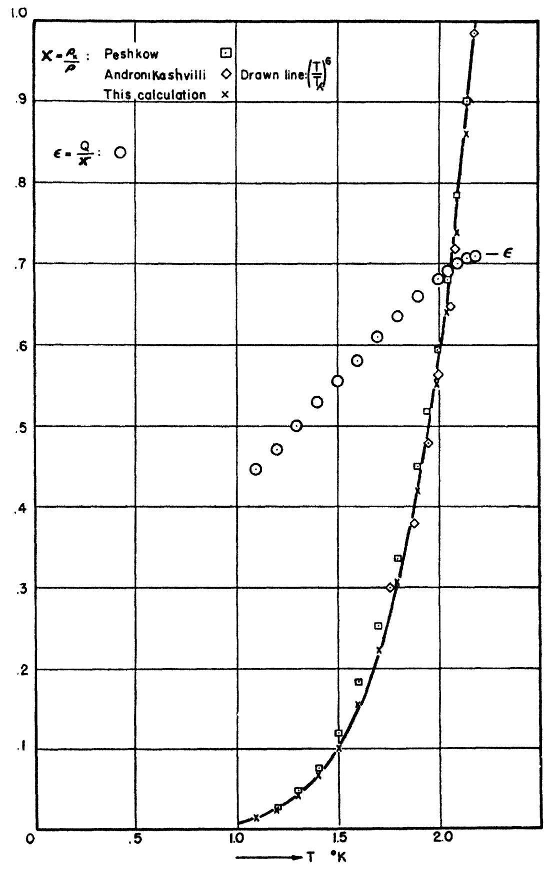

FIG. 1.

To the ordinary Lagrangian is added a dissipation function to take account of the rate at which momentum is being lost by collisions between opposing streams of superfluid and normal part. This loss appears in terms of a relaxation time τ_1 not necessarily identical with the equilibrium relaxation time introduced above in (18). The normal viscosity of the x fluid can also be included in this analysis, but it is found that it contributes only terms which are entirely negligible up to frequencies as high as 10^{11} cycles per sec.

Limiting the solutions of the wave equations to steady oscillations depending on time only through the factor $\exp(2\pi i\nu t)$, the operator $(1 - A_2{}^{-1})$ becomes equivalent finally to the factor

$$(1 - A_2{}^{-1}) = 1/[1 + 1/(2\pi i\nu\tau_2)]. \qquad (23)$$

This factor is directly responsible for a dispersion and attenuation in both ordinary and second sound. The relaxation time τ_1 enters only the second sound propagation, and is identical with the relaxation time previously introduced.[1]

The velocity of normal sound V_n is found to be:

$$V_n{}^2 = V_{n0}{}^2 + K/(1 + 1/2\pi i\nu\tau_2), \qquad (24)$$

where V_{n0} is the ordinary (relaxed) velocity of normal sound and

$$K = (T/\rho^2C)(\partial P/\partial T)_v{}^2/(xC/x'Q - 1). \qquad (25)$$

This is a typical Einstein[5] dispersion, with a peak attenuation given by

$$\alpha_{\max} = \tfrac{1}{2}\pi\nu K/V_{n0}{}^3. \qquad (26)$$

The sharp peak in attenuation observed by Pellam and Squire[6] in the neighborhood of the lambda-temperature may be due to the circumstance that τ_2 suddenly rises from times much smaller in order of magnitude than $1/2\pi\nu$ above T_λ to times considerably longer than this value below T_λ. Pellam and Squire report a peak attenuation "at least 3.5 per cm for 15 Mc." Equation (26) gives $\alpha_{\max}$ equal to 10 per cm. at this frequency.† The wave equation obtained

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[5] A. Einstein, Sitz. Berl. Akad. **1920**, p. 380.

[6] J. R. Pellam and C. F. Squire, Phys. Rev. **72**, 1245 (1947).

† The attenuation in the helium II region well below the lambda-point may be due to the dispersion mentioned in the footnote to Eq. (13).

for second sound is

$$(\partial^2 Z/\partial t^2)+(1/\tau_1)(\partial Z/\partial t)=V_s^2\nabla\nabla\cdot Z, \tag{27}$$

Z being the relative displacement of the x-fluid and V_s^2 is in general complex, depending on $2\pi i\nu\tau_2$. At high frequencies, however,

$$(\partial^2 Z/\partial t^2)=V_{s\infty}^2\nabla\nabla\cdot Z, \tag{28}$$

where

$$V_{s\infty}^2=[x(1-x)/(x')^2](C/T)\times[ST/Q+(1-xc/x'Q)]. \tag{29}$$

At very low frequencies τ_2 again drops out and the second sound wave approaches a diffusion type equation as $\nu\rightarrow 0$,

$$(\partial Z/\partial t)/\tau_1=[x'(1-x)/x^2](SQ^2/C^2)\nabla\nabla\cdot Z. \tag{30}$$

This change-over is expected to set in as soon as $2\pi\nu\tau_1<1$, and occurs for all ordinary frequencies at T_λ. The applications of (30) to experimental evidence will be given in the following paper.

The expression (29) for the velocity of propagation of second sound has been used to compute x as a function of T from known data on C, S, Q[7], †† and the published measurements of V_s by Peshkov[8] and by Lane, Fairbank, and Fairbank.[9] Equation (29) can be written in the form of a differential equation

$$(x'/x)^3-A(1/x-1)(x'/x)+B(1/x-1)=0, \tag{31}$$

where

$$A=C(1+ST/Q)/TV_s^2,$$

and

$$B=C^2/QTV_s^2 \tag{32}$$

are known functions of T between 1.1°K and T_λ. It would be ideally possible to obtain a unique solution for $x(T)$ from this equation if A and B were known over the entire temperature range from 0°K to T_λ, and if it could be assumed that $x(T)$ varied continuously from zero at 0°K to unity at T_λ. Data actually available permit only a selection of a family of curves which behave reasonably near T_λ and appear to go nicely towards zero as temperature falls. Since only half the temperature range is covered by the data, this family of curves is not sufficient to force a unique selection of one curve, although in fact the range of values of x is only a few percent.

TABLE I.

T °K	C cal./g deg.	Q cal./g	S cal./g deg.	V_s m/sec.	x	ϵ cal./g
1.0	0.023	0.00329	0.00383			
1.1	0.0411	0.00645	0.00684	18.2 ±.3	0.0143	0.451
1.2	0.0705	0.01194	0.01160	18.7 ±.1	0.025	0.478
1.3	0.116	0.02109	0.01892	18.95	0.042	0.502
1.4	0.176	0.03560	0.02964	19.82	0.067	0.531
1.5	0.269	0.05737	0.04477	20.09	0.104	0.552
1.6	0.391	0.0902	0.06582	20.33	0.157	0.575
1.7	0.525	0.1359	0.09324	20.35	0.222	0.612
1.8	0.691	0.1964	0.1278	19.90	0.307	0.640
1.9	0.918	0.2784	0.1714	18.68	0.420	0.663
2.0	1.218	0.3745	0.2260	16.69	0.552	0.678
2.05	1.41	0.4407	0.2583	14.94	0.641	0.688
2.10	1.66	0.5162	0.2951	12.25	0.738	0.699
2.15	2.05	0.6073	0.3383	8.05 ±.3	0.860	0.706
2.18	2.85	0.6776	0.3701	4.25 ±.4		
2.186		0.71	0.390	0	1.00	0.71

The final selection of the curve for $x(T)$ was made as follows. From some preliminary selection of the $x(T)$ curve, and the data on Q, we derive the curve for $\epsilon(T)$ by Eq. (10). From the two curves $x(T)$ and $\epsilon(T)$ we obtain graphically the sum $\epsilon'/\epsilon+x'/x$. By Eq. (11) we then have

$$C/Q=\epsilon'/\epsilon+x'/x. \tag{33}$$

By successive approximation, the selection of the $x(T)$ curve was improved until agreement with (33) lay within 1 percent for all temperatures from 1.1°K up to about 1.95°K. Above this temperature the difference rose to about 10 percent at the lambda-point; this may be due to the experimental uncertainties in C and Q near T_λ, or to errors in the graphical determination of x' and ϵ' near T_λ where both curves change their slopes very strongly. The results are shown in Fig. 1, and Table I, where the values of C, S, and Q are also given.

Peshkov[8] has published values for $x(T)$ deduced from Landau's theory of second sound[10] and using his measurements of the velocity. In that theory, entire absence of relaxation was assumed in setting up mass continuity equations, but the fully relaxed (equilibrium) "elastic coefficients" were used.

Andronikashvilli[11] has also deduced values of $x(T)$ from his measurements of the inertia of

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[7] W. H. Keesom, *Helium* (Amsterdam, New York, 1942).

†† Q and S have been calculated by numerical integration of the Leyden measurements on C by A. P. Keesom and W. K. Westmyze (see reference 7).

[8] V. Peshkov, J. Phys. U.S.S.R. **10**, 389 (1946).

[9] C. T. Lane, H. A. Fairbank, and W. M. Fairbank, Phys. Rev. **71**, 600 (1947).

[10] L. Landau, J. Phys. U.S.S.R. **10**, 91 (1947).

[11] E. Andronikashvilli, J. Phys. U.S.S.R. **10**, 201 (1946).

He II. His points lie about 5 to 10 percent below those of Peshkov, which is just about the limit of the experimental errors. The values calculated in the present paper, using non-equilibrium "elastic coefficients" are slightly lower than Andronikashvilli's, but are much nearer to the latter than those of Peshkov, the relative difference from Peshkov's being about 10 percent.[†††] Our results can, to a first approximation, be represented by a $(T/T_\lambda)^6$ curve up to about 1.8°K, and above this temperature they tend rather towards a 6.5 or 7th power curve.

The writers are happy to express their gratitude to Professor Clarence Zener for helpful discussions.

APPENDIX A

Thermodynamics

In general, any function of three variables ρ, x, and σ can be expanded:

$$\Delta F=(\partial F/\partial\rho)_{x\sigma}\Delta\rho+(\partial F/\partial x)_{\rho\sigma}\Delta x+(\partial F/\partial\sigma)_{x\rho}\Delta\sigma+\cdots. \quad \text{(A1)}$$

Under zero relaxation, $\Delta\sigma=0$, which reduces (A1) to its first two terms. Under full relaxation the expansion also reduces to two terms, because here $\Delta\sigma$ has its equilibrium value:

$$\Delta\sigma=(\partial\sigma/\partial\rho)_x\Delta\rho+(\partial\sigma/\partial x)_\rho\Delta x+\cdots. \quad \text{(A2)}$$

In what follows, partial first derivatives with one suffix will indicate equilibrium derivatives; first derivatives with two suffixes indicate that the equilibrium represented by (A2) cannot be *assumed.* In particular for internal equilibrium we also have

$$\Delta F=(\partial F/\partial\rho)_x\Delta\rho+(\partial F/\partial x)_\rho\Delta x+\cdots, \quad \text{(A3)}$$

and this must be identical with the result of using (A2) in (A1). Therefore it follows that

$$(\partial F/\partial x)_{\rho\sigma}=(\partial F/\partial x)_\rho-(\partial F/\partial\sigma)_{x\rho}(\partial\sigma/\partial x)_\rho, \quad \text{(A4)}$$

$$(\partial F/\partial\rho)_{x\sigma}=(\partial F/\partial\rho)_x-(\partial F/\partial\sigma)_{x\rho}(\partial\sigma/\partial\rho)_x. \quad \text{(A5)}$$

Writing E for F in (A4), it is then easy to show that

$$(\partial E/\partial x)_{\rho\sigma}=\mu_x+\sigma T_x, \quad \text{(A6)}$$

where μ_x and T_x have been defined in the text. Also from (A5) with E replacing F, we derive

$$\rho^2(\partial E/\partial\rho)_{x\sigma}=\rho^2(\partial E/\partial\rho)_x+T_x(\partial P/\partial T)_v. \quad \text{(A7)}$$

This last equation yields Eq. (8) in the text, while from (A6), (9) and (8) and the thermodynamic relation

$$P=\rho^2(\partial E/\partial\rho)_T+T(\partial P/\partial T)_v, \quad \text{(A8)}$$

Eq. (7) of the text also follows at once.

To evaluate the second derivatives of E the first derivatives given by (A6) and (A7) are substituted for F in (A4) and (A5). Thus (A4) gives

$$(\partial^2E/\partial x^2)_{\rho\sigma}=(\partial[\mu_x+\sigma T_x]/\partial x)_\rho-(\partial\sigma/\partial x)_\rho(\partial[\mu_x+\sigma T_x]/\partial\sigma)_{\rho x}. \quad \text{(A9)}$$

In the first term, the relaxed derivative, we may already set $\mu_x=0$ and $T_x=T$, so obtaining simply $(\partial\sigma T/\partial x)_\rho$. Again from (A4) with σT_x in place of F we have

$$(\partial\sigma T/\partial x)_\rho-(\partial\sigma/\partial x)_\rho(\partial\sigma T_x/\partial\sigma)_{\rho x}=(\partial\sigma T_x/\partial x)_{\rho\sigma}=\sigma(\partial T_x/\partial x)_{\rho\sigma}.$$

Using this in the second term on the right of (A9) we obtain

$$\begin{aligned}(\partial^2E/\partial x^2)_{\rho\sigma}&=\sigma(\partial T_x/\partial x)_{\rho\sigma}-(\partial\sigma/\partial x)_\rho(\partial\mu_x/\partial\sigma)_{\rho x}\\&=\sigma(\partial T_x/\partial x)_{\rho\sigma}+(\partial\mu_x/\partial x)_{\rho x}-(\partial\mu_x/\partial x)_\rho. \quad \text{(A10)}\end{aligned}$$

The last term here again vanishes because $\mu_x=0$ for the relaxed condition implied in the single suffix, while from (9)

$$(\partial\mu_x/\partial x)_{\rho\sigma}=C[1-(\partial T_x/\partial T)_{\rho\sigma}]/T\cdot(x')^2. \quad \text{(A11)}$$

Finally, therefore, evaluating the expression *at* the origin of the variations—namely, at $T_x=T$, we find

$$(\partial^2E/\partial x^2)_{\rho\sigma}=(S/xx')(\partial T_x/\partial T)_{\rho\sigma}+C[1-(\partial T_x/\partial T)_{\rho\sigma}]/T\cdot(x')^2$$

or

$$\boxed{(\partial^2E/\partial x^2)_{\rho\sigma}=[C/T\cdot(x')^2]\times[ST/Q+(1-xC/x'Q)]. \quad \text{(A12)}}$$

From (8) and (A7) and (A8)

$$\rho^2(\partial^2E/\partial\rho^2)_{\sigma x}=(\partial P/\partial\rho)_T-2P/\rho-[1-(\partial T_x/\partial T)_{x\sigma}](\partial T/\partial\rho)_s(\partial P/\partial T)_v \quad \text{(A13)}$$

when evaluated at $T_x=T$. This can be written

[†††] A comprehensive discussion of the different methods used to compute the fraction x has been given by J. G. Daunt at the Washington Meeting of the American Physical Society, May 1947, which will be published soon. We wish to thank Professor Daunt for sending us a copy of this report.

in the form

$$\boxed{\begin{aligned}\rho^2(\partial^2E/\partial\rho^2)_{x\sigma}=\rho^2(\partial^2E/\partial\rho^2)_x+(\partial T/\partial\rho)_s&\\ \times(\partial P/\partial T)_v/[xC/x'Q-1].&\quad\text{(A14)}\end{aligned}}$$

From (8) we may write

$$\begin{aligned}\rho^2(\partial^2E/\partial x\partial\rho)_\sigma=(\partial P/\partial x)_{\rho\sigma}&\\ -(1/x')[1-(\partial T_x/\partial T)_{\rho\sigma}](\partial P/\partial T)_v.&\end{aligned}$$

But

$$(\partial P/\partial x)_{\rho\sigma}=\sigma(\partial P/\partial S)_\rho=-(S/x)(\partial T/\partial V)_s,$$

so that we have, finally,

$$\boxed{\begin{aligned}\rho^2(\partial^2E/\partial x\partial\rho)_\sigma=-(S/x)(\partial T/\partial V)_s&\\ -(1/x')[1-xC/x'Q](\partial P/\partial T)_v.&\quad\text{(A15)}\end{aligned}}$$

APPENDIX B

Relaxation Theory

As stated in the text, Eqs. (17) and (18), the quantity $\chi=T_x-T$ is a measure of the lack of internal equilibrium, and we assume

$$d\chi/dt=-\chi/\tau_2+(d\chi/dt)_\infty,\qquad\text{(B1)}$$

where $(d\chi/dt)_\infty$ is the value expected at zero relaxation, $\tau_2=\infty$. But we can write

$$\begin{aligned}(d\chi/dt)_\infty=(d\Delta\rho/dt)(\partial\chi/\partial\rho)_{x\sigma}&\\ +(d\Delta x/dt)(\partial\chi/\partial x)_{\rho\sigma},&\quad\text{(B2)}\end{aligned}$$

where $\Delta\rho$ and Δx are the first-order variations present in the wave. Note that $\Delta\sigma=0$ at zero relaxation. However, by definition of χ we have

$$(\partial\chi/\partial\rho)_{x\sigma}=[(\partial T_x/\partial T)_{x\sigma}-1](\partial T/\partial\rho)_s,$$

and

$$(\partial\chi/\partial x)_{\rho\sigma}=[(\partial T_x/\partial T)_{\rho\sigma}-1](\partial T/\partial x)_{\rho\sigma}.\qquad\text{(B3)}$$

For variations at constant x we may therefore write (B1) as

$$\begin{aligned}d\chi_x/dt=-\chi_x/\tau_2+(\partial T/\partial\rho)_s&\\ \times[(\partial T_x/\partial T)_{x\sigma}-1](d\Delta\rho/dt),&\quad\text{(B4)}\end{aligned}$$

and for variations at constant ρ

$$\begin{aligned}d\chi_\rho/dt=-\chi_\rho/\tau_2+(\partial T/\partial x)_\rho&\\ \times[(\partial T_x/\partial T)_{\rho\sigma}-1](d\Delta x/dt).&\quad\text{(B5)‡}\end{aligned}$$

Introduce the Frenkel operator defined by Eq. (19) in the text, such that

$$\tau_2 d/dt=A_2-1,$$

then (B4) and (B5) can be solved formally for χ_x and χ_ρ:

$$\left.\begin{aligned}\chi_x&=(\partial T/\partial\rho)_s[(\partial T_x/\partial T)_{x\sigma}-1](1-A_2^{-1})\Delta\rho,\\ \chi_\rho&=(\partial T/\partial x)_\rho[(\partial T_x/\partial T)_{\rho\sigma}-1](1-A_2^{-1})\Delta x.\end{aligned}\right\}\ \text{(B6)}$$

At full relaxation, $\tau_2\rightarrow0$, $A_2\rightarrow1$ and χ_x, χ_ρ both vanish. At zero relaxation, $\tau_2\rightarrow\infty$, $A_2^{-1}\rightarrow0$, and therefore

$$\left.\begin{aligned}\chi_x&\rightarrow(\partial T/\partial\rho)_s[(\partial T_x/\partial T)_{x\sigma}-1]\Delta\rho,\\ \chi_\rho&\rightarrow(\partial T/\partial x)_\rho[(\partial T_x/\partial T)_{\rho\sigma}-1]\Delta x.\end{aligned}\right\}\ \text{(B7)}$$

It is now clear from a comparison between (B6) and (B7) that in the general (partially relaxed) case $[(\partial T_x/\partial T)-1](1-A_2^{-1})$ replaces $[(\partial T_x/\partial T)-1]$, as stated in the text, no matter whether x or ρ is kept constant.

Using (20) in the equation preceding (A12), we obtain for the general (partially relaxed) case the following expression for the "elastic coefficient" which determines the velocity of second sound propagation:

$$\begin{aligned}(\partial^2E/\partial x^2)_{\rho\sigma}=S/xx'+(C/Tx'^2-S/xx')&\\ \times[1-(\partial T_x/\partial T)_{\rho\sigma}](1-A_2^{-1}).&\quad\text{(B8)}\end{aligned}$$

By (15) this reduces to the expression given in the text, (21). Similarly, using (20) in (A13) yields the third expression in (21) for the elastic coefficient determining the velocity of ordinary sound. Finally, with the same substitution, (A15) gives the cross term coefficient in (21).

APPENDIX C

The Lagrangian Equations of Wave Motion

Using the two velocity field model, the Lagrangian per unit mass in the completely unrelaxed case is written

$$L_u=\tfrac{1}{2}x\bar{u}_x^2+\tfrac{1}{2}(1-x)\bar{u}_z^2-\Delta_uE,\qquad\text{(C1)}$$

where $\bar{u}_x$ and $\bar{u}_z$ are the velocities of the two components, normal, and superfluid, and Δ_uE is obtained from (5) in the text. Following Tisza[2] we use coordinates for the center of mass and relative displacement:

$$\langle\dot{\eta}\rangle=x\bar{u}_x+(1-x)\bar{u}_z,\qquad\text{(C2)}$$

$$\langle\dot{\xi}\rangle=\bar{u}_x-\langle\dot{\eta}\rangle=(1-x)(\bar{u}_x-\bar{u}_z),\qquad\text{(C3)}$$

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‡ As we are not interested eventually in the *shape* of the attenuation curve, we assume for simplicity that the relaxation times for the two processes, (B4) and (B5), are identical.

then the Lagrangian is

$$L_u=\tfrac{1}{2}\langle\dot{\eta}\rangle^2+\tfrac{1}{2}[x/(1-x)]\langle\dot{\xi}\rangle^2-\Delta_u E. \qquad \text{(C4)}$$

To express $\Delta_u E$ in terms of $\bar{\eta}$ and $\bar{\xi}$ we write the mass continuity equations, assuming zero relaxation, in the forms given by Tisza:[2]

$$\Delta\rho/\rho=-\nabla\cdot\bar{\eta}, \qquad \text{(C5)}$$

$$\Delta x/x=-\nabla\cdot(\bar{\eta}+\bar{\xi}). \qquad \text{(C6)}$$

It is then more convenient to use in place of $\bar{\xi}$ the variable

$$\bar{\zeta}=(x/(1-x))^{\frac{1}{2}}\bar{\xi}. \qquad \text{(C7)}$$

Using (C5), (C6), (C7) in (C4) with (5), we finally obtain for this limiting case

$$L_u=\tfrac{1}{2}\langle\dot{\eta}\rangle^2+\tfrac{1}{2}\langle\dot{\zeta}\rangle^2+f_u(\nabla\cdot\bar{\eta})^2+g_u(\nabla\cdot\bar{\zeta})^2 +h_u(\nabla\cdot\bar{\eta})(\nabla\cdot\bar{\zeta}), \qquad \text{(C8)}$$

where

$$\begin{aligned} f_u&=-\tfrac{1}{2}\rho^2(\partial^2E/\partial\rho^2)_{x\sigma}-\tfrac{1}{2}x^2(\partial^2E/\partial x^2)_{\rho\sigma},\\ g_u&=-\tfrac{1}{2}x(1-x)(\partial^2E/\partial x^2)_{\rho\sigma}, \qquad \text{(C9)}\\ h_u&=-(x(1-x))^{\frac{1}{2}}\rho(\partial^2E/\partial x\partial\rho)_\sigma -2x(x(1-x))^{\frac{1}{2}}(\partial^2E/\partial x^2)_{\rho\sigma}. \end{aligned}$$

Turning now to the fully relaxed limit, we can no longer use (C6), but are obliged instead to use the conservation of energy in the form

$$C\dot{T}=-\epsilon\nabla\cdot(x\bar{u}_x) \qquad \text{(C10)}$$

or, from (C3),

$$C\Delta T=-\epsilon x\nabla\cdot(\bar{\xi}+\bar{\eta}), \qquad \text{(C11)}$$

so that

$$\begin{aligned}\Delta x/x&=-(\epsilon x'/C)\nabla\cdot(\bar{\xi}+\bar{\eta}),\\ &=-(x'Q/xC)\nabla\cdot(\bar{\xi}+\bar{\eta}). \qquad \text{(C12)}\end{aligned}$$

This modifies the Lagrangian which will have the same form as (C8) except that the coefficients f, g, h have new values which we may denote as follows:

$$\begin{aligned} f_r&=-\tfrac{1}{2}\rho^2(\partial^2E/\partial\rho^2)_x-\tfrac{1}{2}x^2(\partial^2E/\partial x^2)_\rho(x'Q/xC),\\ g_r&=-\tfrac{1}{2}x(1-x)(\partial^2E/\partial x^2)_\rho(x'Q/xC)^2, \qquad \text{(C13)}\\ h_r&=-(x(1-x))^{\frac{1}{2}}\rho(\partial^2E/\partial x\partial\rho)(x'Q/xC) -2x(x(1-x))^{\frac{1}{2}}(\partial^2E/\partial x^2)_\rho(x'Q/xC)^2, \end{aligned}$$

where it is to be noted the partial derivatives are now all fully relaxed ones.

For the general case intermediate between the above two limits we note first that it is necessary to employ the general expressions given in (21) instead of the limiting forms for the elastic coefficients. In addition to this however, it is necessary to choose the correct generalization of the two limiting expressions of the conservation equation, (C6) and (C12), respectively. From (15) and (20) it is immediately clear that the required generalization is simply

$$\Delta x/x=(\partial T_x/\partial T)_{\rho x}(x'Q/xC)\nabla\cdot(\bar{\eta}+\bar{\xi}), \qquad \text{(C14)}$$

since this reduces to the correct expression at each limit, zero and full relaxation.

Using (C14) in place of (C6) or (C12), the Lagrangian retains the same form as before but now the coefficients f, g, h, must be replaced by the following expressions:

$$\begin{aligned} f^*&=-\tfrac{1}{2}\rho^2(\partial^2E/\partial\rho^2)^*-\tfrac{1}{2}x^2(\partial^2E/\partial x^2)^* \times(x'Q/xC)(\partial T_x/\partial T)_{\rho\sigma}{}^*,\\ g^*&=-\tfrac{1}{2}x(1-x)(\partial^2E/\partial x^2)^* \times(x'Q/xC)^2[(\partial T_x/\partial T)_{\rho\sigma}{}^*]^2, \qquad \text{(C15)}\\ h^*&=-(x(1-x))^{\frac{1}{2}}\rho(\partial^2E/\partial x\partial\rho)^* \times(x'Q/xC)(\partial T_x/\partial T)_{\rho\sigma}{}^* -2x(x(1-x))^{\frac{1}{2}}(\partial^2E/\partial x^2)^* \times(x'Q/xC)^2[(\partial T_x/\partial T)_{\rho\sigma}{}^*]^2. \end{aligned}$$

In general, terms with the asterisk contain the operator A_2.

The cross terms, h^*, represent coupling between the two sounds, but can be formally eliminated by transforming to new coordinates:

$$\begin{aligned}\bar{Y}&=\bar{\eta}\cos\theta+\bar{\zeta}\sin\theta,\\ \bar{Z}&=\bar{\zeta}\cos\theta-\bar{\eta}\sin\theta,\end{aligned} \qquad \text{(C16)}$$

where

$$\tan 2\theta=h^*/(f^*-g^*).$$

The Lagrangian now becomes

$$L^*=\tfrac{1}{2}\langle\dot{Y}\rangle^2+\tfrac{1}{2}\langle\dot{Z}\rangle^2+F^*(\nabla\cdot\bar{Y})^2+G^*(\nabla\cdot\bar{Z})^2, \qquad \text{(C17)}$$

where

$$F^*=f^*+h^{*2}/4f^*, \quad G^*=g^*-h^{*2}/4f^*. \qquad \text{(C18)}$$

From a numerical point of view the differences between F^* and f^*, and G^* and g^*, are practically negligible, so that actually this coupling can be neglected.

The components of the two vectors $\bar{Y}$ and $\bar{Z}$ are six dependent variables, say Y_i and Z_i, $i=1, 2, 3$; and the four independent variables are the position coordinates and time, x_i, t. Write for the partial derivatives

$$Y_{ij}=\partial Y_i/\partial x_j, \quad Z_{ij}=\partial Z_i/\partial x_j; \qquad \text{(C19)}$$

then the Lagrangian has the general form

$$L^*(x_i, t;\ Y_i, Z_i, Y_{ij}, Z_{ij}, Y_i, Z_i) \text{ per cm}^3. \qquad \text{(C20)}$$

The dynamical law, in absence of dissipation, has the variational form

$$\delta\int\int\int\int L^* dx_1 dx_2 dx_3 dt = 0,$$

and the Euler equations solving this are simply

$$\begin{aligned}&\partial L^*/\partial Y_i - \textstyle\sum_j(\partial/\partial x_j)(\partial L^*/\partial Y_{ij})\\&\qquad -(d/dt)(\partial L^*/\partial \dot{Y}_i) = 0 \text{ all } i,\\&\partial L^*/\partial Z_i - \textstyle\sum_j(\partial/\partial x_j)(\partial L^*/\partial Z_{ij})\\&\qquad -(d/dt)(\partial L^*/\partial \dot{Z}_i) = 0 \text{ all } i.\end{aligned} \tag{C21}$$

If there is dissipation, the rate of loss of momentum appears on the right of these equations.

Following the ideas already introduced,[1] we shall now assume that there is a dissipation of relative momentum by relaxation interference between the opposing streams of superfluid and normal part. The relative momentum is $(\partial L^*/\partial Z_i)$, so that we may define the relaxation time as τ_1 where the loss of momentum occurs at the rate $(\partial L^*/\partial Z_i)/\tau_1$. This loss appears only in the equation for $\bar{Z}$, not in that for $\bar{Y}$, in (C21). Writing out this modified form of (C21), using (C17), we find the two wave equations:

$$\begin{aligned}(\partial^2\bar{Y}/\partial t^2) + 2F^*\nabla\nabla\cdot\bar{Y} &= 0,\\(\partial^2\bar{Z}/\partial t^2) + 2G^*\nabla\nabla\cdot\bar{Z} &= -\langle\dot{Z}\rangle/\tau_1.\end{aligned} \tag{C22}$$

The first of these is the wave equation for ordinary sound, and includes dispersion and attenuation effects in the expression for F^*. The second equation is the propagation of second sound, including the relaxation term which is eventually responsible for the degeneration to diffusion.

Consider steady-state solutions $\bar{Y}$, $\bar{Z}$, depending on time only through the factor $\exp(2\pi i\nu t)$, then the operator $(1-A_2^{-1})$ contained in F^* and G^* becomes equivalent to multiplication by the complex factor

$$(1-A_2^{-1}) = 1 - 1/(1+2\pi i\nu\tau_2). \tag{C23}$$

This vanishes when $\tau_2 \to 0$ in full relaxation, and becomes unity when $\tau_2 \to \infty$ for zero relaxation.

Second Sound

Case 1: $2\pi\nu\tau_2 \gg 1$

From (21) we have in this case

$$\begin{aligned}&(\partial^2E/\partial x^2)^* \to (\partial^2E/\partial x^2)_{\rho\sigma}\\&\qquad -[(\partial^2E/\partial x^2)_{\rho\sigma} - S/xx']/2\pi i\nu\tau_2\end{aligned} \tag{C24}$$

gives the partially relaxed coefficient in terms of the unrelaxed one. Also from (15) and (20) we have in this case

$$\begin{aligned}&(\partial T_x/\partial T)_{\rho\sigma}{}^* \to xC/x'Q\\&\qquad +(1-xC/x'Q)/2\pi i\nu\tau_2.\end{aligned} \tag{C25}$$

Now use (C24) and (C25) in (C15) and (A12), the result is

$$-2g^* = V_{s\infty}{}^2(1+1/2\pi i\nu\tau_3)^{-1}, \tag{C26}$$

where $V_{s\infty}$ is given by (29) in the text, and where

$$\begin{aligned}&1/\tau_3 = (1/\tau_2)\{3 - 2x'Q/xC\\&\qquad - S(1-x)/x'V_{s\infty}{}^2\}.\end{aligned} \tag{C27}$$

Evidently the velocity of propagation of second sound at infinite frequency is $V_{s\infty}$ as stated in the text. Dispersion effects can only be expected if the frequency is such that the relaxation time given in (C27) is small enough to compare with $1/2\pi\nu$.

Case 2: $2\pi\nu\tau_2 \to 0$

Equation (21) now gives

$$(\partial^2E/\partial x^2)^* \to S/xx', \tag{C28}$$

and (20) gives, of course,

$$(\partial T_x/\partial T)^* \to 1.$$

Now (C15) yields immediately

$$-2g^* = x(1-x)(S/xx')(x'Q/xC)^2$$

so the wave equations becomes

$$(\partial^2\bar{Z}/\partial t^2) + \langle\dot{Z}\rangle/\tau_1 = V_{s0}{}^2\nabla\nabla\cdot\bar{Z}, \tag{C29}$$

where

$$V_{s0}{}^2 = x'(1-x)SQ^2/x^2C^2. \tag{C30}$$

If now also $2\pi\nu\tau_1 \to 0$, then $(\partial^2\bar{Z}/\partial t^2) \ll \langle\dot{Z}\rangle/\tau_1$ and (C29) reduces to the diffusion Eq. (30) as stated in the text.

Ordinary Sound

Neglecting terms of the order g^*/f^*, or $V_s{}^2/V_n{}^2$, we find for the velocity of propagation V_n of ordinary sound, from (C22) and (C15)

$$V_n{}^2 = -2F^* = \rho^2(\partial^2E/\partial\rho^2)^* \tag{C31}$$

or from (21)

$$V_n{}^2 = V_{n0}{}^2 + K/(1+1/2\pi i\nu\tau_2),$$

where K is as given in the text, Eqs. (24) and (25).

Case 1: $2\pi\nu\tau_2 \gg 1$

Here

$$V_n{}^2 = V_{n0}{}^2 + K + iK/2\pi\nu\tau_2, \qquad \text{(C32)}$$

so that at infinite frequency

$$V_{n\infty}{}^2 = V_{n0}{}^2 + K. \qquad \text{(C33)}$$

The attenuation coefficient arising from (C32), and entering the solution of the wave equation in the form

$$\bar{Y} = \bar{Y}_0 e^{-\alpha x} \exp 2\pi i\nu(t - r/V_n),$$

where r is position, is

$$\alpha = K/2\tau_2 V_{n0}{}^3. \qquad \text{(C34)}$$

Case 2: $2\pi\nu\tau_2 \ll 1$

Here

$$V_n{}^2 = V_{n0}{}^2 + K \cdot 2\pi i\nu\tau_2, \qquad (35)$$

and at zero frequency

$$V_n{}^2 = V_{n0}{}^2, \qquad \text{(C36)}$$

while the attenuation coefficient for this frequency range is

$$\alpha = 2\pi^2\nu^2\tau_2 K/V_{n0}{}^3. \qquad \text{(C37)}$$

The peak attenuation occurs at the frequency given by

$$2\pi\nu\tau_2 = (V_{n0}{}^2/(V_{n0}{}^2 + K))^{\frac{1}{2}},$$

which is practically unity. The peak attenuation coefficient is then

$$\alpha_{\max} = \tfrac{1}{2}\pi\nu K/V_{n0}{}^3, \qquad \text{(C38)}$$

as stated in the text.

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Relaxation Theory of Thermal Conduction in Helium II

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The equation for thermal diffusion derived in the foregoing paper agrees with the relaxation equation for second sound used previously (see reference 3), if the second sound velocity is corrected to zero frequency. The relaxation time τ is calculated to a higher accuracy. The peculiar property of the heat current in being proportional to the cube root of the temperature gradient can be phenomenologically reduced to the relation

$$u_s{}^2\tau_s = J,$$

where u_s is the velocity of the superfluid balancing the mass flow of normal particles transporting the energy. τ_s is a relaxation time derived from τ, the relaxation time responsible for the damping of second sound, by the conditions for frequency balance. J appears to be a simple function of temperature only, approaching $h/m\pi$ at temperatures below 1°K. The equations lead to a qualitative understanding of the reduction of thermal conductivity by mass flow found by Kapitza.

IN the previous paper[1] it has been shown that the wave equation for second sound degenerates into a diffusion equation approaching zero frequency and becomes

$$\langle\dot{Z}\rangle/\tau_1 = V_{s0}{}^2\nabla\nabla\cdot\bar{Z}; \quad V_{s0}{}^2 = x'(1-x)SQ^2/x^2C^2, \qquad (1)$$

where x is the fraction of the liquid in the normal (excited) states, $\bar{Z}$ is the relative displacement vector of the normal fluid, τ_1 is the relaxation time for momentum exchanges between the two component streams. Take the divergence of this equation: assume very small amplitude and absence of ordinary sound, then we shall have, by (C6) of the previous paper,

$$\nabla\cdot\bar{Z} \propto \Delta x/x.$$

Thus:

$$(1/\tau_1)(\partial/\partial t)(\Delta x) = V_{s0}{}^2\nabla\nabla\cdot(\Delta x). \qquad (2)$$

Here there is full relaxation so we have $\Delta x = x'\Delta T$,

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[1] W. Band and L. Meyer, Phys. Rev. **74**, 386 (1948).