The Self-Oscillations of a Charged Particle

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It is found that certain rigid charge distributions can oscillate without radiation even when no forces are present, other than their own retarded fields. The periods are of the order of the time required for light to cross the particle. The energy of these oscillations is always positive, and there are therefore no exponentially increasing unstable motions of the type possessed by the Dirac classical electron.

The frequencies of these oscillations are such, that when quantized, the energy of the first excited state is of the order of the meson self-energy. Hence, it is suggested that some kinds of mesons may be electrons in such an excited state of self-oscillation.

It is indicated that the principle of causality may have to be reformulated in terms of causal connections over finite intervals of time if one wishes to regard the electron plus its associated electromagnetic field as a single system.

I. INTRODUCTION

CURRENT relativistic, quantum field theories are based on the assumption that the elementary charges have no extension in space. It is well known that this assumption leads to infinite results in the higher approximations.[1,2] With the aid of the canonical transformations introduced by Schwinger[3] and Tomonaga,[4] however, it has been possible to classify these infinities in a relativistically invariant way, such that the infinite parts can be uniquely identified as contributions to the mass and to the renormalization of the electronic charge, while the remaining finite terms, which are also unique, give various small but real effects such as the displacements of energy levels observed by Lamb and Retherford.[5] Although this procedure is very satisfactory in that it yields many results which agree with experiment, at least up to second order in $e^2/\hbar c$, it is by no means certain that in higher orders one can continue to isolate the infinite terms in a logically consistent and unambiguous way.[6] It is, in fact, widely felt that some totally new idea is required, which will make the infinite terms finite and unique but at the same time will leave the results of the present theory unaltered wherever these are finite and unambiguous.

A direct way to obtain finite results has always been open, namely, to assume that the electronic charge has a finite extension in space. This idea, however, has never been successfully applied because of the difficulty of specifying the charge distribution in a way that is consistent with the relativistic interpretation of causality. In order to illustrate the problems involved, let us consider, for example, the behaviour of a perfectly rigid electron at which is directed a pulse of electromagnetic radiation. As soon as the pulse strikes the edge of the charge distribution the electron as a whole is set into motion. Thus an impulse is transmitted across the electron instantaneously, contradicting the relativistic law of causality. Although it has often been suggested that the failure of causality for very short intervals of time may be permitted,[7,8] a procedure more in line with our present ideas is to take a model in which the electron is not rigid, but transmits impulses with the speed of light or less. This approach, however, presents problems of its own, for we must then include in our theory a specification of not only the structure of the electron, but also of the dynamical laws governing the transmission of impulses through this structure. In a quantum theory of an extended charge the nonrigid model leads to still another problem, this

[1] R. Serber, Phys. Rev. **49**, 545 (1936).
[2] V. F. Weisskopf, Phys. Rev. **56**, 72 (1939).
[3] J. A. Schwinger, Pocono Conference Notes (by J. A. Wheeler); See also J. A. Schwinger, Phys. Rev. **73**, 416 (1948).
[4] S. Tomonaga, Progress of Theoretical Physics **1**, 27 (1946).
[5] W. E. Lamb, Jr. and R. C. Retherford, Phys. Rev. **72**, 241 (1947).
[6] A. Pais, *Positon Theory* (Princeton University Press, Princeton, to be published).
[7] J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **17**, 157 (1945).
[8] D. Blokhinzev, J. Phys. USSR **10**, 167 (1946).

time when one tries to give a quantum description of the shape of the electron. In a state of steady motion, for example, the electron is flattened in the direction of motion because of the Lorentz contraction. When the electron is scattered, however, it undergoes an indivisible change of momentum. What shape does it have during the process of transition? If it may be said to have any shape at all, that shape must be some symmetrical function of initial and final velocities which corresponds neither to initial nor to final shapes. The way to deal with these problems is certainly not yet understood.

Because the theory of an extended charge involves so many difficulties and ambiguities, there has been little impetus to develop this general line of approach. We wish to show, however, that the motion of a rigid, and therefore non-relativistic, extended charge has some unexpected properties, which suggest that the effort to develop a relativistic theory of the extended charge is worth more intensive study. In particular, we find that these new properties may lead to the possibility of describing mesons and perhaps other kinds of particles as excited states of electrons. In this way, the same step that removes the infinities may also reduce the number of elementary particles.

II. SUMMARY OF PROCEDURE AND RESULTS

In this work we consider only a spinless rigid electron with an arbitrary spherically symmetric distribution of charge, which we assume at the outset to be specified. The equations of motion of the center of mass of the charge are solved, taking into account exactly the action of the retarded fields produced by the charge itself. For a general charge distribution we make the assumption that $(v/c)\ll 1$, i.e., only non-relativistic motions are considered. In the special case of a spherical shell charge, however, we can solve the equations exactly for arbitrary values of v/c between 0 and 1, but one must remember that, despite the appearance of arbitrary v/c, this treatment is still not relativistically invariant because the charge is assumed to be rigid.

In agreement with Markov,[9] who uses a similar treatment for the case of small v/c, we find that there are no "self-accelerated" exponential motions of the type discussed by Dirac.[10] Instead, however, we find that for a certain general class of charge distributions, an electron ***under the action of no external forces***, is capable of undergoing oscillatory motion. The period, τ_0, depends somewhat on the form of the charge distribution, but it is of the general order of magnitude of the time required for light to cross the electron. These "self-oscillations" consist of simple harmonic motion of the center of mass of the particle; they do not represent a dynamical property of the internal structure of the electron. Nor do they constitute a form of instability, as does the "self-acceleration" of the Dirac classical electron, because they grow only in proportion to their excitation by an external source of energy. The amplitude of the oscillations, A, is less than the mean radius of the electron, a, for since the charge can move in the same direction for at most half a period, $A\sim v_{\mathrm{Av}}(\tau_0/2)\sim(v/c)a$. Oscillations are made possible by the fact that each part of the electron moves in a field which was produced by other parts of the electron in the past, so that for certain charge distributions undergoing motion with the proper period the net force can vanish. This implies also that no radiation is emitted,* so that the charge and its own electromagnetic field form a closed system.

These fairly general results were derived on the assumption that $(v/c)\ll 1$. For the special case of a spherical shell charge, however, an exact solution can be obtained. We find that for arbitrary periodic motion of such a charge, with period $\tau_0=2a/c$, the electromagnetic self-force vanishes provided that $v/c<1$. Thus if a spherical shell charge possesses no non-electromagnetic mass, it can continue to move indefinitely in such a periodic orbit, under the action of no external forces and without radiation once it has been set in motion in this way. This type of charge distribution has already been studied by Schott,[11] who shows that for arbitrary spinless periodic motion with the above period there is no radiation. He also shows[12] that for uniform motion in a circular

[9] M. Markov, J. Phys. USSR **10**, 159 (1946).

[10] P. A. M. Dirac, Proc. Roy. Soc. **A167**, 148 (1938).

* The self-force can be regarded as the sum of two terms, one of which yields the rate of radiation while the other leads to forces similar to those of inertia.

[11] G. A. Schott, Phil. Mag. **15**, 752 (1933).

[12] G. A. Schott, Proc. Roy. Soc. **A159**, 570 (1937).

orbit the total self-force vanishes. This is a special case of our result.

In general, for motions which are neither rectilinear nor circular, the retarded fields will produce a couple.[13] In all cases which we consider, however, the self-torque can be shown to vanish, a result necessary for the consistency of the assumption of no spin.

III. EFFECTS OF QUANTIZATION

While we have not, as yet, quantized these self-maintaining oscillations, one can obtain a general idea of the effect of quantization by estimating the energy levels corresponding to the excitation of the first excited states. Let us do this for the special case of the spherical shell charge. The energy of excitation, ΔE, is obtained by multiplying the angular frequency, ω, by $\hbar$, giving

$$\Delta E=\hbar\omega=n\pi(\hbar c/a),$$

where n is an integer. If we take

$$a\sim\frac{e^2}{mc^2}$$

we get

$$\Delta E\sim n\pi\frac{\hbar c}{e^2}mc^2\sim 400nmc^2.$$

Thus the energy of the first excited state with $n=1$ is not far from the rest energy of a π meson. By choosing different values of n we can obtain a spectrum of masses; still other masses and different spectra can be obtained by changing the charge distribution.

The idea then suggests itself that perhaps some kinds of mesons are really excited states of the electron. The decay from one kind of meson to another, or from meson to electron would then correspond simply to the loss of this excitation energy. The exact values of the energy obtained from this crude theory are probably not very significant, first, because they depend on the shape assumed for the charge distribution, and second, because the theory given thus far is not relativistically invariant. The essential point of this work is simply to suggest that the same step which makes the theory finite can also bring in the idea of unifying a whole spectrum of particles into a single particle. Theories involving a range of masses for the same particle have already been suggested on other grounds.[14–16]

One can easily see that for weakly coupled fields, like the electromagnetic, there is little physical significance to the non-relativistic limit in the quantum theory.[6] In order to excite the oscillation with $n=1$ to the first quantum state, for example, it is necessary to give the electron 400 times its own rest energy, and therefore to bring it into the relativistic range of velocities. Before such a theory can be applied at all, it must be extended to the relativistic quantum domain. The spherical shell charge does yield a solution for arbitrary v/c, but does not take into account the need for a non-rigid electron which can change its shape and transmit impulses with the speed of light or less. One possible approach to this problem is to try to make some relativistically invariant assumptions about the properties that we have discussed, and then to quantize the resulting theory. On the other hand, one may attempt to go directly to a quantum theory by guessing a formulation which leads to finite results, being guided by the idea that such a theory should contain the possibility of describing many masses as different states of the same particle. The authors are at present exploring both lines of approach and hope to publish some results soon.

The idea that an extended charge should be capable of self-oscillations is on a considerably more solid footing when applied to a strong-coupling meson theory, for the lowest quantized state of excitation in such a theory need not carry the particle into the relativistic region of velocities. To see this, we note that if the coupling constant is g, the contribution of the meson field to the particle mass is

$$m\sim g^2/ac^2.$$

The ratio of the energy of the first excited state to the self-energy of the particle at rest is then

$$\Delta E/mc^2\sim\pi(\hbar c/g^2),$$

and for large $g^2/\hbar c$ the ratio is small, so that one need not go to a relativistic theory. Thus, in a strong-coupling theory, even with scalar mesons

[13] G. A. Schott, Proc. Roy. Soc. **A159**, 548 (1937).

[14] H. J. Bhabha, Rev. Mod. Phys. **17**, 200 (1945).
[15] F. Bopp, Zeits. f. Naturforschung **1**, 237 (1946).
[16] D. Blokhinzev, J. Phys. USSR **11**, 72 (1947).

one should be able to obtain proton isobars in which the energy of excitation is contained in the motion of the center of mass of the proton. If one combines two such oscillations, which are orthogonal and 90° out of phase, one gets circular motion so that the particles obtain an additional contribution to their intrinsic angular momentum which would be some multiple of $\hbar$ in the quantized theory. In this way one can obtain spin isobars even with a scalar meson theory.

IV. REFORMULATION OF PRINCIPLE OF CAUSALITY

We observe that after the electromagnetic field has been eliminated the equation of motion of the electron (10), is an integral equation which can therefore have, in general, a greater arbitrariness in its boundary conditions than does the usual second order differential equation of motion of mechanics. Let us consider, for example, a spherical shell charge. The solution, as we have pointed out, is an arbitrary periodic function with period $\tau_0 = 2a/c$. Within the basic period of time, τ_0, the position of the center of mass, $\xi(t)$, may therefore be given an arbitrary form. One could do this, for example, by Fourier analyzing the function within a region of width τ_0 and specifying the amplitude of each Fourier component. The solution of the integral equation then consists simply in repeating the arbitrary function periodically, from one basic period to the next.

One can use this result as the basis of an interesting reformulation of the principle of causality as applied to the motion of the electron. Within a block of time of length τ_0 the behaviour of the electron is arbitrary, i.e., it seems to follow no causal laws. The causal laws are constituted by the requirement that from one block of time to the next the motion must repeat itself. (All of this is in the absence of external fields; such fields would cause the motion to be not quite periodic.) As a result, time seems to take on a twofold character. Over a block of time of length τ_0 it acts just like a space-coordinate in that there are no particular relations between the particle coordinates at neighboring instants. It is only between one block of time and the next that such causal connections appear.

The disappearance of causal laws as applied to the electron within a block of time is compensated by the appearance of new degrees of freedom, which may be taken as the amplitudes of the Fourier components of ξ. Strictly speaking, these are not really new degrees of freedom, but are merely a re-description in terms of the particle coordinates of some of the arbitrariness resulting from the old degrees of freedom, i.e., the electromagnetic field, which has been eliminated. When the electromagnetic field is taken into account, of course, the system does obey causal laws, which involve the field quantities as well as the particle coordinates. The apparent lack of causality over a basic period of time, τ_0, merely reflects the fact that if we specify only the particle coordinates, it is necessary to know the motion over the entire length of time needed for light to cross the electron before we can know the total field existing over the electron at any instant, and therefore, before we can predict the subsequent behaviour of the electron.** The essential conclusion to be drawn from this fact is that if one wishes to regard the electron plus its associated field as a single system, one may be forced to reformulate the principle of causality in terms of causal connections over finite intervals of time.

In a sense, this dual property of time may be regarded as a sort of quantization of time, since there will now be a minimum interval which can be involved in causal relations. Although shorter intervals of time can exist, they have new properties. For example, no predictions of the electronic motion can be made over intervals shorter than τ_0. The arbitrariness of the Fourier components of ξ within the period, τ_0, would in a quantized theory, be interpreted as the possibility of creating various kinds of mesons. Since the amplitude of each Fourier component can be specified only after the motion is known over the whole period, one would be unable to give a meaning to the problem of specifying the kind of

** It might be thought, at first sight, that the lack of causality is connected with the assumed rigid structure of the electron, which can transmit impulses faster than light. We are, however, discussing here a non-relativistic theory, within the framework of which one can postulate the transmission of impulses with arbitrary velocity. It is only when we make a Lorentz transformation that the transmission of impulses faster than light leads to a breakdown of causality, since events which are connected by such impulses may have future and past interchanged as a result of the transformation. The attempt to make a relativistic theory of a rigid electron would introduce an element of genuine lack of causality, which could not be explained in terms of eliminated field quantities.

meson which exists in a time less than τ_0. Thus one would obtain a new kind of complementarity between time and the nature of the particle with which one was dealing.

This discussion is intended merely to show how an attempt to devise a systematic relativistic quantum theory of extended charges might readily lead to important revisions of some of our concepts of causality.

V. THE EQUATION OF MOTION AND ITS SOLUTIONS

The equation of motion of a rigid charged particle, including the action of its own electromagnetic field, has already been studied extensively.[9–13,17] For the sake of completeness however, we include a brief derivation which, although it repeats a certain amount of work already existing in the literature, provides the result in a form convenient for our purposes.

We begin by assuming a certain spherically symmetrical charge density which we denote by

$$\rho(\mathbf{x}, t) = ef(|\mathbf{x}-\boldsymbol{\xi}(t)|)$$

where $\boldsymbol{\xi}(t)$ is the position vector of the center of mass and e is the total charge, so that

$$\int d\mathbf{x} f(|\mathbf{x}|) = 1.$$

This, of course, is equivalent to assuming a rigid charge distribution. The current density is then

$$\mathbf{j}(\mathbf{x}, t) = \frac{e}{c}\dot{\boldsymbol{\xi}}(t) f(|\mathbf{x}-\boldsymbol{\xi}(t)|),$$

which implies that the particle has no spin, i.e., currents arise only from the motion of the center of mass.

In deriving the fields produced by this charge-current distribution, it will be convenient to choose the gauge in which the vector potential, $\mathbf{a}_\perp$, is divergenceless. In this case one can easily show that the scalar potential, ϕ, satisfies the equation

$$\Delta\varphi = -4\pi\rho, \tag{1}$$

i.e., it is not retarded. Hence the field resulting from ϕ can produce no net force, and so far as the motion of the center of mass of the particle is concerned, we need pay attention only to the transverse fields arising from $\mathbf{a}_\perp$, which satisfies

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{a}_\perp = -4\pi\mathbf{j}_\perp, \tag{2}$$

where $\mathbf{j}_\perp$ is the divergenceless part of the current density. The next step is to Fourier analyze this equation. In the subsequent work we will denote the kth Fourier component of a function, $g(\mathbf{x}, t)$, by

$$g_{\mathbf{k}}(t) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int g(\mathbf{x}, t)(\exp - i\mathbf{k}\cdot\mathbf{x})d\mathbf{x},$$

where

$$g(\mathbf{x}, t) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int g_{\mathbf{k}}(t)(\exp i\mathbf{k}\cdot\mathbf{x})d\mathbf{k}.$$

Then $\mathbf{a}_{\mathbf{k}\perp}(t)$ must satisfy

$$\left(c^2k^2 + \frac{\partial^2}{\partial t^2}\right)\mathbf{a}_{\mathbf{k}\perp} = 4\pi c^2 \mathbf{j}_{\mathbf{k}\perp}. \tag{3}$$

The solution corresponding to the retarded potential*** is

$$\mathbf{a}_{\mathbf{k}\perp}(t) = \frac{4\pi c}{k}\int_{-\infty}^{t} dt'(\mathbf{j}_{\mathbf{k}}(t'))_\perp \sin ck(t-t'). \tag{4}$$

Now

$$\mathbf{j}_{\mathbf{k}}(t) = \frac{e}{c}\dot{\boldsymbol{\xi}}(t) f_k(\exp - i\mathbf{k}\cdot\boldsymbol{\xi}(t)), \tag{5}$$

where

$$f_k = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d\mathbf{x} f(|\mathbf{x}|)(\exp - i\mathbf{k}\cdot\mathbf{x}),$$

and since

$$(\mathbf{j}_{\mathbf{k}})_\perp = \frac{\mathbf{k}\times(\mathbf{j}_{\mathbf{k}}\times\mathbf{k})}{k^2},$$

the vector potential† becomes

[17] H. A. Lorentz, *Theory of Electrons* (B. G. Teubner, Liepzig, 1909).

*** If one chooses $+\infty$ for the lower limit, one obtains the advanced potential.

† It should be noted that, in general, a convergence factor is necessary to give meaning to the integral defining $\mathbf{a}_{\mathbf{k}\perp}$. This procedure, however, will always yield a unique result with a bounded charge because when one re-Fourier analyzes to find the potentials as functions of x and t, the effect of the k integration is to restrict the time during which contributions are made to the t' integral to a short interval.

$$\mathbf{a}_\perp(\mathbf{x},t)=\frac{4\pi e}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{t}dt'\int\frac{d\mathbf{k}}{k}f_k\,\mathrm{sin}ck(t-t')\times\exp i\mathbf{k}\cdot(\mathbf{x}-\boldsymbol{\xi}(t'))\frac{\mathbf{k}\times(\dot{\boldsymbol{\xi}}(t')\times\mathbf{k})}{k^2}. \quad (6)$$

The fields are given by

$$\boldsymbol{\varepsilon}_\perp=-\frac{1}{c}\frac{\partial\mathbf{a}_\perp}{\partial t},\quad \mathbf{H}=\mathrm{curl}\mathbf{a}_\perp, \quad (7)$$

and the self-force is just

$$\mathbf{F}_{\mathrm{self}}=\int d\mathbf{x}[\rho(\mathbf{x},t)\boldsymbol{\varepsilon}_\perp(\mathbf{x},t)+\mathbf{j}(\mathbf{x},t)\times\mathbf{H}(\mathbf{x},t)]. \quad (8)$$

Inserting the results of Eq. (7) into Eq. (8) and making the substitutions $t-t'=\tau$, $\boldsymbol{\xi}(t)-\boldsymbol{\xi}(t-\tau)=\mathbf{s}$, we obtain

$$\mathbf{F}_{\mathrm{self}}=-4\pi e^2\int_0^\infty d\tau\int\frac{d\mathbf{k}}{k}|f_k|^2\times(\exp i\mathbf{k}\cdot\mathbf{s})\left[k\,\mathrm{cos}ck\tau\frac{\mathbf{k}\times(\dot{\boldsymbol{\xi}}(t-\tau)\times\mathbf{k})}{k^2}-i\,\mathrm{sin}ck\tau\frac{\dot{\boldsymbol{\xi}}(t)}{c}\times(\mathbf{k}\times\dot{\boldsymbol{\xi}}(t-\tau))\right]. \quad (9)$$

The equation of motion for the electron is

$$m\ddot{\boldsymbol{\xi}}=\mathbf{F}_{\mathrm{self}}+\mathbf{F}_{\mathrm{ext}}, \quad (10)$$

where $\mathbf{F}_{\mathrm{ext}}$ is the external force and m is the non-electromagnetic mass.

Approximation of Small Velocities

Let us now investigate the limit in which $(v/c)\ll 1$. In this approximation the magnetic term in Eq. (9) may be neglected in comparison with the first term, which is of order v/c. Now $\mathbf{s}=\boldsymbol{\xi}(t)-\boldsymbol{\xi}(t-\tau)$ is essentially just the distance covered by the particle during the time τ. One can easily see that the integrand of Eq. (9) is large only over a time $\tau\sim 2a/c$, which is of the order of that needed for light to cross the electron, so s is of the order of $2a(v/c)$. Retaining only first order terms in v/c, we can replace $(\exp i\mathbf{k}\cdot\mathbf{s})$ by unity to obtain

$$\mathbf{F}_{\mathrm{self}}=-4\pi e^2\int_0^\infty d\tau\int d\mathbf{k}|f_k|^2\,\mathrm{cos}ck\tau\times\left[\dot{\boldsymbol{\xi}}(t-\tau)-\frac{\mathbf{k}(\mathbf{k}\cdot\dot{\boldsymbol{\xi}}(t-\tau))}{k^2}\right].$$

Note that this equation is linear: small v/c leads to a linear approximation.††

The above expression can be put into a more convenient form by integration by parts over τ. The integrated part vanishes as $\tau\to\infty$ for any form of f_k resulting from a bounded charge distribution,††† while at $\tau=0$ it vanishes because of the appearance of $\mathrm{sin}ck\tau$ as a factor. After angular integration the self force becomes

$$\mathbf{F}_{\mathrm{self}}=\frac{-32\pi^2e^2}{3c}\int_0^\infty d\tau\ddot{\boldsymbol{\xi}}(t-\tau)\int_0^\infty dkk|f_k|^2\,\mathrm{sin}ck\tau.$$

With the substitution

$$G(\tau)=\frac{-32\pi^2}{3c}\int_0^\infty dkk|f_k|^2\,\mathrm{sin}ck\tau \quad (11)$$

the above expression becomes

$$\mathbf{F}_{\mathrm{self}}=e^2\int_0^\infty d\tau G(\tau)\ddot{\boldsymbol{\xi}}(t-\tau), \quad (12)$$

and the equation of motion, in the absence of external forces, is

$$m\ddot{\boldsymbol{\xi}}=e^2\int_0^\infty d\tau G(\tau)\ddot{\boldsymbol{\xi}}(t-\tau)=\frac{-32\pi^2e^2}{3c}\int_0^\infty d\tau\ddot{\boldsymbol{\xi}}(t-\tau)\times\int_0^\infty dkk|f_k|^2\,\mathrm{sin}ck\tau. \quad (13)$$

This equation is accurate for arbitrarily large accelerations, but only for small v/c. It is essentially the same equation obtained by Markov[9] in a similar way.

†† It is readily shown that in the linear approximation the self torque vanishes because of symmetry.

††† This may be seen, for example, by expressing f_k in terms of $f(x)$.

Self-Oscillations

We shall now demonstrate that Eq. (13) possesses oscillatory solutions, and examine the conditions under which these solutions exist. Inserting $\xi=\mathbf{A}(\exp-i\omega t)$ into (13), we get

$$m\omega^2=\frac{-32\pi^2e^2}{3c}\omega^2\int_0^\infty d\tau(\exp i\omega\tau)\times\int_0^\infty dkk|f_k|^2\sin ck\tau=e^2\omega^2\int_0^\infty d\tau(\exp i\omega\tau)G(\tau),\quad(14)$$

which defines the permissible frequencies of self-oscillation, if any. Since m must be real, (14) is equivalent to the two conditions

$$m=\frac{-32\pi^2e^2}{3c}\int_0^\infty d\tau\times\int_0^\infty dkk|f_k|^2\sin ck\tau\cos\omega\tau,\quad(15)$$

$$0=\int_0^\infty d\tau\int_0^\infty dkk|f_k|^2\sin ck\tau\sin\omega\tau.\quad(16)$$

Condition (16) can be put into a more convenient form by performing the τ integration. Since the integrand is an even function of τ, this condition can be rewritten

$$0=\int_{-\infty}^\infty d\tau\int_0^\infty dkk|f_k|^2\times[\cos\tau(ck-\omega)-\cos\tau(ck+\omega)].$$

The integrals over τ are just δ functions, giving

$$f_{\omega/c}=0.\quad(16')$$

It is readily shown that when condition (16′) is satisfied the radiation vanishes to order v/c.

The spectrum of frequencies at which self-oscillations can take place depends strongly on the shape of the charge. First, in order to satisfy condition (16′), it is necessary that at least one Fourier component of the charge distribution shall vanish. In addition, at this frequency, condition (15) must be satisfied, but so long as the distribution processes at least one vanishing Fourier component it will be possible, in general, to choose the non-electromagnetic mass, m, in such a way as to satisfy (15). Those distributions with vanishing Fourier components may be characterized roughly as either sharply varying, or non-monotonic. Both the spherical shell and the uniform sphere, for example, can undergo self-oscillations. On the other hand, smoothly varying, monotonic distributions, such as the Gaussian and exponential, have no vanishing Fourier components, while a smoothly varying but non-monotonic distribution like $xe^{-x/a}$ has one.

The following are a few examples of frequencies of self oscillation.

(*a*) *Spherical Shell*

$$f(x)=\frac{\delta(x-a)}{4\pi a^2};\quad f_k=\frac{1}{(2\pi)^{\frac{3}{2}}}\frac{\sin ka}{ka}.\quad(17)$$

In this case it is most convenient to use Eq. (14) directly. We obtain from (11) and (17),

$$G(\tau)=\begin{cases}-\dfrac{1}{3a^2c}, & \tau<\dfrac{2a}{c},\\ 0, & \tau>\dfrac{2a}{c}.\end{cases}$$

The condition for self oscillation (14) is

$$m=\frac{ie^2}{3\omega ca^2}[(\exp 2i\omega a/c)-1].$$

Solutions are possible only when $m=0$. The frequencies are $\omega=n\pi c/a$, where n is an integer.

(*b*) *Uniform Sphere*

$$f(x)=\begin{cases}\dfrac{3}{4\pi a^3}, & x<a\\ 0, & x>a\end{cases};\quad(18)$$

$$f_k=\frac{3}{(2\pi)^{\frac{3}{2}}}\frac{1}{(ka)^2}\left[\frac{\sin ka}{ka}-\cos ka\right].$$

Although the uniform sphere has an infinite number of vanishing Fourier components, condition (15) limits the spectrum to a single frequency, $\omega^2=2e^2/ma^3$. For a given e and a, however, there are many possible ways of choosing m, given by the solutions of

$$\tan(2e^2/mc^2a)=(2e^2/mc^2a).$$

If $a\sim e^2/mc^2$, the largest value of m which will permit self-oscillation is of the same order as e^2/ac^2.

$$(c)\qquad f(x)=\frac{1}{24\pi a^4}xe^{-x/a};$$

$$f_k=\frac{1}{3(2\pi)^{\frac{3}{2}}}\frac{(3-k^2a^2)}{(1+k^2a^2)^3}. \qquad (19)$$

This distribution has only one frequency, $\omega=\sqrt{3}(c/a)$, for which $f_{\omega/c}$ vanishes. To satisfy condition (15) we must choose $m=e^2/192ac^2$ Since the electromagnetic mass is of the order of e^2/ac^2, in this case the mechanical mass must be much less than the electromagnetic mass.

Absence of "Self-Accelerated" Solutions

Markov[9] has suggested that Eq. (13) should have no exponentially increasing solutions of the type obtained by Dirac.[10] One can obtain this result very simply by attempting to find solutions of the form $\xi=A(\exp(\lambda-i\omega_0)t)$ where ω_0 and λ are real and positive, i.e., by replacing ω in Eq. (14) by $\omega_0+i\lambda$. Because of the appearance of the factor $e^{-\lambda\tau}$ in the integrand, the integral over τ converges, and can be carried out first, yielding

$$m+\frac{32\pi^2e^2}{3}\int_0^\infty dkk^2|f_k|^2 \times\frac{[(\lambda^2-\omega_0{}^2+c^2k^2)+2i\lambda\omega_0]}{(\lambda^2-\omega_0{}^2+c^2k^2)^2+4\lambda^2\omega_0{}^2}=0. \qquad (20)$$

There can be no solution unless the imaginary part of the integral vanishes, but since the imaginary part of the integrand is everywhere positive this cannot happen unless either $\lambda=0$ or $\omega_0=0$. If $\omega_0=0$ the integrand is everywhere positive, and there is a solution only for negative m. Peierls has already shown that the Dirac classical electron is equivalent to one in which an infinite negative mass almost cancels the infinite positive contribution of the point charge. The extended electron, therefore, has no motions of the "self-accelerating" type possessed by the Dirac classical electron so long as the non-electromagnetic mass is positive or zero.

Stability of Motion of Extended Electron

From the above result one can immediately conclude that the motion of the extended electron is stable. In order to investigate the stability of the electron at rest, for example, we must see whether small displacements remain small for all time. Now, the assumption of small v/c leads immediately to the validity of Eq. (13). Thus if we assume a displacement, $\xi=Ae^{\lambda t}(\exp-i\omega_0t)$, which is small at the time t, it will be smaller still throughout all the earlier time during which contributions are made to Eq. (13). On the other hand, it is inadmissible to test for the stability with the aid of the Lorentz expansion in a series of derivatives of ξ, just because the ratio $(d^{n+1}\xi/dt^{n+1})/(d^n\xi/dt^n)$ may not be small, even though $\dot{\xi}$ itself is small.

Another way to investigate the stability of the resting electron is to see whether the energy of a moving electron is always greater than that of a fixed electron.‡ Remembering that the electrostatic potential is not retarded, since the electron does not change its shape‡‡ one can conclude that the electrostatic energy is independent of the velocity. It can be shown,[18] furthermore, that the total electromagnetic energy is just the sum of the electrostatic contribution and that due to $\mathbf{a}_\perp$. Now, $\mathbf{a}_\perp=0$ when the electron is at rest, so the increase in energy which is due to the motion is given by

$$W=\frac{1}{8\pi}\int d\mathbf{x}\left[\frac{1}{c^2}\left(\frac{\partial\mathbf{a}_\perp}{\partial t}\right)^2+(\nabla\times\mathbf{a}_\perp)^2\right], \qquad (21)$$

and since $\mathbf{a}_\perp$ cannot vanish everywhere for a moving electron this is clearly a positive quantity. The state of rest is therefore a stable one.

‡ We have seen that the instability of the Dirac classical electron can, in fact, be traced to the assumption of an infinite negative mechanical mass which almost cancels the electromagnetic mass. It is this negative mass which supplies the electromagnetic energy generated in the exponential "self-accelerated" motion.

‡‡ One assumes that the charge is rigid, which means that the tendency of the electrostatic force to blow up the charge is automatically counteracted by some other force which holds the electron together. Since this force does no work, it need not be considered in any study of the motion of the electron.

[18] W. Heitler, *Quantum Theory of Radiation* (Clarendon Press, Oxford, 1936).

It is instructive to evaluate W for the case of small v/c. If one expresses $\mathbf{a}_\perp$ in terms of its Fourier components one obtains

$$W=\frac{1}{8\pi c^2}\int d\mathbf{k}[|\dot{a}_{\mathbf{k}\perp}|^2+c^2k^2|a_{\mathbf{k}\perp}|^2], \quad (22)$$

where $a_{\mathbf{k}\perp}$ is given in Eq. (4). In the linear approximation we can neglect ξ in the exponential, and with this simplification one obtains

$$W=\frac{16\pi^2e^2}{3}\int_0^\infty d\tau\int_0^\infty d\tau'\int_0^\infty dkk^2|f_k|^2 \times\cos ck(\tau-\tau')\dot{\xi}(t-\tau)\cdot\dot{\xi}(t-\tau'). \quad (23)$$

As an example we consider the spherical shell charge, for which f_k is given in Eq. (17). We get, for a self-oscillation,

$$W=\frac{e^2}{6a^2c}\int_0^{2a/c} d\tau\dot{\xi}^2(t-\tau)=\frac{M\bar{v}^2}{2}, \quad (24)$$

where $M=(2/3)(e^2/ac^2)$ is the electromagnetic mass.

Exact Treatment for Self-Force of Spherical Shell Charge

Let us begin with the exact Eq. (9) for the self-force. We shall consider a typical term and show that it vanishes for an arbitrary periodic motion of period $2a/c$.

The electric term in the self-force is

$$\mathbf{F}_\epsilon=-4\pi e^2\int_0^\infty d\tau\int d\mathbf{k}|f_k|^2 \times(\exp i\mathbf{k}\cdot\mathbf{s})\cos ck\tau\left[\dot{\xi}(t-\tau)-\frac{\mathbf{k}(\mathbf{k}\cdot\dot{\xi}(t-\tau))}{k^2}\right].$$

Integration over angles in $\mathbf{k}$ space yields

$$\mathbf{F}_\epsilon=-16\pi^2e^2\int_0^\infty d\tau\left[\dot{\xi}(t-\tau)-\frac{3\mathbf{s}}{s^2}\mathbf{s}\cdot\dot{\xi}(t-\tau)\right] \times\int_0^\infty\frac{dk}{s^2}|f_k|^2\cos ck\tau\left(\cos ks-\frac{\sin ks}{ks}\right) -16\pi^2e^2\int_0^\infty d\tau\left[\dot{\xi}(t-\tau)-\frac{\mathbf{s}}{s^2}\mathbf{s}\cdot\dot{\xi}(t-\tau)\right] \times\int_0^\infty dkk|f_k|^2\frac{\sin ks}{s}\cos ck\tau.$$

Writing

$$K(\tau)=\int_0^\infty\frac{dk}{s^2}|f_k|^2\cos ck\tau\left(\cos ks-\frac{\sin ks}{ks}\right), \quad (25)$$

$$M(\tau)=\int_0^\infty\frac{dk}{s}k|f_k|^2\cos ck\tau\sin ks. \quad (26)$$

$\mathbf{F}_\epsilon$ can be rewritten

$$\mathbf{F}_\epsilon=-16\pi^2e^2\int_0^\infty d\tau \times\left(K(\tau)\left[\dot{\xi}(t-\tau)-\frac{3\mathbf{s}}{s^2}\mathbf{s}\cdot\dot{\xi}(t-\tau)\right] +M(\tau)\left[\dot{\xi}(t-\tau)-\frac{\mathbf{s}}{s^2}\mathbf{s}\cdot\dot{\xi}(t-\tau)\right]\right). \quad (27)$$

We first consider the term containing $K(\tau)$. Now, when dealing with a distribution of infinitesimal thickness, like a spherical shell, one must define the integrals as limits of distributions of finite thickness. Since it is not always possible to permit these distributions to approach zero thickness before the results have been obtained, we shall at the outset spread the distribution by writing

$$f_k=\frac{1}{(2\pi)^{\frac{3}{2}}}\frac{i\lambda}{k+i\lambda}\frac{\sin ka}{ka}, \quad (28)$$

which approaches a spherical shell in the limit $\lambda\to\infty$. With this value for f_k, $K(\tau)$ becomes

$$s<c\tau<2a-s:\ K(\tau)=\frac{1}{64\pi^2a^2s^2\lambda} \times[(\exp-\lambda(2a-c\tau))-2(\exp-\lambda c\tau)] \times\left(\cosh\lambda s-\frac{\sinh\lambda s}{\lambda s}\right) +\text{terms proportional to }(\exp-\lambda(2a+c\tau\pm s)),$$

$$c\tau>2a+s:\ K(\tau)=\frac{1}{64\pi^2a^2s^2\lambda} \times(\exp-\lambda(c\tau-2a))\left(\cosh\lambda s-\frac{\sinh\lambda s}{\lambda s}\right) +\text{terms proportional to }(\exp-\lambda(c\tau+2a\pm s)).$$

We first observe that the terms in $K(\tau)$ proportional to $(\exp-\lambda(2a+c\tau\pm s))$ will make no contribution to the integral in the limit $\lambda\rightarrow\infty$. This follows from the fact that $s=|\xi(t)-\xi(t-\tau)|$ is always less than or equal to $c\tau$.¶ The remaining terms make contributions only near the points at which the arguments of the exponentials in $K(\tau)$ vanish, i.e., only near $c\tau\pm s=0$ and $2a-c\tau\pm s=0$. The only two points satisfying these relations are $\tau=0$, and $\tau=2a/c$. This is because, from the fact that $s<c\tau$, and from the periodicity, one can conclude also that $s<|2a-c\tau|$. This means that contributions to the τ integral come only from the infinitesimal regions near $\tau=0$ and $\tau=2a/c$, where s is small. In these regions we can therefore expand s in powers of τ and $|\tau-2a/c|$:

$$s\sim v(t)\tau \qquad \text{near} \quad \tau=0,$$
$$s\sim v(t)\left|\tau-\frac{2a}{c}\right| \qquad \text{near} \quad \tau=\frac{2a}{c}. \tag{29}$$

With these values for s,

$$(1/\lambda s^2)[\cosh\lambda s-(\sinh\lambda s)/\lambda s]$$

can be replaced by $\lambda/3$,¶¶ so that $K(\tau)$ becomes

$$K(\tau)=\frac{\lambda}{3.64\pi^2a^2}\times\begin{cases}(\exp-\lambda(2a-c\tau))-2(\exp-\lambda c\tau), & 0\leqslant\tau\leqslant\dfrac{2a}{c},\\ (\exp-\lambda(c\tau-2a)), & \tau\geqslant\dfrac{2a}{c}.\end{cases} \tag{30}$$

Thus $K(\tau)$ clearly approaches

$$K(\tau)=\frac{1}{3\times32\pi^2a^2}(\delta(c\tau-2a)-\delta(c\tau)), \tag{31}$$

and since

$$\dot{\xi}(t-\tau)-\frac{3\mathbf{s}}{s^2}\mathbf{s}\cdot\dot{\xi}(t-\tau)$$

approaches $-2\dot{\xi}[t-(2a/c)]$ and $-2\dot{\xi}(t)$ as τ approaches zero and $2a/c$ respectively, we obtain for the exact value of the force resulting from the term containing $K(\tau)$,

$$\frac{e^2}{3a^2c}\left[\dot{\xi}\left(t-\frac{2a}{c}\right)-\dot{\xi}(t)\right]$$

which vanishes if $\xi(t)$ is a function with period $2a/c$. It is readily verified that $M(\tau)$ leads to the same set of δ functions, so that the second term in $\mathbf{F}_\epsilon$ also vanishes. One can treat the magnetic term and the self-torque in the same way and show that these are also zero.

¶ Since s is essentially just the distance covered by the particle in time τ, $s<c\tau$ follows from the fact that $(v/c)<1$.

¶¶ One must let $\lambda\rightarrow\infty$ in such a way that $(3/\lambda^2s^2)\times[\cosh\lambda s-(\sinh\lambda s)/\lambda s]$ remains close to unity in the region where the integrand is appreciable. Because $|2a-c\tau|$ and $c\tau$ are both greater than s, the exponential factors will then make the result small, except near $\tau=0$ and $\tau=2a/c$. The contributions of the regions where the above factor is not unity are negligible for large λ, because the exponential factor far more than makes up for the increase in the $\cosh\lambda s$ and $\sinh\lambda s$ functions.