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Phenomenological Quantum Electrodynamics. Part II. Interaction of the Field with Charges

J. M. JAUCH AND K. M. WATSON
Department of Physics, State University of Iowa, Iowa City, Iowa
and
Institute for Advanced Study, Princeton, New Jersey
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The quantization of the phenomenological field is carried through with the presence of charged particles described by a Dirac Hamiltonian. The elimination of the longitudinal field presents some features which are qualitatively different from the vacuum case. In the first place the Coulomb energy has an angular dependent term. Further the elimination procedure introduces an additional term in the interaction Hamiltonian which is proportional to the medium velocity. The physical significance of this term is clarified in the discussion of the Čerenkov radiation which is carried through in the two coordinate systems: medium at rest, particle moving and particle at rest, medium moving. The result for the total radiation emitted is in agreement with the corresponding classical calculation.

A. CLASSICAL FIELD WITH CHARGES

In the first part we have developed the classical and quantum theory of the pure radiation field in a refractive medium. In this part we shall study the interaction of charges with the field. In classical field theory the charges and current density together form a four-vector density which we may call j^μ. Here $j^0=\rho$ is the charge density and $j^k=s^k$ ($k=1, 2, 3$) is the current density. Maxwell's field equations are now in rationalized units of the charge (such that $e^2=4\pi/137$)

$$\nabla\times\mathbf{E}=-\mathbf{B},\quad \nabla\times\mathbf{H}=\dot{\mathbf{D}}+\mathbf{s}, \tag{1}$$

$$\nabla\cdot\mathbf{B}=0,\quad \nabla\cdot\mathbf{D}=\rho, \tag{2}$$

$$\mathbf{D}=\epsilon\mathbf{E},\quad \mathbf{B}=\mu\mathbf{H}. \tag{3}$$

Charges and currents are connected by the continuity equation

$$\dot{\rho}+\nabla\cdot\mathbf{j}=0. \tag{4}$$

Introducing the tensors $F^{\lambda\mu}$ and $G^{\lambda\mu}$ of Part I (Eqs. (4) and (5), Part I) we may write Eqs. (1), (2), and (3) in the form

$$\partial_\lambda F_{\mu\nu}+\partial_\mu F_{\nu\chi}+\partial_\nu F_{\lambda\mu}=0, \tag{5}$$

$$\partial_\mu G^{\lambda\mu}=\mu j^\lambda, \tag{6}$$

$$G_{\lambda\mu}=F_{\lambda\mu}+\kappa(F_{\mu\sigma}v_\lambda-F_{\lambda\sigma}v_\mu)v^\sigma. \tag{7}$$

It is seen that in Eq. (6) the constant μ enters explicitly in contradistinction to the radiation field where ϵ and μ entered only in the combination $n^2=\epsilon\mu$. In order to have the radiation field only dependent on n^2 we have defined the tensor $G_{\lambda\mu}$ different from the usual way[1] by multiplying it with μ. For the radiation field the equations for $G_{\lambda\mu}$ are thereby not changed since they are homogeneous. However, when sources are present the μ shows up explicitly either in the definition of the $G_{\lambda\mu}$ or on the right-hand side of (6). We have maintained here the definition of the $G_{\lambda\mu}$ for the pure radiation field and correspondingly find the inhomogeneous field equations in the form (6) and therefore in the following all the expressions for the energy will occur multiplied with μ. The continuity equation for charge and current density is then

$$\partial_\lambda j^\lambda=0.$$

Equation (5) is the same as for the radiation field and permits the introduction of the vector potential ϕ_λ

$$F_{\lambda\mu}=\partial_\lambda\phi_\mu-\partial\phi_\lambda. \tag{9}$$

The field equations for the potentials are then from (6):

$$(-\partial^\mu\partial_\mu+\partial_\mu v^\mu\partial_\sigma v^\sigma)\Psi^\lambda=\mu j^\lambda. \tag{10}$$

Here we have used again the subsidiary condition

$$\chi\equiv\partial^\rho\phi_\rho-\kappa v^\sigma\partial_\sigma\phi^\rho v_\rho=0, \tag{11}$$

[1] See, for instance, W. Pauli, *Encyklopädie der math. Wiss.* (B. G. Teubner, Leipzig, 1921), V. 19, p. 655.

and for Ψ^λ we have, as in Part I,

$$\Psi^\lambda = \phi^\lambda - \kappa\phi^\sigma v_\sigma v^\lambda. \tag{12}$$

B. LAGRANGEAN FORM OF THE FIELD EQUATIONS

If we consider the current density vector j^μ as a given space-time function, then the classical field equations (9), (10) can be derived from the variational principle.

$$\delta\int L dx^0 = 0 \tag{13}$$

where

$$L = L^0 + L' = \int \mathcal{L} d^3x. \tag{14}$$

L^0 is given by Eqs. (15), (16) of Part I. For L' we have

$$L' = \int \mathcal{L}' d^3x, \tag{15}$$

$$\mathcal{L}' = +\mu j_\nu \phi^\nu. \tag{16}$$

The Euler-Lagrange equations which follow from (13) are

$$\partial^\lambda(\partial/\partial\partial^\lambda\phi^\mu) - (\partial/\partial\phi^\mu) = 0$$

or

$$\partial^\lambda G_{\mu\lambda} - \partial^\lambda(g_{\lambda\mu} - \kappa v_\lambda v_\mu)\chi = \mu j_\mu. \tag{17}$$

These equations are identical with (10). If we impose the subsidiary condition (11), they become identical with (6). From (17) and the continuity equation (8) follows

$$(\partial^\lambda\partial_\lambda - \kappa\partial^\lambda v_\lambda \partial^\mu v_\mu)\chi = 0, \tag{18}$$

exactly as for the radiation field. Just as in the latter case one concludes from this that the subsidiary condition (11) is identically satisfied if it is valid together with its time derivative at a given value of the time $t=0$.

C. HAMILTONIAN FORMALISM

The canonically conjugate variables Π_μ are defined by

$$\Pi_\mu = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi^\mu)} = G_\mu{}^0 - (g_\mu{}^0 - \kappa v^0 v_\mu)\chi \tag{19}$$

and the Hamiltonian is then

$$H = \int \mathcal{H} d^3x$$

with

$$\mathcal{H} = \Pi_\mu \dot{\phi}^\mu - \mathcal{L}. \tag{20}$$

We may decomposite it into the two

$$H = H^0 + H', \tag{21}$$

where H^0 is the expression given by Eq. (25), Part I, and

$$H' = -\mu\int d^3x j_\nu \phi^\nu. \tag{22}$$

The Eqs. (7), (9), and (17) are then equivalent with the Hamiltonian equations

$$\dot{\Pi}_\mu = -\delta H/\delta\phi^\mu, \quad \dot{\phi}^\mu = \delta H/\delta\Pi_\mu. \tag{23}$$

Since this Hamiltonian differs from the Hamiltonian H^0 of the radiation field only through the presence of the interaction term H' (22), we may find an explicit expression for it in exactly the same way as the expression (30) in Part I was derived. Thus for the Maxwell case (that is, subsidiary condition $\chi=0$ (11) satisfied) we may write

$$\mathcal{H}^0 = \frac{1}{2}\frac{1}{1+\kappa v_0{}^2}\Big\{\mathbf{\Pi}\cdot\mathbf{\Pi} + 2\kappa v^0(\mathbf{v}\times\mathbf{W}\cdot\mathbf{\Pi}) + \frac{\kappa}{1+\kappa}(\mathbf{\Pi}\cdot\mathbf{v})^2 + \kappa(\mathbf{v}\times(\mathbf{v}\times\mathbf{W})\cdot\mathbf{W})\Big\} + \tfrac{1}{2}\mathbf{W}\cdot\mathbf{W} + \mathbf{\Pi}\cdot\nabla\phi_0. \tag{24}$$

The last term may be transformed in the following way. We have

$$\mathbf{\Pi}\cdot\nabla\phi_0 = \nabla\cdot(\mathbf{\Pi}\phi_0) - (\nabla\cdot\mathbf{\Pi})\phi_0.$$

Here the first term is a space divergence and may be omitted from the density function since it contributes nothing to the integrated Hamiltonian. For the second term we get from (19) and (17)

$$\nabla\cdot\mathbf{\Pi} = -\mu j^0 + 2\kappa(\mathbf{v}\cdot\nabla)v^0\chi. \tag{25}$$

Since the last term is zero, if (11) holds we may replace the last term in (24) by $\mu j_0\varphi_0$. For the total Hamiltonian density we obtain thus

$$H = H_0 + H_1, \tag{26}$$

$$H_0=\frac{1}{2}\frac{1}{1+\kappa v_0^2}\int\left\{\mathbf{\Pi}\cdot\mathbf{\Pi}+2\kappa v^0(\mathbf{v}\times\mathbf{W}\cdot\mathbf{\Pi})+\frac{\kappa}{1+\kappa}(\mathbf{\Pi}\cdot\mathbf{v})^2+\kappa(\mathbf{v}\times(\mathbf{v}\times\mathbf{W})\cdot\mathbf{W}\right\}d^3x+\tfrac{1}{2}\int\mathbf{W}\cdot\mathbf{W}d^3x, \quad (27)$$

$$H_1=-\mu\int d^3x\mathbf{j}\cdot\boldsymbol{\phi}. \quad (28)$$

D. QUANTIZATION

The quantization procedure for the field with charges can be carried through in exactly the same way as for the radiation field. For the charge and current we shall assume here that they are described by the charge and current operators of the Dirac theory for particles of spin $\frac{1}{2}$. Furthermore, we shall introduce the Hamiltonian for the particles alone. In order to simplify the procedure and also to avoid the well-known divergence difficulties of the hole theory, we describe the particle system in configuration space.

For the charge and current density operator we have in the Dirac theory

$$\rho(\mathbf{x})=\sum_n e_n\delta(\mathbf{x}-\mathbf{x}_n),\quad j(\mathbf{x})=\sum_n e_n\boldsymbol{\alpha}_n\delta(\mathbf{x}-\mathbf{x}_n). \quad (29)$$

Here the index n is the particle number with the charge e_n. $\boldsymbol{\alpha}_n$ is the Dirac matrix vector and $\mathbf{x}_n$ is the position vector of the nth particle. With these expressions we obtain for the interaction Hamiltonian (28)

$$H_1=-\mu\sum_n e_n\boldsymbol{\alpha}_n\cdot\boldsymbol{\phi}(\mathbf{x}_n). \quad (30)$$

For the particle Hamiltonians we have the well-known expressions

$$H^P=\sum_n(\boldsymbol{\alpha}_n\cdot\mathbf{p}_n+m_n\beta_n) \quad (31)$$

and for the total Hamiltonian of the system we write therefore

$$H=H_0+H_1+H^P. \quad (32)$$

The Schrödinger functional depends now besides the field variables q also on the particle variables which we symbolically denote with $q_1, q_2, \cdots, q_k, \cdots$. Thus the Schrödinger equation may be written

$$H\Omega(q;q_1,q_2,\cdots t)=i\dot{\Omega}(q;q_1,q_2,\cdots t) \quad (33)$$

with H defined by Eqs. (32), (31), (30), and (27).

E. ELIMINATION OF THE LONGITUDINAL FIELD

The elimination of the longitudinal components of the field in ordinary quantum electrodynamics leads to the Coulomb interaction potential between the charged particles. We shall now carry out the analogous calculation for the field in a refractive medium. To that end we go into momentum space with the transformation formula

$$\left.\begin{aligned}\mathbf{\Phi}(\mathbf{x})&=(2\pi)^{-\frac{3}{2}}\int d^3k\mathbf{Q}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}\\ \mathbf{\Pi}(\mathbf{x})&=(2\pi)^{-\frac{3}{2}}\int d^3k\mathbf{P}(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\end{aligned}\right\}. \quad (34)$$

In order to distinguish the longitudinal and transverse components we introduce again the special coordinate system $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ for each $\mathbf{k}$ defined in Eq. (61) in Part I.

$$\left.\begin{aligned}\mathbf{P}&=P^{(1)}\mathbf{e}_1+P^{(2)}\mathbf{e}_2+P^{(3)}\mathbf{e}_3\\ \mathbf{Q}&=Q^{(1)}\mathbf{e}_1+Q^{(2)}\mathbf{e}_2+Q^{(3)}\mathbf{e}^3\end{aligned}\right\}. \quad (35)$$

The Hamiltonian for the field may then be written as

$$H_0=H^{\mathrm{long}}+H^{\mathrm{tr}} \quad (36)$$

with

$$H^{\mathrm{tr}}=\tfrac{1}{2}\int d^3k\left[\frac{1}{1+\kappa v_0^2}\left\{P^{(1)+}P^{(1)}+P^{(2)+}P^{(2)}\left(1+\frac{\kappa v^2\sin^2\alpha}{1+\kappa}\right)\right\}+k^2\left(1-\frac{\kappa^2v^2\cos^2\alpha}{1+\kappa v_0^2}\right)Q^{(1)+}Q^{(1)}+k^2\left(1-\frac{\kappa v^2}{1+\kappa v_0^2}\right)Q^{(2)+}Q^{(2)}-2\kappa i v^0(\mathbf{v}\cdot\mathbf{k})(Q^{(1)}P^{(1)}+Q^{(2)}P^{(2)})\right], \quad (37)$$

$$H^{\text{long}}=\tfrac{1}{2}\int d^3k\frac{1}{1+\kappa v_0^2}\left\{P^{(3)+}P^{(3)}\left(1+\frac{\kappa v^2\cos^2\alpha}{1+\kappa}\right)-2\kappa i v^0 kv|\sin\alpha|Q^{(2)}P^{(2)}-\frac{\kappa v^2}{1+\kappa}(P^{(2)+}P^{(3)}+P^{(3)+}P^{(2)})\times|\sin\alpha|\cos\alpha\right\}. \quad (38)$$

This expression may be transformed with the help of Eq. (25) which may be written (since $\chi\Omega=0$)

$$(\nabla\cdot\mathbf{\Pi}+\mu\rho)\Omega=0. \quad (39)$$

If we develop $\rho(\mathbf{x})$ into a Fourier series according to the formula,

$$\rho(\mathbf{x})=(2\pi)^{-\frac{3}{2}}\int d^3k\rho(\mathbf{k})e^{-i\mathbf{x}\cdot\mathbf{k}}, \quad (40)$$

Eq. (39) may be written

$$\{-iP^{(3)}(\mathbf{k})k+\mu\rho(\mathbf{k})\}\Omega=0, \quad (41)$$

which must hold for all values of $\mathbf{k}$. This means that for a Maxwell field we get an equivalent H^{long} if we replace everywhere $P^{(3)}(\mathbf{k})$ according to the equation

$$P^{(3)}(\mathbf{k})=i(\mu/k)\rho(\mathbf{k}). \quad (42)$$

In this way we obtain for H^{long} an expression which we conveniently separate into two parts as follows:

$$H^{\text{long}}=H_c+H_2, \quad (43)$$

$$H_c=\frac{1}{2}\frac{1}{1+\kappa v_0^2}\int d^3k\left(1+\frac{\kappa v^2\cos^2\alpha}{\kappa+1}\right)\times\frac{\mu^2\rho^+(\mathbf{k})\rho(\mathbf{k})}{k^2}, \quad (44)$$

$$H_2=\frac{v\kappa}{1+\kappa v_0^2}\int d^3k\frac{|\sin\alpha|}{k}\mu\rho(\mathbf{k})\times\left\{\frac{i(\mathbf{v}\cdot\mathbf{k})}{k(\kappa+1)}P^{(2)+}(\mathbf{k})-kv^0Q^{(2)}(\mathbf{k})\right\}. \quad (45)$$

The expression H_c is the generalization of the Coulomb energy for moving media. For the density corresponding to point charges we have Eq. (29) and therefore for $\rho(\mathbf{k})$ we obtain from (40)

$$\rho(\mathbf{k})=(2\pi)^{-\frac{3}{2}}\textstyle\sum_n e_n e^{i\mathbf{k}\cdot\mathbf{x}_n}. \quad (46)$$

Inserting this in (44) we obtain

$$H_c=\frac{1}{2(2\pi)^3}\frac{\mu^2}{1+\kappa v_0^2}\sum_{n,n'}e_ne_{n'}\int\frac{d^3k}{k^2}\times\left(1+\frac{\kappa}{\kappa+1}\frac{(\mathbf{v}\cdot\mathbf{k})^2}{k^2}\right)\exp[i\mathbf{k}\cdot(\mathbf{x}_n-\mathbf{x}_{n'})]. \quad (47)$$

These integrals can be evaluated if we drop the divergent self-energy terms and we find thus for the Coulomb energy

$$H_c=\frac{1}{4\pi}\sum_{n>n}\frac{e_ne_{n'}}{|\mathbf{x}_n-\mathbf{x}_{n'}|}\left\{1+\frac{1}{2}\frac{\kappa}{1+\kappa}\times\frac{(\mathbf{v}\times(\mathbf{x}_n-\mathbf{x}_{n'}))^2}{|\mathbf{x}_n-\mathbf{x}_{n'}|^2}\right\}\frac{\mu^2}{1+\kappa v_0^2}. \quad (48)$$

The second term in (43) given by expression (45) is a non-static interaction term. We write it by introducing the emission and absorption operators (Eq. (70), Part I)

$$H_2=\frac{v\kappa\mu}{\sqrt{2}(1+\kappa v_0^2)}\int d^3k|\sin\alpha|\left[\frac{1}{\beta}\frac{\mathbf{v}\cdot\mathbf{k}}{k(\kappa+1)}-\beta v^0k\right]\times[a_2(\mathbf{k})\rho(\mathbf{k})+a_2^+(\mathbf{k})\rho^+(\mathbf{k})], \quad (49)$$

where β is given by Eq. (73), Part I.

This result differs in two respects from the corresponding result in the vacuum theory. Equation (48) shows that the Coulomb law for two particles at rest in a moving medium is modified by the presence of an angular dependent term. The second difference is the appearance of an additional interaction term H_2 which depends on the medium velocity and goes to zero with the latter. The physical significance of this term will be discussed in connection with the problem of the Čerenkov radiation.

The elimination of the longitudinal field can now be completed by introducing the Schrödinger functional

$$\Omega^{\text{tr}}=S\Omega, \quad (50)$$

with

$$S=\exp\left[-\int\frac{\mu}{k}Q^{(3)}(\mathbf{k})\rho(\mathbf{k})d^3k\right]. \quad (51)$$

The transformation operator S has the property that it commutes with all the variables which occur in H^{tr}, H^{c}, H^{P}, H_1, and H_2 except with the particle momentum $\mathbf{p}_n$ which occurs in H^P, the Hamiltonian for the particles. One verifies easily that this momentum operator satisfies the commutation rule

$$[\mathbf{p}_n, S] = -e_n\mu\boldsymbol{\phi}^{\mathrm{long}}(\mathbf{x}_n)S, \tag{52}$$

where $\boldsymbol{\phi}^{\mathrm{long}}$ is defined as

$$\boldsymbol{\phi}^{\mathrm{long}}(\mathbf{x}) = (2\pi)^{-\frac{3}{2}}\int \mathbf{e}_3 Q^{(3)}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}d^3k. \tag{53}$$

Writing further

$$\boldsymbol{\phi}^{\mathrm{tr}} = \boldsymbol{\phi} - \boldsymbol{\phi}^{\mathrm{long}}.$$

We obtain finally for the Schrödinger equation for Ω^{tr}

$$H\Omega^{tr} = i\dot{\Omega}^{tr}, \tag{54}$$

$$H = H^{tr} + H^{c} + H^{P} + H_1 + H_2. \tag{55}$$

With the exception of H_1 the various terms are defined in Eqs. (37), (44), (31), and (49).

H_1 is now given by

$$H_1 = -\mu\sum e_n\boldsymbol{\alpha}_n\cdot\boldsymbol{\phi}^{\mathrm{tr}}(\mathbf{x}_n). \tag{56}$$

F. THE CLASSICAL FIELD OF A POINT-CHARGE AT REST

In the following sections we shall discuss a few applications of the theory so far developed. As a first example we shall study the classical field around a particle at rest in a moving medium. Since the classical equations hold also for the expectation values of the field quantities, we may consider the Eqs. (9)–(12) as the differential equations for the problem.

Let e be the charge of a particle at rest located at the origin. The density function is then $j^0 = e\delta(\mathbf{x})$. The currents j^k vanish ($k = 1, 2, 3$). We want stationary solutions of the differential equations (1) which vanish at infinity. They are of the form $\Psi^k = 0$, $\Psi^0 \neq 0$. Ψ^0 is the solution of

$$\{-\nabla^2 + \kappa(\mathbf{v}\cdot\nabla)^2\}\Psi^0 = e\mu\delta(\mathbf{x}). \tag{57}$$

We write the solution as a Fourier integral,

$$\Psi^0 = (2\pi)^{-3}\int F(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}d^3k. \tag{58}$$

From (57) follows then for $F(\mathbf{k})$

$$\{k^2 - \kappa(\mathbf{v}\cdot\mathbf{k})^2\}F(\mathbf{k}) = e\mu$$

or

$$\Psi^0 = (2\pi)^{-3}\int \frac{e\mu}{k^2 - \kappa(\mathbf{v}\cdot\mathbf{k})^2}e^{i\mathbf{k}\cdot\mathbf{x}}d^3k. \tag{59}$$

In order to evaluate the integral we assume first

$$\kappa v^2 < 1. \tag{60}$$

By a convenient choice of the coordinate system we may assume the direction of velocity to be the 3-direction. Introducing further the new variables

$$\begin{aligned} k_1' &= k_1, \quad k_2' = k_2, \quad k_3' = k_3(1-\kappa v^2)^{\frac{1}{2}},\\ x_1' &= x_1, \quad x_2' = x_2, \quad x_3' = x_3/(1-\kappa v^2)^{\frac{1}{2}}, \end{aligned} \tag{61}$$

we obtain for (59)

$$\Psi^0 = (2\pi)^{-3}[e\mu/(1-\kappa v^2)^{\frac{1}{2}}]\int e^{i\mathbf{k}'\cdot\mathbf{x}'}/k'^2 d^3k'$$

or

$$\Psi^0 = (1/4\pi)[e\mu/(1-\kappa v^2)^{\frac{1}{2}}](1/|x'|). \tag{62}$$

For a general coordinate system this may be written

$$\Psi^0 = (e\mu/4\pi)\{x^2(1-\kappa v^2) + \kappa(\mathbf{x}\cdot\mathbf{v})^2\}^{-\frac{1}{2}}. \tag{63}$$

On the other hand, if $\kappa v^2 > 1$ the integrand of (59) has two poles for real values of $\mathbf{k}$. We shall again choose the notation (61) and carry out first the integration over k_3'. In order to make k_3' and x_3' real they are now defined as

$$k_3' = k_3(\kappa v^2 - 1)^{\frac{1}{2}}, \quad x_3' = x_3/(\kappa v^2 - 1)^{\frac{1}{2}}$$

and the integration over k_3' gives

$$I = \int dk_3' \frac{e^{ik_3'x_3'}}{k_1'^2 + k_2'^2 - k_3'^2}. \tag{64}$$

There are two different solutions possible which are real according to whether we displace the path in the upper or lower half-plane around the two poles $k_3' = \pm(k_1'^2 + k_2'^2)^{\frac{1}{2}}$. In the rest system of the medium these two solutions correspond to the advanced and retarded potentials, respectively. The principle value, on the other hand, gives a linear combination of these two solutions. Just as in the case of the advanced potentials, we must exclude the first solution on physical grounds. We obtain in this way for I

with the residue theorem

$$I=\begin{cases}0 & \text{for } x_3'<0\\ +2\pi(\sin kx_3'/k) & \text{for } x_3'>0\end{cases} \tag{65}$$

with $k=(k_1^2+k_2^2)^{\frac{1}{2}}$, and, consequently,

$$\Psi^0=(2\pi)^{-2}\frac{e\mu}{(\kappa v^2-1)^{\frac{1}{2}}}\int d^3k\frac{e^{i\mathbf{k}\cdot\mathbf{x}}\sin kx_3'}{k}. \tag{66}$$

Integrating over the angles first this gives

$$\Psi^0=(e\mu/2\pi)[1/(\kappa v^2-1)^{\frac{1}{2}}]\times\int_0^\infty J_0(kr)\sin kx_3'dk, \tag{67}$$

$$(r=(x_1^2+x_2^2)^{\frac{1}{2}}).$$

The last integral can be evaluated with the help of the formula[2]

$$\int_0^\infty J_0(at)\sin bt dt=\begin{cases}0 & b<a\\ 1/(b^2-a^2)^{\frac{1}{2}} & b>a\end{cases}$$

Hence after transforming the result back to the general coordinate system,

$$\Psi^0=\begin{cases}0 & \text{for } \mathbf{x}\cdot\mathbf{v}<vr(\kappa v^2-1)^{\frac{1}{2}}\\ (e\mu/2\pi)\{|x|^2(1-\kappa v^2)+\kappa(\mathbf{v}\cdot\mathbf{x})^2\}^{-\frac{1}{2}} & \\ & \text{for } \mathbf{x}\cdot\mathbf{v}>vr(\kappa v^2-1)^{\frac{1}{2}}\end{cases}. \tag{68}$$

We see from this, the solution Ψ^0 vanishes everywhere except inside the cone given by

$$\mathbf{x}\cdot\mathbf{v}=rv(\kappa v^2-1)^{\frac{1}{2}}. \tag{69}$$

On the cone itself the solution (68) becomes infinite.* From (68) and with the help of Eqs. (12) and (9) it is easy to obtain the field strengths. Since

$$\begin{aligned}\Phi^\lambda&=\Psi^\lambda+[\kappa/(1+\kappa)]\Psi^\sigma v_\sigma v^\lambda,\\ \Phi^i&=[\kappa/(1+\kappa)]v_0\Psi^0v^i,\\ \Phi^0&=\Psi^0[1+(\kappa/(1+\kappa))v^0v_0],\end{aligned} \tag{70}$$

the field strengths are given by

$$\begin{aligned}F_{ij}&=[\kappa/(1+\kappa)]v_0(\partial_iv_j-\partial_jv_i)\Psi^0,\\ F_{i0}&=\partial_i\Psi_0[1+(\kappa/(1+\kappa))v^0v_0].\end{aligned} \tag{71}$$

[2] See G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, 1922).

* We notice that the expression (68), insofar as it is different from zero, is twice as large as (63). This means that the field simply "folds over" to the inside of the cone (69) for velocities v larger than $\kappa^{-\frac{1}{2}}$.

G. THE SELF-FORCE ON A PARTICLE AT REST IN A MOVING MEDIUM

We consider now, still in the classical case, the net force which the field exerts on a particle at rest when the medium is moving. In vacuum electrodynamics this force is, of course, always equal to zero since the field is spherically symmetrical around a point particle. However, this is, no longer the case when the particle is at rest in a moving diffracting medium.

According to Lorentz the four-vector density of the force is given by**

$$f_\mu=\mu F_{\mu\nu}j^\nu. \tag{72}$$

For a particle at rest this reduces to

$$f_i=\mu F_{i0}\rho;\quad f_0=0. \tag{73}$$

The total force on the charge is then

$$F_i=\mu\int F_{i0}\rho(\mathbf{x})d^3x. \tag{74}$$

In order to evaluate this expression we assume a charge distribution $\rho(\mathbf{x})$ with the Fourier expansion

$$\rho(\mathbf{x})=(2\pi)^{-\frac{3}{2}}\int d^3k\rho(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{75}$$

The field strengths are then by the obvious generalization of Eqs. (59), (71) for a continuous charge distribution

$$F_{i0}(\mathbf{x})=-\left(1-\frac{\kappa}{1+\kappa}v_0^2\right)(2\pi)^{-3}i\int d^3k\times\int d^3y\frac{k_i\mu^2}{k^2-\kappa(\mathbf{v}\cdot\mathbf{k})}e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\rho(\mathbf{y}),$$

and hence for the force (74),

$$F_i=\left(\frac{\kappa}{1+\kappa}v_0^2-1\right)i\int d^3k\frac{k_i\mu^2}{k^2-\kappa(\mathbf{v}\cdot\mathbf{k})^2}\times\rho(\mathbf{k})\rho(-\mathbf{k}). \tag{76}$$

Since the integrand in this expression is odd, we see that $F_i=0$ if the denominator has no singularity, that is, if $\kappa v^2<1$. In the other case

** The explicit appearance of the factor μ in Eq. (72) comes from our definition of energy and momentum of the field.

($\kappa v^2>1$), we must evaluate it again with the path of integration chosen according to the solution (68). In this way we obtain again with the residue theorem the expression:

$$F_i=\frac{2\pi^2}{1+\kappa}\frac{\kappa v^2-1}{\kappa v^3}v_iI, \tag{77}$$

with

$$I=\int_0^\infty kdk\mu^2\rho(\mathbf{k})\rho(-\mathbf{k}) \tag{78}$$

for spherically symmetrical charge distribution.

For a point charge $\rho(k)=e(2\pi)^{-\frac{3}{2}}$, independent of k and the integral (78) is divergent. This means in a medium with constant ϵ and μ a point particle can never be accelerated to a velocity such that $\kappa v^2>1$. The situation is different, however, in a dispersive medium. In that case the factors in front of I will be frequency-dependent and will therefore go under the integral sign. The integral is then always convergent provided $\kappa\rightarrow 0$ for $k\rightarrow\infty$.

H. THE ČERENKOV RADIATION

With Čerenkov radiation one denotes the radiation emitted by a charged high speed particle traversing a refractive medium with a speed so high that $\kappa v^2>1$. This radiation is essentially different from and must not be confused with the bremsstrahlung. It was discovered experimentally by Čerenkov in 1934 and subsequently studied by other workers.[3] The classical theory of the radiation was given by Frank and Tamm.[4] We shall here develop the quantum-mechanical theory of the radiation by using the two representations: medium at rest, particle moving and medium moving, particle at rest.[4a]

In the first case the interaction Hamiltonian is given entirely by the term H_1 in Eq. (55). We consider an electron of rest mass m and momentum $\mathbf{p}$, such that $p^2\kappa>m$. This electron can carry out a transition to a state with momentum $\mathbf{p}'$ under emission of a photon of momentum and polarization $\mathbf{k}$, r and energy k/n. The probability per unit time for such a process is given by quantum-mechanical perturbation theory as

$$dP=2\pi d\rho_F\langle|H_1|^2\rangle_{\text{Av}}. \tag{79}$$

Here $\langle|H_1|^2\rangle_{\text{Av}}$ is the square of the matrix element of H_1 for the transition in question averaged over all the spin states of the initial and final states of the electron. $d\rho_F$ represents the density of the final states, per unit energy range for photons emitted into the solid angle $d\Omega$. By denoting the spinor amplitudes of the plane waves of the electron with $u(\mathbf{p})$ and $u(\mathbf{p}')$ for initial and final states, respectively, we obtain from Eq. (56) the following expression for the matrix element

$$\langle|H_1|^2\rangle_{\text{Av}}=[e^2\mu^2/16\pi^3k(1+\kappa)^{\frac{1}{2}}] \times\tfrac{1}{2}\sum|u^*(\mathbf{p})(\boldsymbol{\alpha}\cdot\mathbf{e}_r)u(\mathbf{p}')|^2. \tag{80}$$

The summation in this expression extends over all the spin states of the initial and final state of the electron with positive energy. This sum may be evaluated in the following way. We introduce the annihilation operators

$$\begin{aligned}\Lambda(\mathbf{p})&=[(\boldsymbol{\alpha}\cdot\mathbf{p})+\beta m+E_p]/2E_p,\\ \Lambda(\mathbf{p}')&=[\boldsymbol{\alpha}\cdot(\mathbf{k}-\mathbf{p}')+\beta m+E_{p'}]/2E_{p'},\end{aligned} \tag{81}$$

with

$$E_p=+(p^2+m^2)^{\frac{1}{2}},\quad E_{p'}=+(p'^2+m^2)^{\frac{1}{2}}.$$

We obtain then for this sum

$$\begin{aligned}S&=\tfrac{1}{2}\sum|u^*(\mathbf{p})(\boldsymbol{\alpha}\cdot\mathbf{e}_r)u(\mathbf{p}')|^2\\ &=\tfrac{1}{8}tr[(\boldsymbol{\alpha}\cdot\mathbf{e}_r)\Lambda_{p'}(\boldsymbol{\alpha}\cdot\mathbf{e}_r)\Lambda_p]\\ &=\frac{1}{2E_p'E_p}\{E_pE_{p'}-(\mathbf{p}\cdot\mathbf{p}')+2(\mathbf{p}\cdot\mathbf{e}_r)^2-m^2\}.\end{aligned} \tag{82}$$

In the applications the only case of importance is that for which the photon quanta have momentum very much smaller than the momentum of the particle $k\ll p$. For this case (82) may be developed in powers of k/p In this way we obtain for the sum (82)

$$S=(1/E_p^2)(\mathbf{p}\cdot\mathbf{e}_r)^2. \tag{83}$$

From this expression it may be seen that the Čerenkov radiation is polarized with the electric

[3] P. A. Čerenkov, C. R. Acad. Sci. U.R.S.S. **2**, 451 (1934); **3**, 413 (1936); **14**, 99, 103 (1937); **20**, 651 (1938); **21**, 116, 319 (1938); Phys. Rev. **52**, 378 (1937). G. B. Collins and V. G. Reiling, Phys. Rev. **54**, 499 (1938). H. O. Wyckoff and J. E. Henderson, Phys. Rev. **64**, 1 (1943).

[4] I. M. Frank and Ig. Tamm, C. R. Acad. Sci. U.R.S.S. **14**, 107 (1937). Ig. Tamm, J. Phys. U.R.S.S. **1**, 439 (1938).

[4a] A quantum mechanical treatment of the Čerenkov radiation has also been given by V. L. Ginsburg, J. Phys. U.R.S.S. **2**, 441 (1940). However, the relativistic invariance and the elimination of the longitudinal field are not discussed in this paper.

vector in the plane of the vectors $\mathbf{p}$ and $\mathbf{k}$. The total photon-emission probability is then proportional to

$$S=(1/E_p^2)\{p^2-[(\mathbf{p}\cdot\mathbf{k})^2/k^2]\}. \quad (84)$$

For the density of final states we have

$$d\rho_F=k^2(dk/dE_F)d\Omega. \quad (85)$$

The conservation of energy gives for the final energy the expression

$$E_F=(k/n)+[(\mathbf{p}-\mathbf{k})^2+m^2]^{\frac{1}{2}}. \quad (86)$$

From this we get

$$dE_F/dk=(1/n) +(k-p\cos\theta)[(\mathbf{p}-\mathbf{k})^2+m^2]^{-\frac{1}{2}}, \quad (87)$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{k}$. This angle is entirely determined by the conservation of momentum and energy. For we obtain for the parallel and perpendicular components of momentum the equations

$$p=p_{\parallel}+k\cos\theta,\quad 0=p_{\perp}+k\sin\theta, \quad (88)$$

and for the energy

$$E_p=E_{p'}+(1/n)k. \quad (89)$$

The sum of the squares of the first two equations gives

$$p'^2=p^2+k^2-2pk\cos\theta \quad (90)$$

and the square of the last

$$p'^2=p^2-(2/n)kE_p+(1/n^2)k^2. \quad (91)$$

Equating the last two expressions gives

$$\cos\theta=[E_p/pn]+[\kappa/(\kappa+1)][k/2p], \quad (92)$$

which is the desired relationship between θ and k. We mention here that in the classical theory[4] only the first term occurs on the right-hand side of Eq. (92). That the quantum theoretical treatment leads to the additional term on the right was pointed out before by Cox.[5] Inserting (84), (85), (87), and (92) in (79) gives us the desired cross section for the photon emission per unit time into the solid angle $d\Omega$.

We shall carry out the discussion of the result in the approximation $k\ll p$ which was already used to get the result (84). In this case we obtain $[u=v/(1+v^2)^{\frac{1}{2}}]$:

$$dE_F/dk\sim[k/2E_p][\kappa/(\kappa+1)], \quad (93)$$

$$d\rho_F\sim 2kE_p[(\kappa+1)/\kappa]d\Omega, \quad (94)$$

$$S\sim u^2[1-(1/n^2)u^2], \quad (95)$$

and therefore from (79)

$$dP=E_p[(\kappa+1)/\kappa][e^2\mu^2/4\pi^2(1+\kappa)^{\frac{1}{2}}] \times u^2(1-(1/u^2n^2)). \quad (96)$$

The emission probability as a function of the angle is then obtained by expressing k as a function of θ by Eq. (92).

In order to obtain the total energy radiated per unit time dw/dt we write

$$d\Omega=\sin\theta d\theta d\varphi=-d\varphi[\kappa/(\kappa+1)](dk/2p), \quad (97)$$

$$dw/dt=\int dP\cdot(k/n) =(e^2/4\pi)\int(kdk/n^2)\mu^2[1-(1/n^2)u^2]. \quad (98)$$

It is seen that this expression diverges if no dispersion is assumed. It can be made finite with any kind of dispersion law which assumes only $n\rightarrow 1$ for $k\rightarrow\infty$. Incidentally, the expression (98) is identical with the classical expression given by Frank and Tamm.[4]

We shall finish the discussion of the Čerenkov radiation by considering the same problem in the coordinate system for which the particle is at rest. In this case H_1 does not contribute anything for $k\ll m$ since the matrix element s of α are proportional to k/m. At first hand it seems, therefore, that no radiation will be emitted. However, the part H_2 given in Eq. (49) will now contribute the necessary interaction terms. We recall that H_2 was introduced into the Hamiltonian by the process of eliminating the longitudinal part of the field. Since in ordinary quantum electrodynamics this term does not appear, its physical significance was not immediately obvious. We see now better what its physical implications are.

The probability for the emission of a photon of momentum $\mathbf{k}$ and a corresponding change of the electron momentum from 0 to $-\mathbf{k}$ is then given by

$$dP'=2\pi\rho_F\langle|H_2|^2\rangle_{\text{Av}}. \quad (99)$$

[5] R. T. Cox, Phys. Rev. 66, 106 (1944).

A straightforward calculation along the lines of the previous case leads to the following result for the probability of photon emission in a direction **k** into the solid angle $d\Omega$.

$$dP' \simeq [e^2\mu^2/2][1/(2\pi)^2] \times [(\kappa v^2 - 1)/\kappa][(\kappa)^{\frac{1}{2}}/(1+\kappa)]d\Omega. \quad (100)$$

The conservation of momentum and energy give for the angle α of the direction of the emitted photon with the velocity v

$$\cos\alpha \simeq -[v^0/2mv]k - [1/(\kappa v^2)^{\frac{1}{2}}].$$

Thus for $d\Omega$ we may write

$$d\Omega = \sin\alpha d\alpha d\varphi = (v^0/2vm)dkd\varphi.$$

If we calculate the total momentum transferred to the electron per unit time, we find the expression

$$F = -\int k \cos\alpha dP'$$

or for the ith component,

$$F_i = [e^2\mu^2/4\pi][1/(1+\kappa)] \times [(\kappa v^2 - 1)/\kappa v^3]v_i \int kdk. \quad (101)$$

This expression is identical with the classical expression obtained in Eq. (77) for the static self-force of an electron at rest in a moving medium.

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The Application of the Bethe-Peierls Method to Ferromagnetism*

P. R. WEISS
Physics Department, Rutgers University, New Brunswick, New Jersey
(Received June 11, 1948)

The method introduced by Bethe and Peierls for treating the problem of order-disorder in alloys is applied to the problem of ferromagnetism. The method is first applied to the Ising model of the spin in which case the treatment is much the same as it is for the alloy problem, as has already been pointed out by Peierls. The correct treatment of the spin is used for spin values of $\frac{1}{2}$ and 1 per atom. The critical temperatures of different types of lattices are investigated. The method gives results in agreement with the rigorous results of the Bloch spin-wave theory in that only three-dimensional lattices are found to be ferromagnetic. The values of the critical temperatures of these lattices are found to lie between the values predicted by the two Heisenberg approximations. The discontinuity of the specific heat at the critical temperature is computed for a body-centered lattice and for the two values of the spin. The magnitude of the discontinuity is larger than that predicted by Heisenberg's first approximation. The magnitude for the spin 1 is 3.4 k per atom and compares favorably with the experimental value for iron. The susceptibility is computed as a function of the temperature above the critical point. The variation of the susceptibility with temperature does not obey the Curie-Weiss Law but displays some curvature. This curvature explains qualitatively the difference between the "paramagnetic" and ferromagnetic critical temperatures and also helps remove some of the discrepancy between the number of Bohr magnetons per atom as measured at high and low temperatures.

I. INTRODUCTION

THE calculations described below are based on the physical model of ferromagnetism first introduced by Heisenberg.[1] This model can be described briefly as follows. Each atom in a domain has a spin **S** which is the resultant of the spins of individual electrons (or holes) residing in an incomplete inner shell. The orbital moment is quenched so that the magnetization arises entirely from the spins and is, in the first approximation, isotropic. The exchange interaction integral is significant only when it refers to electrons in neighboring atoms and is the same for all such pairs of atoms of the domain. The exchange integral, J, is positive. The incomplete inner shell referred to is, in iron, nickel, and cobalt, the 3-d shell and, in gadolinium, the 4-f shell. It is also assumed in this model that all

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* Some of the developments reported here are contained in a thesis submitted in 1940 in partial fulfillment of the requirements for the degree of Doctor of Philosophy at Harvard University. These developments are also contained in a review article, J. H. Van Vleck, Rev. Mod. Phys. **17**, 27 (1945).

[1] W. Heisenberg, Zeits. f. Physik **49**, 619 (1928).