the form

$$x_m = a[\eta_m(\partial/\partial\eta_5) - \eta_5(\partial/\partial\eta_m)], \quad (4)$$

where a has the dimension of length.

It appears a natural assumption that this factor a should be simply related to the fundamental length l_0 already introduced. Bearing in mind that $\partial/\partial\eta_5 = 2\pi i/l_0$, according to the assumption that η_5 appears only in the exponential factor, it is suggested that $a = l_0/2\pi i$. Thus,

$$x_m = (l_0/2\pi i)[\eta_m(\partial/\partial\eta_5) - \eta_5(\partial/\partial\eta_m)] \quad (5)$$

and

$$M_{m5} = m_0 c x_m. \quad (6)$$

Let the case of a free particle be considered and let its wave function be ψ.

The observed value of the coordinate x_m is $\bar{x}_m = \psi^+ x_m \psi$. Thus,

$$\bar{x}_m = (l_0/2\pi i)[\psi^+ \eta_m(\partial\psi/\partial\eta_5) - \psi^+\eta_5(\partial\psi/\partial\eta_m)];$$

and since

$$\partial\psi/\partial\eta_5 = 2\pi i\psi/l_0,$$

$$\bar{x}_m = \psi^+\eta_m\psi - (l_0/2\pi i)\psi^+\eta_5(\partial\psi/\partial\eta_m)$$

$$= \bar{\eta}_m - (l_0/2\pi i)\psi^+\eta_5(\partial\psi/\partial\eta_m). \quad (7)$$

The quantity $\bar{\eta}_m$ is the average value of the coordinate η_m, but, according to the formula (7), the observed value of the coordinate operator x_m differs from it by the second term on the right-hand side. According to the assumption that the value to be attached to η_5 is l_0, and since $\partial\psi/\partial\eta_m = (2\pi i/h)p_m\psi$, where p_m is the momentum conjugate to η_m, this term becomes $l_0{}^2\psi^+ p_m\psi/h$.

If the particle is moving with velocity v in the direction of η_m, this term has the value $h\beta/m_0c(1-\beta^2)^{\frac{1}{2}}$, where $\beta = v/c$. This may be interpreted as meaning that in any attempt to measure the coordinate of the particle there is an error of magnitude $h\beta/m_0c(1-\beta^2)^{\frac{1}{2}}$. It is thus impossible to locate the particle with greater accuracy than this.

By applying the formula (7) to the case $m=4$ it follows, according to the same assumption, that it is impossible to locate the particle in time with a greater accuracy than $h/m_0c^2(1-\beta^2)^{\frac{1}{2}}$. Both these results can be combined in the statement that there is no physical significance in associating any interval of proper time less than h/m_0c^2 with a particle of rest mass m_0.

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The Threshold for Spark Development by Streamer Mechanism in Uniform Fields

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THE streamer mechanism of the ordinary spark required a quantitative criterion for streamer advance for its promulgation. A semi-empirical condition for streamer advance was simultaneously and quite independently proposed by J. M. Meek[1] and by H. Raether.[2] The writer as well as Meek[3] and Raether[2] accepted the proposed condition that a streamer could advance when the positive space charge tip field created by the avalanche was approximately equal to the imposed field only as a temporary makeshift in order to launch the theory, since the more correct solution including photoelectric ionization appeared to lie well in the future. In a recent article the writer[4] indicated that the correct solution must retain a formal analogy to the well-known Townsend criterion for the spark at lower pressure, i.e., that the condition would have to have the form $Kfe^{\alpha\delta} = 1$. Recent analysis of the pre-onset positive burst pulse corona threshold has now shown the way to a proper procedure for calculating the threshold for streamer advance.[5] While the present lack of

[1] J. M. Meek, Phys. Rev. **57**, 722 (1940); L. B. Loeb and J. M. Meek, J. App. Phys. **11**, 438, 459 (1940).

[2] H. Raether, Zeits. f. Physik **117**, 375, 524 (1941); Archiv. f. Elektrotech. **34**, 49 (1940).

[3] L. B. Loeb and J. M. Meek, *Mechanism of the Electric Spark* (Stanford University Press, Stanford University, California, 1941), pp. 40 and 42.

[4] L. B. Loeb, Rev. Mod. Phys. **20**, 151 (1948).

[5] L. B. Loeb, Phys. Rev. **73**, 798 (1948).

fundamental data precludes a detailed study and test at the present time, it was felt important to indicate the line of reasoning in the hope that it will guide further work on the problem. It must also be indicated that while the problem may perhaps better be solved by some other approach, the present approach is the only one now obvious. In this, as in all threshold problems, one must work with equations involving average values of quantities subject to considerable fluctuations.

Consider a uniform electrical field of strength X. As shown elsewhere[6] an electron proceeding a distance δ from the cathode to close to the anode will produce an avalanche of $e^{\alpha\delta}$ electrons and positive ions lying within an average distance ρ of the avalanche axis. The value of ρ depends on the coefficient of electron diffusion D and is

$$\rho=(4Dt)^{\frac{1}{2}}, \tag{1}$$

with

$$t=(\delta/\bar{v}) \tag{2}$$

representing the time of avalanche advance at an average drift velocity $\bar{v}$. In a distance dx of advance, when the avalanche has progressed as far as δ, the number of new ions formed is

$$n=\alpha e^{\alpha\delta}dx, \tag{3}$$

where α is Townsend's first coefficient in the field X. The density of positive ions left behind by the electrons created in dx at δ is then given by

$$N=n/\pi\rho^2 dx=\alpha e^{\alpha}/\pi\rho^2. \tag{4}$$

If ρ is not too large compared to $1/\alpha$, one can approximate the space charge field X^1 produced by the density of positive ions N, at any distance x from the center of the charge of radius ρ at the end of the run δ by assuming the ions distributed uniformly in a sphere of radius ρ and volume $(4/3)\pi\rho^3$. Then

$$X^1=q\epsilon/x, \tag{5}$$

with

$$q=(4/3)\pi\rho^3 N \tag{6}$$

and ϵ the electron. Inserting N from (4) into (6),

$$q=(4/3)\rho\alpha e^{\alpha\delta} \tag{7}$$

and

$$X^1=(4/3)\epsilon\rho\alpha e^{\alpha\delta}/x^2=E/x^2. \tag{8}$$

[6] See reference 3, pp. 34–37; L. B. Loeb and J. M. Meek, J. App. Phys. 11, 440 (1940).

The streamer theory ascribes spark breakdown of the gas to the advance of the positive space charge initiated by the avalanche, from the anode backward towards the cathode, producing a conducting filament of plasma that bridges the gap. The advance takes place as a result of new avalanches initiated by photoelectrons created *in the gas* from the cathode side and which move towards the positive space charge in the combined fields $X+X^1$. Accompanying the creation of q positive ions within the sphere of radius ρ, the electron impacts also produced fq high energy photons[4, 5] of absorption coefficient μ, which on absorbtion ionize the gas surrounding the sphere. Now if one of these photons is absorbed far enough from the surface of the sphere of radius ρ on the cathode side so as to produce an electron and a new avalanche in the vector field

$$X_1=X+X^1=X+\frac{E}{x^2}, \tag{9}$$

which extends the space charge, the streamer will advance. Thus the threshold for streamer advance is fixed[4] as

$$Kqf=(4/3)K\rho\alpha Kfe^{\alpha\delta}=1. \tag{10}$$

Here the coefficient K is a complicated expression determining the chance of absorption and photoionization which may be regarded as follows. Assume a critical distance x_1 from the center of the space charge such that a photoelectron produced beyond x_1 towards the cathode in the field X_1 is capable of yielding the needed ionization. Then K will be given by[5]

$$K=\frac{\underline{a}}{4\pi}\exp(-\mu x_1). \tag{11}$$

The quantity $\underline{a}$ defines a solid angle within which the photon beam must lie in order that the streamer be extended effectively towards the cathode. Rigorously, an evaluation of such a solid angle would require a complicated averaging process involving the radius of x_1 and the variation of the field X_1 off the axis of the avalanche and relative to the direction of X. Roughly, since it simplifies calculation to set X_1 as the algebraic sum of X and X^1, one could evaluate $\underline{a}$ as the solid angle subtended at the center of the space charge by the intersection of a cylinder of radius ρ in the direction of streamer advance and a sphere of radius x_1 drawn about the center of the space charge. With $\underline{a}$ designated, it requires only the evaluation of x_1 to complete the solution. With this approach the Meek-Raether[1,2] condition for streamer advance is replaced by defining x_1 in such a way that the *new*, photoelectrically initiated avalanche, in moving from x_1 to ρ, must *create the same number* q of ions as *did the initiating avalanche*. This is indicated by equating q in Eq. (7) to the ionization produced by the new avalanche in going from x_1 to ρ in the field X_1, where Townsend's coefficient is now represented by α^1, yielding the relation

$$\rho^1\bar{\alpha}^1 \exp\left(\int_{x_1}^{\rho}\alpha^1 dx\right) = \rho\alpha e^{\alpha\delta}. \quad (12)$$

The quantity $\bar{\alpha}^1$ is the value of α^1 in the field X_1 at a distance ρ. The quantity ρ^1 is the value of ρ for the new avalanche, which by Eq. (1) is $\rho^1 = [4D(x_1/\bar{v})]^{\frac{1}{2}}$, such that with $\bar{v}$ sensibly constant

$$\rho/\rho^1 = (\delta/x_1)^{\frac{1}{2}}. \quad (13)$$

Equation (12) then becomes

$$\int_{x_1}^{\rho}\alpha^1 dx = \alpha\delta + \log_e(\alpha/\bar{\alpha}^1)(\delta/x_1)^{\frac{1}{2}}. \quad (14)$$

Now in the region at which sparking takes place in a given gas at a field X and pressure p, experimental data yield the relation between α, p, and X as

$$\alpha/p = F(X/p). \quad (15)^7$$

In the limited region of X/p, involved in any solution, the observed curves can be fitted by empirical analytical functions which give satisfactorily $\alpha = pF(X/p)$ for use in the solution of Eq. (14).[7] For example, in the case of air near atmospheric pressure, $F(X/p)$ is a quadratic which for simplicity of illustration in this case can be set as

$$\alpha = (A/p)(X^2). \quad (16)$$

Then replacing X of (16) by its value X_1 in this analysis, given by $X_1 = X + E/x^2$ as in Eqs. (8) and (9), and placing this in turn into Eq. (14), one has

$$(A/p)\int_{x_1}^{\rho}[X + (E/x^2)]dx = \alpha\delta + \log_e(\alpha/\bar{\alpha}^1)(x_1/\rho)^{\frac{1}{2}}. \quad (17)$$

Solution of this equation yields x_1.

To utilize this theory for the evaluation of a sparking threshold the procedure is as follows (assuming that δ and p are given, with $\alpha = pF(X/p)$, D, $\bar{v}$, f, and μ known): From relations (1) and (2) evaluate ρ. Choose a likely value of X for the spark. Evaluate α from (15) or (16) and solve for X^1 and E by means of (8). Calculate $e^{\alpha\delta}$. With these data solve (17) graphically, for x_1 using the value of α at X and of $\bar{\alpha}^1$ as derived from (8) and (15) with $x = \rho$. With x_1 evaluated, determine $\underline{a}$ as indicated and proceed to evaluate K from (11) using μ. With the values of f, K, and $e^{\alpha\delta}$ solve Eq. (10). If $(4/3)\rho\alpha Kfe^{\alpha\delta}$ as computed is greater than unity repeat with a lower value of X. Solution for the sparking field strength and sparking potential is then accomplished by trial and error as with Townsend's or Meek's equations,[1] with obviously a much more elaborate procedure. It is, however, a procedure that is in principle correct and now includes the effects of pressure on the absorption coefficient μ as well as on ionization density. The inclusion of this further pressure variation will increase the deviations from Paschen's law and should bring theory more into line with observations than does the Meek equation.[8] This derivation applies primarily to the calculation of breakdown in a uniform gap, presupposing that the streamer progresses primarily through the ionization by one effective avalanche primarily in the field direction. For the case of the pre-onset streamers in a highly divergent field, it is probable that streamer advance can be determined initially by the simultaneous influx of several electron avalanches. The condition for streamer onset on this basis, using the same procedure, has been developed by R. A. Wijsman, and leads to an equation which has the same general form as Eq. (10) with, however, a different set of constants.

[7] L. B. Loeb, *Fundamental Processes of Electrical Discharge in Gases* (John Wiley and Sons, Inc., New York, 1939), pp. 344 and 347.

[8] C. G. Miller and L. B. Loeb, Phys. Rev. **73**, 84 (1948).