Calculations were also made for the deuteron in a closely parallel form to serve as a control. The analogous S state calculation gives 76 percent for the analytic joining and 92 percent for the new type of joining. With tensor forces, the results are 66 percent and 77 percent for analytic and mere continuity joining, respectively.

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Scattering of Particles in Air Showers*

L. EYGES**
Cornell University, Ithaca, New York

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The radial distribution of singly scattered particles in air showers is discussed. The probability $\mathcal{P}(x')dx'$ that a singly scattered particle is found in the annular ring between x' and $x'+dx'$ is $\mathcal{P}(x')dx'=0.16(dx'/x'^3)$, where x' is the "Moliere unit." In the deduction of this formula Belenky's expression for the mean square length of a shower, which remains valid below the critical energy, is evaluated. The main process contributing to particles at large distances from the shower axis involves the radiation, and subsequent rematerialization, of electrons below the critical energy.

I. INTRODUCTION

EXTENSIVE cosmic-ray showers are initiated at the top of the atmosphere by particles with energies up to 10^{17} ev. In traversing the atmosphere the electrons and positrons in the shower are scattered by the electrostatic fields of the air nuclei, and over most of the shower's length can be deflected to distances of the order of hundreds of meters from the shower axis. The particles at the largest distances from the axis will be predominantly those which have been scattered once through a large angle (single scattering). For experiments such as those of Skobeltzyn *et al.*[1] it is important to know the radial distribution of these particles. Moliere[2] has calculated this, but the details of the calculation are not given. Moreover, he uses the Arley approximation, neglecting radiation processes below the critical energy, which gives too few low energy electrons, as Moliere himself recognizes. In this paper we have independently calculated the distribution function for singly scattered electrons without using the Arley approximation. It turns out that the most important process contributing to the electrons at large distances is radiation and rematerialization of electrons below the critical energy. Surprisingly enough, however, our result does not differ greatly from Moliere's. It therefore seems clear that Skobeltzyn's[1] assumption that the radial distribution falls off exponentially at large distances is not correct.

II. DERIVATION OF THE SINGLE-SCATTERING FORMULA

In this section we will derive a formula for the distribution of particles which have been once scattered through a sufficiently large angle that the probability of a second scattering through an angle of the same order is small. Hence these particles will retain their original direction except for small deviations resulting from multiple scattering. In this derivation several approximations will be made, without a discussion of their validity. The justification for these approxi-

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** Present address: Department of Mathematical Physics, The University, Birmingham, 15, England.

[1] D. V. Skobeltzyn, G. T. Zatsepin, and V. V. Miller, Phys. Rev. 71, 315 (1947).

[2] G. Moliere, *Cosmic Radiation* ed. Heisenberg (Dover Publications, New York, 1946), p. 26.

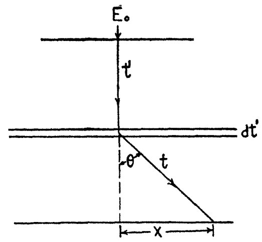

FIG. 1.

mations can only be made after evaluation of the formula, which is done in Sections III and IV. The validity of the approximations is then taken up in Section V.

The probability that an electron of energy E is scattered into the solid angle $2\pi \sin\theta d\theta$ in travelling a distance dt' (radiation units) is:[3]

$$d\sigma = \frac{1}{8\ln(181Z^{-\frac{1}{3}})}\frac{E_s^2}{E^2}\frac{\sin\theta d\theta}{(1-\cos\theta)^2}dt', \quad (1)$$

where $E_s = 21$ Mev is the 'characteristic scattering energy,' and Z is the average atomic number of the scatterer, air in our case. We will assume that although θ is large it is still small enough that the approximations $\sin\theta \approx \theta$, $1-\cos\theta \approx \theta^2/2$ are reasonably good. Using them (1) becomes:

$$d\sigma = \frac{1}{2\ln(181Z^{-\frac{1}{3}})}\frac{E_s^2}{E^2}\frac{d\theta}{\theta^3}dt'. \quad (2)$$

Consider now a shower initiated by a particle of energy E_0. Let the number of particles of energy E at depth t' be $\pi(E_0, E, t')$.

The number of particles of energy E scattered in the layer dt' is $\pi(E_0, E, t')d\sigma$. In traversing an additional thickness t of material these particles suffer a lateral displacement (Fig. 1)

$$x = t\sin\theta \approx t\theta \quad (3)$$

and produce a shower of their own with a total number of particles $\Pi(E, 0, t)$. The number of particles in x, dx at the thickness T is then given by the integral

$$\frac{E_s^2}{2\ln(181Z^{-\frac{1}{3}})}\frac{dx}{x^3}\int_0^{E_0}\frac{dE}{E^2} \times\int_0^T \pi(E_0, E, t')\Pi(E, 0, T-t')(T-t')^2 dt', \quad (4)$$

[3] B. Rossi and K. Greisen, Rev. Mod. Phys. **13**, 262 (1941).

where we have used $(d\theta/\theta^3) = (dx/x^3)t^2 = (dx/x^3)(T-t')^2$. This integral would be very difficult to evaluate; in fact, it seems impossible to do it analytically. We are saved by one fact; the radial distribution is almost independent of T. The reason is that the distribution depends mainly on the energy distribution of electrons a few radiation lengths back in the shower, and this energy distribution is itself an insensitive function of T. The electrons at more than a few radiation lengths back have no effect on the radial distribution at T, since if they have high enough energy to produce particles at T they will not be scattered, and if they are low enough in energy to be scattered appreciably, will produce no particles at T.

From these arguments we may expect that the radial distribution, *averaged over the shower length,* will be a good approximation to the radial distribution at any point in the shower except near the very beginning and end. This average radial distribution is a much easier quantity to find than the distribution defined by (4). Thus consider the expression for the total number of particles at x integrated over the shower length. This number is:

$$\frac{E_s^2}{2\ln(181Z^{-\frac{1}{3}})}\cdot\frac{dx}{x^3}\int_0^{E_0}\frac{dE}{E^2} \times\int_0^\infty \pi(E_0, E, t')dt'\int_0^\infty t^2\Pi(E, 0, t)dt. \quad (4')$$

The *probability* that a particle is found at x is then this number divided by the total number of particles, integrated over t, which is:

$$\int_0^\infty \Pi(E_0, 0, t)dt = E_0/\beta, \quad (5)$$

where β is the "critical energy." If we use two quantities familiar in shower theory: the "track length,"

$$z_\pi(E_0, E) = \int_0^\infty \pi(E_0, E, t)dt, \quad (6)$$

and the "mean square length" of a shower

$$\langle t^2(E)\rangle_{Av} = \frac{\int_0^\infty \Pi(E, 0, t)t^2 dt}{E/\beta} \quad (7)$$

the probability distribution is

$$\mathcal{P}(x)dx=\frac{dx}{x^3}\cdot\frac{E_s^2}{E_0\cdot 2\ln(181Z^{-\frac{1}{3}})}\times\int_0^{E_0}dE\cdot\frac{z_\pi(E_0,E)}{E}\langle t^2(E)\rangle_{\text{Av}}.$$

If we use the 'Moliere unit' of length $x'=E_s x/\beta$ and for convenience in later work introduce the dimensionless variables $\epsilon=2.4E/\beta$, $\epsilon_0=2.4E_0/\beta$ into the integral we get

$$\mathcal{P}(x')dx'=\frac{dx'}{x'^3}\frac{\beta^2}{9.06E_0}\int_0^{\epsilon_0}\langle t^2(\epsilon)\rangle_{\text{Av}}\frac{z_\pi(\epsilon_0,\epsilon)}{\epsilon}d\epsilon,$$

where $9.06=2\ln(181\cdot 7.22^{-\frac{1}{3}})$. We will now anticipate a future result by defining a new function $G(\epsilon)$ by the equation

$$\frac{z_\pi(\epsilon_0,\epsilon)}{\epsilon}=\frac{\epsilon_0}{\beta}G(\epsilon)=\frac{2.4E_0}{\beta^2}G(\epsilon). \tag{8}$$

$\mathcal{P}(x')$ is then:

$$\mathcal{P}(x')dx'=\frac{dx'}{x'^3}\cdot .265\int_0^{\epsilon_0}\langle t^2(\epsilon)\rangle_{\text{Av}}G(\epsilon)d\epsilon. \tag{9}$$

It is clear that the normalization of $\mathcal{P}(x')$ is such that the number of particles in the annular ring between x' and $x'+dx'$ at any depth in the shower is the *total* number of particles at that depth times the probability $\mathcal{P}(x')$ given by (9).

The dependence of the distribution (9) on x' is the same as that given by Moliere. To find the numerical coefficient one must have expressions for z_π and $\langle t^2(\epsilon)\rangle_{\text{Av}}$ which remain valid for energies considerably below the critical energy, which is the region where most of the scattering occurs. Such expressions have been given by Belenky[4] in the form of integrals. These are evaluated in the next section.

III. EVALUATION OF $\langle t^2(\epsilon)\rangle_{\text{Av}}$

Belenky's expression for $\langle t^2(\epsilon)\rangle_{\text{Av}}$ is a sum of five integrals

$$\langle t^2(\epsilon)\rangle_{\text{Av}}=\tau_1(\epsilon)+\tau_2(\epsilon)+\tau_3(\epsilon)+\tau_4(\epsilon)+\tau_5(\epsilon). \tag{10}$$

The integral $\tau_1+\tau_3$ which we shall call τ_{13}, is simpler than either τ_1 or τ_3 taken separately, hence we will work with this. The expressions for the τ's are:

$$\tau_{13}(\epsilon)=\frac{2}{q^2}\left(1+\frac{1}{\sigma}\right)^2\int_0^\epsilon\times\int_{z'}^\epsilon\psi_p(\epsilon,z)\psi_p(\epsilon,z')dzdz', \tag{11a}$$

$$\tau_2(\epsilon)=\frac{2}{q\sigma^2}\int_0^\epsilon\psi_p(\epsilon,z)dz, \tag{11b}$$

$$\tau_4(\epsilon)=\frac{2}{q^2}\left(1+\frac{1}{\sigma}\right)\frac{1}{\sigma}\int_0^\epsilon\int_{z'}^\epsilon\psi_p(\epsilon,z)\times[\psi_\Gamma(z,z')-\psi_p(z,z')]dzdz', \tag{11c}$$

$$\tau_5(\epsilon)=\frac{2}{q^2}\left(1+\frac{1}{\sigma}\right)\int_0^\epsilon\int_{z'}^\epsilon z\psi_p(\epsilon,z)\times\frac{\partial\psi_p(z,z')}{\partial z}dzdz', \tag{11d}$$

where $q=2.4$ and

$$\psi_p(\xi,\eta)=\eta e^\eta\int_\eta^\xi\frac{e^{-x}}{x^2}dx \tag{12}$$

$$\psi_\Gamma(\xi,\eta)=\psi_p(\xi,\eta)-\frac{\eta}{\xi^2}[1-(\exp-\xi+\eta)]. \tag{13}$$

σ is the photon absorption coefficient per radiation unit, i.e., the sum of the absorption coefficients for pair-production and the Compton effect. For high energies $\sigma=\text{constant}=7/9$. We will be interested mainly in the energy range 10^7 to 10^8 ev, where σ varies somewhat and is in fact less than 7/9. We shall assume for the present that it is constant, however, and correct for the variation later.

Now we define four new quantities, I_{13}, I_2, I_4 and I_5 as the integrals which appear in Eqs. (11). Thus,

$$\tau_{13}=\frac{2}{q^2}\left(1+\frac{1}{\sigma}\right)^2I_{13}, \tag{14a}$$

$$\tau_2=\frac{2}{q\sigma^2}I_2, \tag{14b}$$

$$\tau_4=\frac{2}{q^2}\left(1+\frac{1}{\sigma}\right)\frac{1}{\sigma}I_4, \tag{14c}$$

[4] S. Belenky, J. Phys. U.S.S.R. 8, 305 (1944).

TABLE I.

	Small ϵ	Large ϵ
$I_2(\epsilon)$	$\epsilon/2-\epsilon^2/12$	$\ln\epsilon-0.425$
$I_{13}(\epsilon)$	$\epsilon^2/12$	$\frac{1}{2}(\ln^2\epsilon-0.85\ln\epsilon-0.43)$
$I_4(\epsilon)$	$-\epsilon^2/36$	$-\frac{1}{2}(\ln\epsilon-1.735)$
$I_5(\epsilon)$	$\epsilon^2/12$	$\ln\epsilon-1.425$

and

$$\tau_5=\frac{2}{q^2}\left(1+\frac{1}{\sigma}\right)I_5. \tag{14d}$$

The integrals can be evaluated in terms of elementary functions, exponential integrals, and rapidly converging series of incomplete Γ-functions.

The expressions for the four integrals I_{13}, I_2, I_4, I_5 can be put simply in terms of three new functions P, Q, and R. Let

$$\ln\gamma=0.5772\cdots=\int_0^1(1-e^{-t}/t)dt-\int_1^\infty e^{-t}/tdt, \tag{15}$$

$$P(\epsilon)=(1/\epsilon)-(e^{-\epsilon}/\epsilon)-1, \tag{16}$$

$$Q(\epsilon)=\ln\epsilon+\ln\gamma+\int_\epsilon^\infty e^{-x}/xdx, \tag{17}$$

$$R(\epsilon)=(1-e^{-\epsilon})\left(\frac{\ln^2\epsilon}{2}+\frac{\ln\epsilon}{\epsilon}\right)-\ln\epsilon-\frac{1}{2}\int_0^\epsilon\ln^2xe^{-x}dx. \tag{18}$$

Then,

$$I_2(\epsilon)=P(\epsilon)+Q(\epsilon), \tag{19a}$$

$$I_{13}=\ln\gamma[P+Q]+R+2\int_0^\epsilon e^{-x}\times\left[\frac{1}{2!2}+\frac{x^2}{4!4}+\frac{x^4}{6!6}\cdots\right]dx+\int_0^\epsilon e^{-x}\left[\frac{1}{2!}-\frac{x}{3!2}+\frac{x^2}{4!3}+\cdots\right]dx+\int_0^\epsilon\left(\frac{e^{-x}}{x^2}-\frac{1}{x^2}+\frac{1}{x}\right)dx\int_\epsilon^\infty\frac{e^{-y}}{y}dy, \tag{19b}$$

$$I_4=-\frac{3}{2}P-\frac{Q}{2}-\int_0^\epsilon e^{-x}\times\left[\frac{1}{2!2}+\frac{x}{3!3}+\cdots\right]dx, \tag{19c}$$

$$I_5=2P+Q. \tag{19d}$$

There are two independent partial checks on the correctness of these expressions. For ϵ large enough so that $e^{-\epsilon}$ and $1/\epsilon$ are negligible, our expressions agree with those of Belenky.[4] For small ϵ they agree with results obtained by first expanding the integrand in the original integrals and then integrating. The asymptotic forms for small and large ϵ are given in Table I.

As we shall see later, the main contribution to our formula for single scattering will come from small ϵ, and hence from I_2 which varies as ϵ. The other integrals I_{13}, I_4 and I_5 vary as ϵ^2; this is easy to understand. Consider

$$\langle t^2(\epsilon)\rangle_{\text{Av}}=\frac{2.4}{\epsilon}\int_0^\infty\Pi(E,0,t)t^2dt. \tag{20}$$

For small ϵ, i.e., energies below the critical energy, where the range R is proportional to ϵ, $R=k\epsilon$ say, if we assume that the particle loses energy mainly by collision loss, we will have one particle for $t<k\epsilon$ and no particles for $t>k\epsilon$. Thus

$$\Pi(\epsilon,0,t)=1\quad t<k\epsilon,$$
$$\Pi(\epsilon,0,t)=0\quad t>k\epsilon.$$

Then (20) becomes

$$\langle t^2(\epsilon)\rangle_{\text{Av}}=\frac{2.4}{\epsilon}\int_0^{k\epsilon}t^2dt\propto\epsilon^2. \tag{21}$$

I_{13}, I_4, I_5 are thus the contribution to $\langle t^2(\epsilon)\rangle_{\text{Av}}$ of this collision process. It is clear that I_2 must correspond to a different kind of process. I_2 represents the contribution to $\langle t^2(\epsilon)\rangle_{\text{Av}}$ of a particle which produces a photon very soon after its single scattering, which photon travels a long distance and then materializes into a slow electron. This can be seen from Belenky's derivation of $\langle t^2(\epsilon)\rangle_{\text{Av}}$. His result is in the form of integrals over products of track lengths. $I_2(\epsilon)$ comes from the track length γ_r i.e., the track length of photons, in a shower initiated by a photon, and in particular from that part of the expression which corresponds to the original photon continuing for a considerable distance and then materializing. The importance of this process can be seen from the fact that the photon mean free path remains roughly constant (between 1.2 and 1.6 radiation lengths) down to 10^7 ev, whereas the electron mean free path decreases very rapidly for electrons below the critical energy.

We have calculated numerically $\langle t^2(\epsilon)\rangle_{Av}$ for the region in which the asymptotic forms do not hold. The results are listed in Table II. The dominant integral τ_2 is listed separately.

IV. NUMERICAL INTEGRATION

We must now calculate

$$\int_0^{\epsilon_0} \langle t^2(\epsilon)\rangle_{Av} G(\epsilon) d\epsilon. \tag{22}$$

Belenky has given as the expression for z_π

$$z_\pi(\epsilon_0, \epsilon) = -\frac{\epsilon_0}{\beta}\frac{\partial\psi_p(\epsilon_0, \epsilon)}{\partial\epsilon}. \tag{23}$$

From Eq. (8) then

$$G(\epsilon) = -\frac{1}{\epsilon}\frac{\partial\psi_p(\epsilon_0, \epsilon)}{\partial\epsilon}, \tag{24}$$

where ψ_p is given by (12). After some transformation we get

$$\frac{\partial\psi_p}{\partial\epsilon} = 1-(\epsilon e^\epsilon + e^\epsilon)\int_\epsilon^\infty \frac{e^{-x}}{x}dx - (\epsilon e^\epsilon + e^\epsilon)\int_{\epsilon_0}^\infty \frac{e^{-x}}{x^2}dx.$$

Since ϵ_0 is at least 10^{12} ev for air showers and the main contribution to the integral (22) comes from around 10^8 ev, we can neglect the last term and get:

$$G(\epsilon) = -\frac{1}{\epsilon}\left[1 - e^\epsilon(1+\epsilon)\int_\epsilon^\infty \frac{e^{-x}}{x}dx\right]. \tag{25}$$

Using our tabulated values for $\langle t^2(\epsilon)\rangle_{Av}$ the integral was done numerically. It is important to know what ranges of ϵ contribute most; Table III gives this information. For $\epsilon<2$ a correction was made for the fact that $\sigma(\epsilon)$, the photon absorption coefficient, varies somewhat.

If we use this result in (9) we get as the final distribution:

Table II.

ϵ	$\langle t^2(\epsilon)\rangle_{Av}$	$\tau_2(\epsilon)$	ϵ	$\langle t^2(\epsilon)\rangle_{Av}$	$\tau_2(\epsilon)$
0.20	0.141	0.133	2.00	1.503	1.037
0.40	0.286	0.259	3.00	2.241	1.388
0.60	0.435	0.377	4.00	2.939	1.674
0.80	0.586	0.487	5.00	3.589	1.913
1.00	0.738	0.592	6.00	4.197	2.119
1.20	0.890	0.690	7.00	4.764	2.299
1.40	1.044	0.784	8.00	5.295	2.459
1.60	1.197	0.873	9.0	5.795	2.602
1.80	1.350	0.957	10.0	6.627	2.732
2.00	1.503	1.037			

$$\mathcal{P}(x')dx' = 0.162dx'/x'^3. \tag{26}$$

This is to be compared with Moliere's distribution, in which the numerical coefficient is ≈ 0.1 instead of 0.162.

V. VALIDITY OF THE APPROXIMATIONS

We turn now to a discussion of the assumptions and approximations on which this result is based. The first point to consider—it is at the base of the whole calculation—is whether the large angle scattering we have assumed is really possible. Williams has shown that the finite size of the nucleus limits the scattering probability so that for angles greater than a certain θ_{max} the scattering probability goes rapidly to zero. For air $(Z=7.22)$, θ_{max} is given by[3]

$$\theta_{max} \approx (66/E(\text{Mev})). \tag{27}$$

For $\epsilon=2$, $E=72$ and $\theta_{max}=52°$. From Table III we see that roughly 2/3 of the contribution to the integral (22) comes from $\epsilon<2$, i.e., from energies which admit angles greater than 52°. It seems plausible then, especially in view of the fact that the angle defined by (27) is not a perfectly sharp cut-off, that the assumption of large angle single scattering is justified.

We now consider the angular approximations in the scattering formula (1). We have used $(4d\theta/\theta^3)$ as an approximation to the correct expression $[\sin\theta d\theta/(1-\cos\theta)^2]$. In terms of $x/t=\zeta=\sin\theta$ we have replaced the exact expression

$$\frac{d(\cos\theta)}{(1-\cos\theta)^2} = \frac{\zeta d\zeta}{[1-(1-\zeta^2)^{\frac{1}{2}}]^2(1-\zeta^2)^{\frac{1}{2}}}$$

by the approximate expression $(4\zeta d\zeta/\zeta^4)$. The

TABLE III.

a	b	$\int_a^b \langle t^2(\epsilon)\rangle_{Av} G(\epsilon)d\epsilon$
0	0.2	0.028
0.2	0.4	0.121
0.4	1	0.154
1	2	0.105
2	10	0.167
10	∞	0.026
0	∞	0.601

ratio r of the exact expression to the approximate one is:

$$r=\zeta^4/4(1-\zeta^2)^{\frac{1}{2}}[1-(1-\zeta^2)^{\frac{1}{2}}]^2. \tag{28}$$

Introducing $(1-\zeta^2)^{\frac{1}{2}}=\cos\theta=\eta$ we get

$$r=\frac{\left(\dfrac{1+\eta}{2}\right)^2}{\eta}. \tag{29}$$

r thus turns out to be

$$\left(\frac{\text{arithmetic mean of } \eta \text{ and } 1}{\text{geometric mean of } \eta \text{ and } 1}\right)^2.$$

We might expect from this that r remains close to 1 over a large region, and this turns out to be the case. Thus for $\theta=53°$, $r=1.07$, $\theta=66°$, $r=1.22$ and for $\theta=78°$, $r=1.80$. Beyond 80° the approximation rapidly becomes worse, and blows up completely at 90°. For large angles we see that $r>1$, i.e., that our approximation gives too small a result. This error is partially compensated by another approximation we have made. It is clear from Fig. 1 that t must always be greater than x. This means that the last integral in (4) should not be

$$\int_0^\infty t^2\Pi(E,0,t)dt \tag{30}$$

but

$$\int_x^\infty t^2\Pi(E,0,t)dt. \tag{31}$$

The error involved in this replacement is hard to estimate, but obviously is an increasing function of x. It is true however that most of the contribution to (31) comes from large t, hence the approximation represented by (30) is probably not intolerable. We can see that large t contribute most by considering the expressions for $\langle t^2(\epsilon)\rangle_{Av}$ and $\langle t(\epsilon)\rangle_{Av}^2$.

For small ϵ Belenky has found $\langle t(\epsilon)\rangle_{Av}=0.94I_2(\epsilon)$ where I_2 is given by (15). Thus

$$\langle t(\epsilon)\rangle_{Av}\approx 0.47\left(\epsilon-\frac{\epsilon^2}{12}+\cdots\right).$$

We get from Table I and Eqs. (11) that

$$\langle t^2(\epsilon)\rangle_{Av}=0.69\left(\epsilon+\frac{\epsilon^2}{18}+\cdots\right).$$

We see that for ϵ as large as 2, which includes the most important energy range for our integral (22) that $\langle t^2(\epsilon)\rangle_{Av}$ is considerably larger than $\langle t(\epsilon)\rangle_{Av}^2$, confirming that most of the contribution to the integral comes from large t.

It is a pleasure to thank Professor H. A. Bethe for suggesting this problem and for essentially deriving Eq. (9), as well as for several helpful discussions.