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Thermal Voltages of a Quartz Crystal

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The thermal charges and voltages which arise in a piezoelectric material because of its heat energy have been computed. The calculations were made for a quartz bar, and include also some of the effects of the measuring circuit. Measurements of the thermal noise voltage of a quartz bar having a natural frequency of 36.6 kc/sec. were made with a specially shielded and filtered amplifier. Completely successful separation of the thermal voltage from the first-stage tube noises was not achieved. However, fairly good agreement between measured and computed crystal thermal voltage was obtained with one amplifier when it was adjusted to have a high input impedance. Measurement of the thermal voltage appears to offer a means of securing rough values of piezoelectric coefficients at room temperature and higher.

I. INTRODUCTION

THE Debye theory of specific heats regards the internal heat energy of an elastic solid as being distributed among the various normal modes of oscillation. For a piezoelectric material such as crystalline quartz, the elastic stresses associated with each normal mode will produce electric charges and voltages having the same frequency as that of the normal mode. In what follows, these will be referred to as thermal charges and voltages.

In a recent article, Lawson and Long[1] proposed that a quartz crystal be used as a thermometer, for which the thermal voltages would serve as the thermometric property. They suggested that it would be possible to measure very low temperatures (of the order of 0.0001°K) with such a thermometer.

Lawson and Long computed the thermal voltage for a long, thin quartz bar, by assuming that the long-time average potential energy of the fundamental longitudinal mode was $kT/2$. Subsequently, Brown and MacDonald[2] pointed out that the same result could be got by applying Nyquist's theorem to the equivalent circuit of the bar. Near its natural frequency, the motional part of the equivalent circuit can be represented by a series resonant circuit.

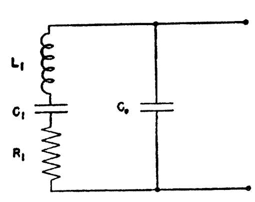

FIG. 1. Equivalent circuit of unloaded quartz bar near a natural frequency.

Gerjuoy and Forrester[3] studied how the components of a resonant circuit would have to be selected in order that its thermal voltage would be large in comparison with the noise voltages of a typical commercially available amplifier tube. They concluded that temperatures as low as 2°K could be measured provided the resonant circuit had a capacitance of 0.02 μf and a natural frequency of 1600 c.p.s.

We have undertaken a study of these thermal voltages, and have made quantitative measurements of them, with a view toward possible applications to the measurement of piezoelectric constants as a function of temperature, as well as toward investigating the feasibility of use for thermometric purposes. Since an amplifier is required for measurement of the voltages, we centered our attention on methods of allowing for the amplifier noise. This noise voltage was comparable with the thermal voltages of the quartz crystal bar which we used in all our measurements.

II. CALCULATION OF THERMAL VOLTAGES

1. Parallel *R*—*C* Circuit

The mean-square thermal voltage, $|v|^2$, which exists between any two points of an electrical

[1] A. W. Lawson and E. A. Long, Phys. Rev. **70**, 220, 977 (1946).
[2] J. B. Brown and D. K. C. MacDonald, Phys. Rev. **70**, 976 (1946).
[3] E. Gerjuoy and A. T. Forrester, Phys. Rev. **71**, 375 (1947).

network can be obtained from Nyquist's theorem:

$$d|v|^2=(2kT/\pi)R(\omega)d\omega, \qquad (1)$$

in which k is Boltzmann's constant, T the absolute temperature of the network, ω the circular frequency (hereinafter called "pulsance"), and $R(\omega)$ the real (resistive) component of the impedance between the points in question.

Of especial interest is the case of a resistance R and a capacitance C in parallel;[4] the thermal voltage of this circuit is useful for checking the performance of noise-measuring apparatus. Application of Eq. (1) to such a circuit gives for the total thermal voltage (caused by all frequencies from 0 to ∞)

$$|v|_t^2=kT/C,$$

a result which can also be obtained by equating the thermal voltage energy of the condenser, $(1/2)C|v|_t^2$, to $(1/2)kT$ in accordance with the principle of equipartition of energy. If a filter consisting of a simple circuit in parallel resonance at pulsance ω_0, is used at least one stage of amplification (with a flat amplifier) beyond the $R-C$ combination, the thermal voltage is reduced to

$$|v|_f^2=\frac{kT}{C}\frac{\omega_0RC}{q_f[1+(\omega_0RC)^2]+\omega_0RC}, \qquad (2)$$

where q_f is the Q of the filter circuit at ω_0. Here the overall gain from the amplifier input to the amplifier output (including the filter) has been normalized to unity at ω_0.

2. Quartz-Crystal Bar

At a natural frequency of a mechanically unloaded quartz crystal bar, an equivalent[5] lumped electrical circuit is that shown in Fig. 1. The motional inductance L_1 and capacitance C_1 depend on the type of cut, mode of vibration, and dimensions of the bar; the motional resistance R_1 represents the losses in and from the bar and its associated mount. The shunt branch C_0 is the sum of C_0', the clamped capacitance of the bar with its electrodes, and C_0'', the input capacitance of the measuring circuit (see Fig. 2). The "capacitance ratio," $c=C_0'/C_1$, is independent of the dimensions of the bar and is about 125 for the fundamental mode of an X-cut plate in thickness vibration or an X-cut Y-bar in longitudinal vibration.

The impedance looking into the crystal circuit is

$$Z_c=\frac{1}{j\omega C_0}\cdot\frac{R_1+j\omega L_1+1/j\omega C_1}{R_1+j\omega L_1+\dfrac{1}{j\omega}\left(\dfrac{1}{C_0}+\dfrac{1}{C_1}\right)}. \qquad (3)$$

Substitution of the real part of Z_c into Eq. (1) and integration gives

$$|V|_t^2=\frac{kT}{C_0}\cdot\frac{C_1}{C_0+C_1}. \qquad (4)$$

The derivative, $(d|v|^2/d\omega)$, varies with frequency in the same manner as the voltage squared across the capacitance of a simple series resonant circuit (having the same Q) driven at a constant voltage. The maximum value of $(d|v|^2/d\omega)$ for the crystal occurs at $\omega=\omega_0$, where

$$\omega_0^2L_1\frac{C_0C_1}{C_0+C_1}=1,$$

and equals

$$\left(\frac{d|v|^2}{d\omega}\right)_{\max}=\frac{2q_0}{\pi\omega_0}|V|_t^2,$$

where $q_0=\omega_0L_1/R_1$. At

$$\omega\approx\left(1\pm\frac{1}{2q_0}\right)\omega_0,\quad \frac{d|v|^2}{d\omega}=\frac{1}{2}\left(\frac{d|v|^2}{d\omega}\right)_{\max}.$$

Thus the thermal voltage output of the crystal bar is localized in a very narrow band of frequencies centered at the anti-resonant pulsance ω_0.

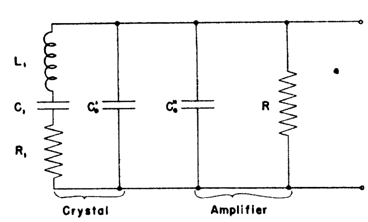

FIG. 2. Equivalent circuit of crystal connected to amplifier.

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[4] E. B. Moullin, *Spontaneous Fluctuations of Voltage* (Clarendon Press, Oxford, 1938), Chap. I.

[5] W. P. Mason, *Electromechanical Transducers and Wave Filters* (D. Van Nostrand Company, Inc., New York, 1942).

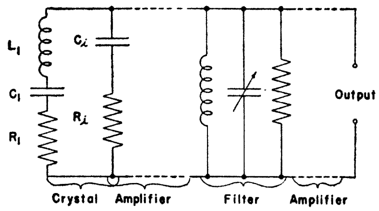

FIG. 3. The circuit of Fig. 2 rearranged as a series circuit and connected to filter.

The total voltage-squared can be expressed in terms of the constants of a rotated X-cut bar by substituting for C_1 and C_0 in Eq. (4). The value of C_1 is

$$C_1=\frac{8}{\pi^2}\cdot\frac{|d_{12}'|^2}{s_{22}'^E}\cdot\frac{l_w l}{l_t},$$

where l_w, l, and l_t are the width, length, and thickness of the bar and d_{12}' and $s_{22}'^E$ are the appropriate piezoelectric and elastic constants (see for example Mason[5]). If the crystal is not shunted by external capacitance C_0'', then

$$C_0=C_0'=\frac{\epsilon l_w l}{4\pi l_t},$$

where ϵ is the clamped dielectric constant. With these substitutions Eq. (4) becomes

$$|V|_t^2=128\cdot\frac{|d_{12}'|^2}{\epsilon^2 s_{22}'^E}\cdot\frac{l_t}{l_w l}\cdot kT\cdot\frac{C_0}{C_0+C_1}, \tag{5}$$

which differs from the result of Lawson and Long,[1] who found the numerical factor $32\pi^2$ rather than 128, and who also omitted the factor $C_0/(C_0+C_1)\approx 1$.

3. Effect of Amplifier Input

When interpreting the results of measurements of thermal voltage of a crystal it is necessary to allow for the effect of the input characteristics of the amplifier. In what follows it will be assumed that the input resistance of the amplifier is at the same temperature as the crystal.

Ordinarily one would consider the amplifier input as a parallel $R-C$ circuit (Fig. 2), and the R and C would be reasonably independent of frequency. When the noise caused by a resistance connected in parallel with a capacitance is being measured, the effect on the noise of these input characteristics is easily computed. However, computation of the thermal voltage of the crystal circuit of Fig. 2 is difficult. It is easier to consider the circuit of Fig. 3, in which the equivalent series characteristics C_i and R_i are computed at the anti-resonant pulsance, ω_0, of the crystal and are considered to be independent of frequency. This is permissible since during all measurements the output of the amplifier is filtered (to increase the signal-to-noise ratio). Application of Nyquist's formula to the circuit of Fig. 3, in which the filter is a parallel resonant circuit tuned to ω_0, gives

$$|V|_f^2=\frac{kT}{C_i}\cdot\frac{C_1}{C_1+C_i}\cdot\frac{q}{q+q_f}(1+\omega_0^2R_1R_iC_i^2). \tag{6}$$

where $q=\omega_0L_1/(R_1+R_i)$. Whenever $q\gg q_f$ and $\omega_0R_iC_i$ is small, Eq. (6) reduces to the simpler Eq. (4). Here the over-all gain from the amplifier input to the amplifier output (including the filter) has been normalized to unity at ω_0, and q_f is the Q of the filter at ω_0, as before.

4. Voltage Sensitivity of Crystal

Equation (4) shows that the integrated mean-square voltage developed across the crystal is very small, even on open circuit. Furthermore $|V|_t^2$ cannot be increased indefinitely by designing the crystal for larger C_1. With the notation $C_0=C_0'+C_0''$ (Fig. 2), differentiation with respect to C_0' of Eq. (4) shows that $|V|_t^2$ has a maximum approximately at $C_0'=C_0''$. Thus if C_0'' cannot be reduced below say 10 $\mu\mu$f, then, since $C_0'/C_1\approx 125$,

$$|V|_{t(\max)}^2\approx\frac{kT}{50}\times 10^{10}\ \text{(mks units)}.$$

At room temperature, $|V|_{t(\max)}\approx 0.93$ μv. and at 0.01°K, $|V|_{t(\max)}\approx 0.0054$ μv. Such small mean voltages may be expected to suffer considerable fluctuation of instantaneous value.

From Eq. (4) we have

$$|q|^2\approx kTC_1$$

for the mean-square charge on C_0. At room temperature for a crystal for which $C_0=10$ $\mu\mu$f ($C_1=0.08$ $\mu\mu$f), $|q|$ amounts to about 160 electrons and at 0.01°K to about 1 electron. At least some part of the fluctuations of galvanometer readings described in Section III can be ascribed to the small numbers of electrons involved in developing the voltages across C_0.

5. Background Noise

The thermal voltages considered above are so small that special attention must be given to amplifier noise. That part of the mean-square noise voltage which originates in the plate circuit of the first tube and in subsequent stages of the amplifier system will be called the residual noise. It is usually assumed to be independent of what impedance is connected between grid and cathode of the first tube. By connecting the grid directly to the cathode, the residual noise can be measured and then subtracted from all other mean-square noise-voltages measured with the same amplifier system.

The noises which originate in the grid circuit of the first tube are due to (1) thermal agitation in the resistive component of the tube's input impedance, (2) shot effect, (3) flicker effect. It is clear that the total background noise at the grid will depend on the external impedance (such as that of the quartz bar) connected between grid and cathode.

Equation (6) in effect, allows for the thermal agitation referred to in (1) above, provided the resistive component of the tube's input impedance is at the same temperature as the quartz bar. This condition can be realized approximately if a resistance is shunted across the quartz bar and considered to be part of the tube's input impedance.

It does not appear feasible to compute the noise caused by shot effect and flicker effect. For the measurements reported in Section III attempts were made to select the operating condition of the tube so that the grid current would be as small as possible during measurement of thermal voltages.

The usual method of allowing for background noise[6] during measurements of thermal voltages in impedances which are mostly resistive is by the introduction of a calibrating voltage in series with the unknown voltage. This method is difficult to apply in the present case because of the very rapid variation of the crystal impedance with frequency indicated in Eq. (3). The method was not used.

III. RESULTS OF MEASUREMENTS

Measurements of the thermal voltage generated at room temperature were made on a quartz crystal bar having a natural frequency of 36.6 kc/sec. when vibrating longitudinally in its fundamental mode. The bar had dimensions of $7.0\times0.70\times0.101$ cm. The thickness dimension (0.101 cm) was parallel to the x axis (electric axis). The two rectangular faces perpendicular to

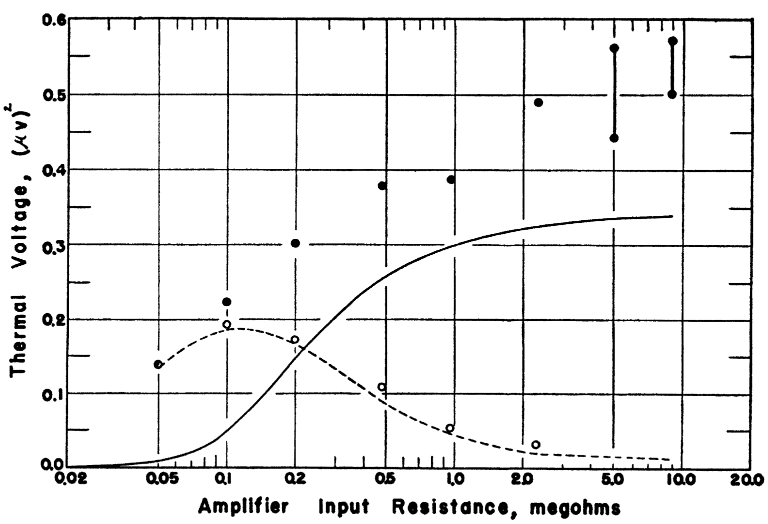

FIG. 4. Thermal voltages using cathode-follower input with 9002 tube. The open circles represent measurements for about 36.2-$\mu\mu$f in parallel with the input resistances shown. The dotted line is computed from Eq. (2). The solid circles represent measurements on the crystal for $C_0\approx40.7$-$\mu\mu$f and for the input resistances shown. The solid line is computed from Eq. (6).

[6] E. B. Moullin, *Spontaneous Fluctuations of Voltage* (Clarendon Press, Oxford, 1938), Chap. VII.

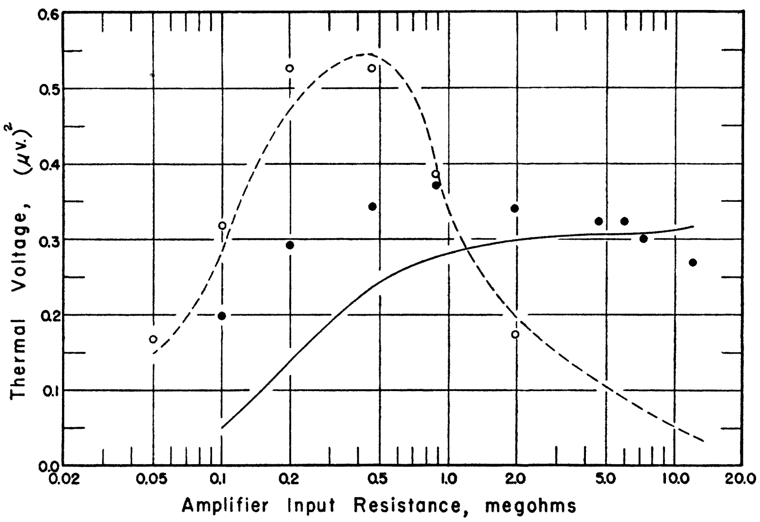

FIG. 5. Thermal voltages using 6AK5-tube input. The open circles represent measurements for about 11.9-$\mu\mu$f in parallel with the input resistances shown. The dotted line is computed from Eq. (2). The solid circles represent measurements on the crystal for $C_0 \approx 42.3$-$\mu\mu$f and for the input resistances shown. The solid line is computed from Eq. (6).

the electric axis were plated with gold. The length dimension (7.0 cm) made an angle of $-18.5°$ with the y axis (mechanical axis). For a bar of given size having its large flat faces perpendicular to the x axis, this orientation yields the lowest frequency for the fundamental mode.

The bar was mounted in air between spherical contacts located at the geometrical centers of its flat faces. Direct measurement (with a Q-meter technique) of the series-chain capacitance yielded $C_1 = 0.142$ $\mu\mu$f at a room temperature of 23°C. This value is in good agreement with the value computed from the known physical constants of quartz and the dimensions of the bar. Direct measurements of the Q of the bar gave values near 20,000.

The calculated clamped capacitance of the bar is about 19 $\mu\mu$f. However, the bar was mounted in a metal box and was always connected to an amplifier when measurements were made, and so the capacitance C_0 was always greater than 19 $\mu\mu$f.

Several different first-stage amplifiers were used. In every case, the crystal was connected between the grid and ground leads. The arrangement of the remaining stages and filter network of the amplifier system was the same for all measurements. It consisted of a four-stage battery-operated amplifier, followed by the filter network. The filter consisted of a high Q inductor and variable air capacitor, both connected in parallel with a high impedance a.c.-operated amplifier. The filter was tuned to pass various frequencies by adjustment of the variable capacitor. The output of the a.c.-operated amplifier was applied to a thermojunction which was connected to a wall galvanometer.

Special attention was paid to the shielding and disposition of ground wires for the system. A heavy iron box, having separate compartments for the crystal holder, first-stage amplifier, two of the battery-operated stages, and batteries, was used to provide electrical and mechanical shielding.

The amplifier system was calibrated by applying known voltages to the first-stage amplifier terminals and reading the corresponding galvanometer deflections. These measurements also showed that the system had accurately a square-law response. Impedance measurements, whenever required, were made with a Q-meter technique.

All measurements were made at a room temperature near 296°K, and for a filter Q of about 300. The filter was always tuned to the natural frequency (36.6 kc/sec.) of the bar.

For the first attempts at measurements of thermal voltage, a type 9002 tube was used with a cathode-follower hook-up for the first-stage amplifier. Figure 4 shows the results in this case.

There is agreement between the measured and computed thermal voltages when the input consisted of a resistance and capacitance in parallel (no crystal). However, when the crystal was also connected to the input, the measured thermal voltage was much greater than expected, and it was surmised that this was due to the shot effect of the d.c. component of the grid current. It did not seem feasible to make quantitative allowance for this effect.

The second measurements were made with a type 6AK5 tube as the first-stage amplifier. A pentode connection with a "floating" control grid was used.[7] The results are shown in Fig. 5.

As before, there is agreement between the computed and measured thermal voltages when no crystal is in the circuit. When the crystal was connected to the input, the measured thermal voltage was greater than the computed value when the amplifier input resistance was less than one megohm, but fairly good agreement obtained for input resistances greater than one megohm. Since this amplifier had a very small control grid current, it is difficult to explain the excessive noise on the basis of shot effect at the grid. In addition, if the excess noise were due to shot effect, the mean-square voltage should have become greater as the input resistance was increased, whereas experimentally the opposite occurred.

[7] V. L. Parsegian,. 6AK5 and 954 tubes in ionization chamber pulse amplifiers. Rev. Sci. Inst. **17**, 39 (1946).

IV. SUMMARY

Calculations have been made of the thermal charges and voltages which, in a piezo-electric material such as crystalline quartz, arise because of the heat energy distributed among the various normal modes of oscillation. The calculations were made for a quartz bar, and include also some of the effects of the electrical measuring circuit to which the crystal might be connected.

Measurements of the thermal noise voltage of a quartz bar having a natural frequency of 36.6 kc/sec. were made with a specially shielded and filtered amplifier. Two different pre-amplifier tubes were tried in an effort to separate the crystal thermal voltage from the inevitable tube noises. Although completely successful separation of the thermal voltage from the tube noises was not achieved, fairly good agreement between measured and computed crystal thermal voltage was obtained with one pre-amplifier when it was adjusted to have a high input impedance.

Measurement of the thermal voltage appears to offer a means of securing rough values of piezoelectric coefficients at room temperature and higher. However, it does not appear that the thermal voltages are a suitable thermometric property for measurement of very low temperatures. The precision of measurement could probably be improved if crystal filters were used.