The Coulomb Scattering of Relativistic Electrons by Nuclei

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The cross section for the Coulomb scattering of relativistic electrons by atomic nuclei has been evaluated. The exact results obtained by Mott have been expanded in a power series in $Z/137$. where Z is the nuclear charge, the coefficients depending principally on the angle of scattering. The coefficients have been evaluated numerically. The resultant cross sections, taken together with those evaluated previously by Bartlett and Watson for Hg, yield the cross section for all Z to within 1 percent accuracy. A new approximate formula valid for $Z/137<0.2$ is obtained.

INTRODUCTION

THE experimental study of the single scattering of electrons by atomic nuclei is of the greatest fundamental importance. In the energy range in which the electron wave-length is considerably larger than the target nucleus, the Coulomb law of force is under scrutiny. For the higher energies which are now becoming available, the electron wave-length will be of the same size, or smaller, than the nucleus, so that scattering experiments should furnish information about the current and charge distribution within the nucleus.

The comparison of theory and experiment has been greatly hampered by the complexity of the theoretical formulas which have been obtained by Mott[1] in the form of a conditionally convergent infinite series. He has also derived an approximate formula, the "Mott formula,"[2] which is valid when

$$\alpha/\beta \ll 1, \quad \alpha = Ze^2/\hbar c, \quad \beta = v/c, \tag{1}$$

where Z is the nuclear charge. Sexl[3] and Urban[4] have also derived a formula valid in this range, which is in disagreement with Mott's result. Bartlett and Watson[5] have summed Mott's series for the case of Hg($Z=80$) for a range of energies up to $\beta \cong 1$. In this paper, the Mott series has been expanded in a power series in α and α/β the coefficients depending upon the angle of scattering θ. The series is accurate for middle Z elements. Used however in conjunction with the results of Bartlett and Watson, the cross section for scattering may be computed for all Z with an error of at most a few percent.

THEORY

Mott gives as the differential scattering cross section the result:

$$\sigma = q^2(1-\beta^2)F^*F \csc^2\theta/2 + G^*G \sec^2\theta/2, \tag{2}$$

where $q=\alpha/\beta$. The functions F and G are given by:

$$\begin{aligned} &F = F_0 + F_1, \quad G = G_0 + G_1, \\ &F_0 = i/2 \exp(iq \ln \sin^2\theta/2)\Gamma(1-iq)/\Gamma(1+iq), \\ &G_0 = -iq \cot^2\theta/2 F_0, \\ &F_1 = i/2 \sum_{k=0}^{\infty} [kD_k + (k+1)D_{k+1}](-)^k P_k(\cos\theta), \\ &G_1 = i/2 \sum_{k=0}^{\infty} [k^2 D_k - (k+1)^2 D_{k+1}] \times (-)^k P_k(\cos\theta), \end{aligned} \tag{3}$$

where Γ is the gamma-function and P_k the Legendre polynomial of order k. Finally

$$D_k = \frac{e^{-\pi i k}}{(k+iq)} \frac{\Gamma(k-iq)}{\Gamma(k+iq)} - \frac{e^{-i\pi\rho_k}}{(\rho_k+iq)} \frac{\Gamma(\rho_k - iq)}{\Gamma(\rho_k+iq)}, \quad \rho_k = (k^2-\alpha^2)^{\frac{1}{2}}. \tag{4}$$

[1] N. F. Mott, Proc. Roy. Soc. **A124**, 426 (1929); N. F. Mott, Proc. Roy. Soc. **A135**, 429 (1932).
[2] See reference 1.
[3] T. Sexl, Zeits. f. Physik **81**, 178 (1933).
[4] P. Urban, Zeits. f. Physik **119**, 67 (1942).
[5] J. H. Bartlett and R. E. Watson, Proc. Amer. Acad. Arts and Sci. **74**, 53 (1940).

TABLE I. The angular functions of Eq. (8).

θ	$A(\theta)$ Re	$A(\theta)$ Im	$B(\theta)$ Re	$B(\theta)$ Im	$C(\theta)$ Re	$C(\theta)$ Im
30°	−0.362	0.064	−0.086	0.498	0.580	−0.010
45	−0.510	0.114	−0.201	0.375	0.404	−0.069
60	−0.637	0.167	−0.339	0.209	0.313	−0.167
80	−0.780	0.235	−0.537	−0.033	0.289	−0.351
90	−0.840	0.266	−0.636	−0.150	0.305	−0.448
100	−0.893	0.295	−0.729	−0.263	0.332	−0.553
120	−0.980	0.344	−0.897	−0.455	0.408	−0.750
135	−1.028	0.373	−0.995	−0.568	0.464	−0.876
150	−1.062	0.394	−1.072	−0.650	0.511	−0.971
180	−1.089	0.411	−1.133	−0.720	0.551	−1.052
θ	$D(\theta)$ Re	$D(\theta)$ Im	$E(\theta)$ Re	$E(\theta)$ Im	$H(\theta)$ Re	$H(\theta)$ Im
30°	−0.107	−0.129	2.249	−0.676	1.483	1.044
45	−0.217	−0.221	1.267	−0.480	1.221	1.371
60	−0.344	−0.310	0.785	−0.347	0.953	1.174
80	−0.525	−0.417	0.437	−0.221	0.643	0.817
90	−0.614	−0.464	0.325	−0.173	0.514	0.658
100	−0.699	−0.505	0.240	−0.133	0.401	0.514
120	−0.849	−0.574	0.122	−0.072	0.219	0.281
135	−0.940	−0.612	0.065	−0.040	0.123	0.158
150	−1.007	−0.638	0.028	−0.017	0.051	0.065
180	−1.062	−0.657	0.000	0.000	0.000	0.000
θ	$I(\theta)$ Re	$I(\theta)$ Im	$J(\theta)$ Re	$J(\theta)$ Im		
30°	3.105	0.471	0	1.711		
45	1.261	0.733	0	1.199		
60	0.354	0.781	0	0.851		
80	−0.110	0.678	0	0.532		
90	−0.200	0.591	0	0.413		
100	−0.206	0.495	0	0.315		
120	−0.174	0.305	0	0.167		
135	−0.085	0.173	0	0.091		
150	−0.071	0.085	0	0.040		
180	−0.000	0.000	0	0		

The procedure used in this paper involves the expansion of F_1 and G_1 in a power series in α and α/β. The expansion will thus be valid only for relativistic electrons $\beta\sim 1$. The first term, the "α approximation" should yield the correct Mott formula. The expansion of the coefficients D_k yields

$$D_k=(-)^{k+1}\left\{\alpha^2\left(\frac{i\pi}{2k^2}+\frac{1}{2k^3}\right)+\alpha^3\left(\frac{i\psi_2}{\beta k^2}+\frac{\pi\psi_1}{\beta k^2}+\frac{\pi}{2\beta k^3}-\frac{i\psi_1}{\beta k^3}-\frac{i}{\beta k^4}\right)+\alpha^4\left(\frac{2\psi_1\psi_2}{\beta^2k^2}-\frac{i\pi\psi_1{}^2}{\beta^2k^2}-\frac{\pi^2}{8k^3}+\frac{\psi_2}{\beta^2k^3}-\frac{i\pi\psi_1}{\beta^2k^3}-\frac{\psi_1{}^2}{\beta^2k^3}+\frac{3}{8}\frac{i\pi}{k^4}-\frac{i\pi}{2\beta^2k^4}-\frac{2\psi_1}{\beta^2k^4}+\frac{3}{8k^5}-\frac{3}{2\beta^2k^5}\right)\right\}+0(\alpha^5). \quad (5)$$

Here $\psi_n(k)$ are the polygamma-functions. The function

$$\psi_1=(d/dk)\ln\Gamma(k),$$

while

$$\psi_n=(d/dk)\psi_{n-1}(k).$$

This expansion of D_k must now be substituted in (3) to obtain F_1 and G_1. Only the sums over k involved in the α^2 term may be performed exactly, the other sums must be performed numerically.

The α^2 term will now be discussed. It should lead to the Mott formula which has been the subject of controversy inasmuch as Mott and Urban have obtained conflicting answers. We take

$$D_k\simeq\frac{(-)^{k+1}\alpha^2}{2}\left(\frac{i\pi}{k^2}+\frac{1}{k^3}\right).$$

Then

$$\begin{aligned}G_1&\simeq\alpha^2/4\sum_{k=1}^{\infty}(\pi-i/k)(P_k+P_{k-1}),\\ G_0&\simeq\tfrac{1}{2}q\cot^2\theta/2+0(\alpha^2),\\ F_0&\simeq i/2+0(\alpha^2).\end{aligned} \quad (6)$$

To evaluate G_1 we use the sum formulas given by Mott,

$$\sum_{k=1}^{\infty}(P_k+P_{k-1})=\csc\theta/2-1,$$

$$\sum_{k=1}^{\infty}\frac{1}{k}(P_k+P_{k-1})=\ln\csc^2\theta/2.$$

Introducing (6) into (3) and (3) into (2) yields for R the ratio of the scattering to Rutherford scattering*

$$R=1-\beta^2\sin^2\theta/2+\pi\alpha\beta\sin\theta/2(1-\sin\theta/2). \quad (7)$$

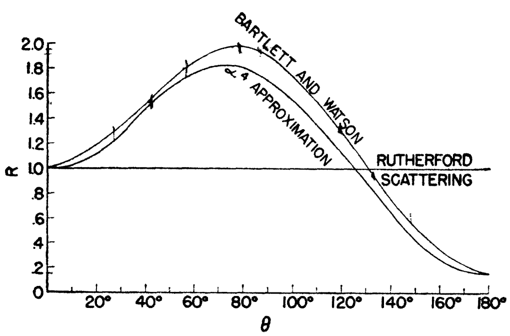

FIG. 1. Comparison between α^4-approximation and exact calculation of Bartlett and Watson for Hg. The energy of the electron is 2 Mev. R is the ratio of the scattering cross section to Rutherford scattering.

* Rutherford scattering yields cross section

$$\sigma=\left(\frac{Ze^2}{2m_0c^2}\right)^2\frac{1-\beta^2}{\beta^4}\csc^4\theta/2.$$

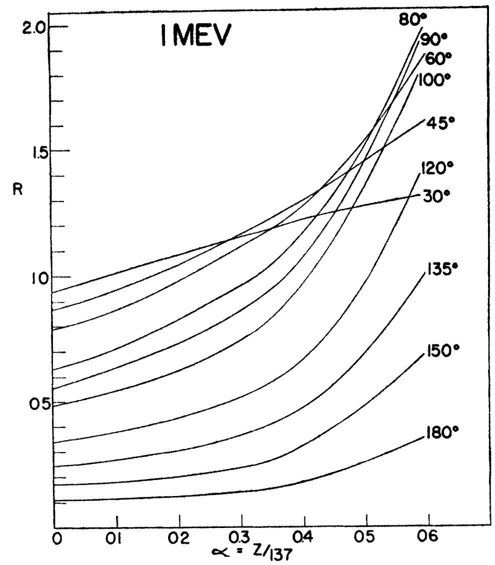

FIG. 2. The ratio R of the scattering cross section to Rutherford scattering as a function of $Z/137$ for the various scattering angles labelling each curve. Electron energy 1 Mev.

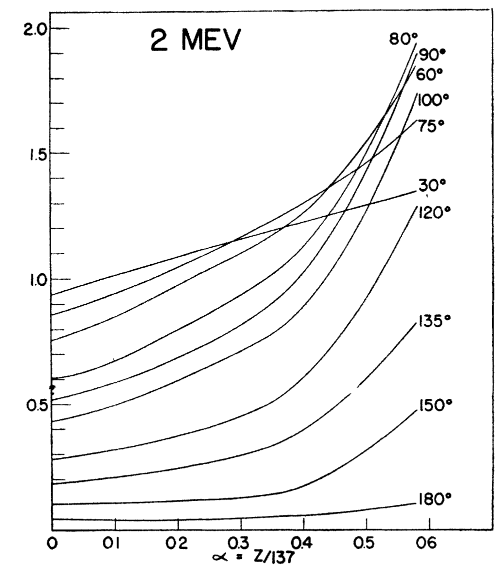

FIG. 3. The ratio R of the scattering cross section to Rutherford scattering as a function of $Z/137$ for the various scattering angles labelling each curve. Electron energy 2 Mev.

This result has been obtained by Julian Schwinger[6] by the variational-iterational procedure. In Mott's formula the factor $(1-\sin\theta/2)$ is replaced by $\cos^2\theta/2$ while Urban replaces this factor by 1. Thus both Mott's and Urban's formulas will be correct for $\theta\sim 0°$, Mott's for $\theta\sim 180°$. Both are incorrect in the intermediate angular range.

When the complete expression for D_k given by (5) is substituted into (3) one obtains expressions for F_1 and G_1 in the following form:

$$\begin{aligned} F&=F_0+A(\theta)\alpha^2+B(\theta)\alpha^3/\beta \\ &\qquad +C(\theta)\alpha^4/\beta^2+D(\theta)\alpha^4, \\ G&=G_0+E(\theta)\alpha^2+H(\theta)\alpha^3/\beta \\ &\qquad +I(\theta)\alpha^4/\beta^2+J(\theta)\alpha^4. \end{aligned} \tag{8}$$

The function F_0 has been tabulated by Bartlett and Watson. Only $E(\theta)$ may be summed analytically (see Eq. (6)). The remaining sums must be performed numerically.[7] The important difficulties which occur are discussed in the appendix. The real and imaginary parts of the functions A through J are given in Table I.

[6] Private Communication.

[7] The polygamma-functions are tabulated in H. T. Davis, *Tables of Higher Functions*, Vol. II (University of Indiana Press, Bloomington, 1935). The Legendre polynomials are tabulated up to P_{32} by H. Tallqvist, Acta. Soc. Scient. Fennica, Vol. II (1938).

Since we have neglected terms of the order of α^5 the error in our results should be of this order. This has been verified by comparing the α^4 approximation of this paper with the results of the Bartlett and Watson for Hg for which $\alpha^5=0.068$. Figure 1 shows this comparison. Since our results extrapolate smoothly to those of Bartlett and Watson, it is possible to combine their results with ours to obtain scattering cross sections which are valid for all Z. These are summarized in the curves given in Figs. 2, 3 and 4, where we have plotted the ratio R as a function of Z for different values of θ. This ratio is independent of the electron energy for energies above 4 Mev within the accuracy of these calculations. In the regions of α in which these plots deviate from the straight line, the approximate formula (6) will be invalid. The value of α at which this occurs varies with angle. Roughly speaking formula (6) should be valid for $\alpha\leq 0.2$.

COMPARISON WITH EXPERIMENT

We have compared our results with those of the most recent experiments,[8] performed with

[8] R. J. Van de Graaff, W. W. Buechner and H. Feshbach, Phys. Rev. **69**, 452 (1946); Buechner, Van de Graaff, Sperduto, Burrill, Jr. and Feshbach, Phys. Rev. **72**, 678 (1947).

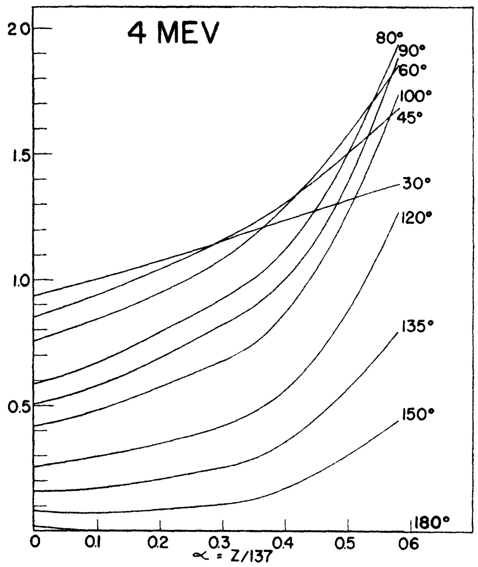

FIG. 4. The ratio R of the scattering cross section to Rutherford scattering as a function of $Z/137$ for the various scattering angles labelling each curve. Electron energy 4 Mev. The energy dependence of the ratio R may be neglected within the accuracy of these calculations above 4 Mev.

the electrostatic generator. The average of the ratios of experimental to the theoretically expected scattering is 1.01 ± 0.04. A typical example of the good agreement is plotted in Fig. 5.

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APPENDIX

The convergence of the infinite series for the angular functions $A(\theta)-J(\theta)$ (Eq. (8)) may be examined by using the asymptotic expressions:

$$P\,(\cos\theta)\xrightarrow[k\to\infty]{}\left(\frac{2}{\pi k\sin\theta}\right)^{\frac{1}{2}}\sin[(k+\tfrac{1}{2})\theta+\pi/4],$$
$$\psi_1(k)\xrightarrow[k\to\infty]{}\ln k, \qquad \text{(A1)}$$
$$\psi_2(k)\xrightarrow[k\to\infty]{}1/k.$$

All of the series involved converge absolutely except for

$$\sum_1^\infty P_k,\quad \sum_1^\infty \psi_1(k)P_k,\quad \sum_1^\infty \psi_1(k)^2P_k,$$

and, similarly,

$$\sum_1^\infty P_{k-1},\quad \sum_1^\infty \psi_1(k)P_{k-1},\quad \sum_1^\infty \psi_1^2(k)P_{k-1}.$$

The latter series converge only conditionally. Fortunately some of them may be summed:

$$\sum_{k=1}^\infty P_k=\tfrac{1}{2}\csc\theta/2-1,\quad \sum_{1=k}^\infty\frac{P_k}{k}=\ln\frac{\csc^2\theta/2}{1+\csc\theta/2},$$
$$\sum_{k=1}^\infty P_{k-1}=\tfrac{1}{2}\csc\theta/2,\quad \sum_{k=1}^\infty\frac{P_{k-1}}{k}=\ln(1+\csc\theta/2),$$
$$\sum_1^\infty\frac{P_{k+1}}{k}=-2\sin\theta/2(1-\sin\theta/2)+\cos\theta\ln\frac{\csc^2\theta/2}{1+\csc\theta/2}. \qquad \text{(A2)}$$

In addition we have summed

$$\sum_{k=1}^\infty\psi_1(k)P_k=-\gamma(\tfrac{1}{2}\csc\theta/2-1)+\tfrac{1}{2}\csc\theta/2\ln\tfrac{1}{4}(\csc\theta/2+1)-\ln\frac{\csc^2\theta/2}{4\csc\theta/2},$$
$$\sum_{k=1}^\infty\psi_1(k)P_{k-1}=-\gamma/2\csc\theta/2+\tfrac{1}{2}\csc\theta/2\ln\tfrac{1}{4}(\csc\theta/2+1), \qquad \text{(A3)}$$

where γ is the Mascheroni constant, 0.5772.

In general, all the series involved, including those which converged absolutely, converged rather poorly. Because of the slow convergence it was found useful to derange the series so as to permit the use of the Euler transformation.[9] The validity of the derangement for the conditionally convergent series is guaranteed by the theorem of Levi.[10] In addition for the $\Sigma\,\psi_1^2(k)P_k$ series it was necessary to use asymptotic expressions for the sum for sufficiently large k:

$$\sum_N^\infty P_k\psi_1^2(k)\simeq\tfrac{1}{2}\csc^2\theta/2\Big\{-4\csc^2\theta/2\sum_{N-1}^\infty\frac{P_k\psi_1(k)}{k}-\tfrac{1}{2}\cos\theta\sum_N^\infty\frac{P_{k-1}\psi_1^2(k)}{k}+\tfrac{1}{2}\sum_{N-2}^\infty\frac{P_k\psi_1^2(k)}{k}-\sum_{N-2}^\infty\frac{P_k\psi_1^2(k)}{k^2}+\frac{2P_{N-1}\psi_1(N-1)}{N-1}-(\cos\theta+\tfrac{1}{2})P_{N-1}\psi_1^2(N-1)-\tfrac{1}{2}P_{N-2}\psi_1^2(N-2)\Big\}+0\left(\sum_N^\infty\frac{P_k\psi_1(k)}{k^2}\right)+0\left(\sum_N^\infty\frac{P_k}{k^2}\right),\quad \text{for }\theta\simeq 90^\circ. \qquad \text{(A4)}$$

$$\sum_N^\infty P_k\psi_1^2(k)\simeq\csc^2\theta/2\Big\{-2(1+\cos^2\theta/2)\sum_{N-1}^\infty\frac{P_k\psi_1(k)}{k}-\frac{\cos\theta}{4}\sum_N^\infty\frac{P_{k-1}\psi_1^2(k)}{k}+\tfrac{1}{4}\sum_{N-2}^\infty\frac{P_k\psi_1^2(k)}{k}-\tfrac{1}{2}\sum_{N-2}^\infty\frac{P_k\psi_1^2(k)}{k^2}-\cos^2\theta/2P_N\psi_1^2(N)-P_{N-2}\frac{\psi_1(N-2)}{(N-2)}-\tfrac{3}{4}P_{N-1}\psi_1^2(N-1)-\tfrac{1}{4}P_{N-2}\psi_1^2(N-2)\Big\}+0\left(\sum_N^\infty\frac{P_k\psi_1(k)}{k^2}\right)+0\left(\sum_N^\infty\frac{P_k}{k^2}\right)+0\left(\sum_N^\infty\frac{P_k\psi_1^2(k)}{k^3}\right),\quad \text{for }\theta\simeq 180^\circ. \qquad \text{(A5)}$$

[9] T. J. I'A. Bromwich, *Introduction to the Theory of Infinite Series* (The MacMillan Company, New York, 1947).

[10] Levi, Duke Math. J. **13**, 579 (1946).

The value chosen for N was 20. These procedures were checked by applying them to series (A2) and (A3) which could be summed analytically. In addition, some internal checks were provided by the recurrence relations

$$\begin{aligned}\psi_1(k+1)-\psi_1(k)&=1/k,\\ \psi_2(k+1)-\psi_2(k)&=-1/k^2,\end{aligned}\tag{A6}$$

which lead to the relations:

$$\begin{aligned}
&\sum_1^\infty \psi_1(k)P_k=\sum_1^\infty \psi_1(k)P_{k-1}+\gamma-\ln\frac{\csc^2\theta/2}{1+\csc\theta/2},\\
&\sum_1^\infty \psi_1^2P_k=\sum_1^\infty \psi_1^2(k)P_{k-1}-\psi_1^2(1)\\
&\qquad\qquad-\sum_1^\infty\frac{P_k}{k^2}-2\sum_1^\infty\frac{\psi_1(k)P_k}{k^2},\\
&\sum_1^\infty \psi_1(k)\psi_2(k)P_k=\sum_1^\infty \psi_1(k)\psi_2(k)P_{k-1}-\psi_1(1)\psi_2(1)\\
&\qquad\qquad+\sum_1^\infty\frac{\psi_1(k)P_k}{k^2}-\sum_1^\infty\frac{\psi_2(k)P_k}{k}+\sum_1^\infty\frac{P_k}{k^3},\\
&\sum_1^\infty \psi_2P_k=\sum \psi_2P_{k-1}-\psi_2(1)+\sum P_k/k^2.
\end{aligned}\tag{A7}$$

Similar relations using the recurrence relations for the Legendre polynomials but were not found as useful except as used in (A4) and (A5). Another check was also provided by the asymptotic expansion for $\psi_2\rightarrow 1/k+1/2k^2+1/6k^3+\cdots$. It is estimated that the error in the computation of R is 0.1 percent.

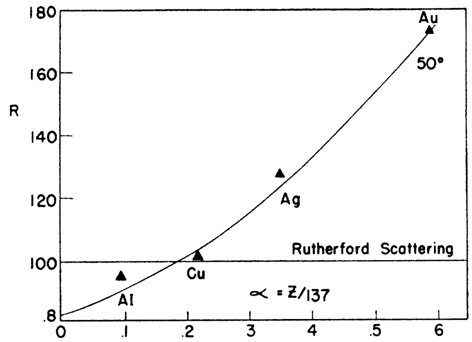

FIG. 5. Comparison of theory and experiment at electron energy of 2 Mev. The solid line is given by the theory. The triangles give the experimental points as obtained by Van de Graaff, Buechner *et al.*

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Variational Methods in Nuclear Collision Problems*

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Variational methods, similar to the Rayleigh-Ritz method for bound state calculations, are developed for the phase shifts and elements of the scattering matrix in nuclear collisions. Numerical applications to neutron-proton and neutron-deuteron scattering involving trial functions with undetermined coefficients are described. Another variational principle, for scattering amplitudes, is shown to lead to the Born approximations and a formula recently derived by Schwinger. It may also be used in conjunction with the method of undetermined coefficients.

I. INTRODUCTION

A GREAT deal of information about the nature of nuclear forces has been derived from a comparison of experimental and theoretical studies of simple nuclear systems.

The bound states of nuclei comprising up to four particles have been theoretically treated with considerable accuracy[1] and have yielded very important results. For nuclear collisions involving more than two particles, Breit and Wigner,[2] Wheeler,[3] Heisenberg,[4] and Wigner[5] have developed general, phenomenological theories. However, no very satisfactory scheme for treating such collisions in detail has so far been given. All calculations up to the present[6] are based on an approximation of the wave function by an

* This paper is based on Part I of a thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, at Harvard University, June 1948.

** Parker Fellow, 1947–8.

[1] See, for example, W. Rarita and R. D. Present, Phys. Rev. **51**, 788 (1937); H. Margenau and D. T. Warren, Phys. Rev. **52**, 790 (1937).

[2] G. Breit and E. Wigner, Phys. Rev. **49**, 519 (1936).

[3] J. A. Wheeler, Phys. Rev. **52**, 1107 (1937).

[4] W. Heisenberg, Zeits. f. Physik **120**, 513 (1942).

[5] E. P. Wigner, Phys. Rev. **70**, 15 and 606 (1946).

[6] For example, L. Motz and J. Schwinger, Phys. Rev. **58**, 26 (1940); R. A. Buckingham and H. S. W. Massey, Proc. Roy. Soc. **179**, 123 (1941).