conversion. Such a calculation for gold, and an electron energy of 2.35 Mev, leads to a value of 7.4 percent for the fraction of the incident power that is converted into radiation. The theory under these conditions predicts a value of 8.3 percent. The agreement between the two figures is quite good, considering the uncertainties involved in the calculations from the experimental measurements and the approximations made in applying a theory developed for thin targets to those in which the electrons are completely stopped. A similar agreement between experiment and theory has been found by other observers in the case of copper,[3] lead,[5] and gold.[2]

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[5] Ivanov, Walter, Sinelnikov, Taranov, and Abramovich, J. Phys. U.S.S.R. 4, 319 (1941).

Further Calculations on the Cascade Theory

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The paper summarizes the previous work, and briefly gives the steps by which the mathematical solution of the cascade equations given in our previous papers can be established rigorously.

This solution has been used to calculate the number of shower particles for thicknesses between 0.25 characteristic unit and 20 characteristic units, and for primary energies from 2.7 times the critical energy ($y_0 = 1$) to 2.7×10^{10} times the critical energy ($y_0 = 24$). For this purpose, the second term of our series solution has also been calculated. The results are given in Table III and Fig. 3 in a form suitable for comparison with, and analysis of, experiment. These show that the second and higher terms of our series are negligible compared with the first for thicknesses less than three to four times that at which the shower reaches its maximum.

A method has been developed which allows the integrals to be evaluated at very small thicknesses where transition effects are still of importance, so that it is now possible to trace a shower from its very beginning to large depths. It is shown that for very small showers started by particles of two to three times the critical energy the shower must penetrate to depths which are three times the maximum depth to which a single particle could penetrate as a result of collision loss alone. This is possible because part of the path of the shower is covered by photons alone which then materialize at a subsequent depth.

A simple formula has been given (37) which enables one to calculate with considerable accuracy the spectrum of shower electrons of energy much below the critical energy. It is proved that this spectrum increases monotonically with decreasing energy at all thicknesses. Its form is approximately that of a modified inverse square law at the maximum of the shower, the power of the law becoming higher with increasing depth.

A KNOWLEDGE of the generation of cascades by electrons and gamma-rays is of great importance in the interpretation of cosmic-ray phenomena. Since cascades are produced so readily by the electronic component, they have often served to mask the effects of the other components of cosmic rays. For example, an accurate knowledge of the number of electrons to be expected in a cascade produced by an electron of known energy in a plate of given thickness, of the fluctuations in this number, and of the energy spectrum of the electrons in the cascade, would enable one to decide by experiment whether the cascades in cosmic radiation conform entirely to the theoretical pattern, or if not, to unravel other effects which may be mixed up with them, such as, for example, the creation of mesons, or of particles having masses a few times that of the electron, as has been suggested by Auger and his co-workers.[1] Knowledge of the

[1] Auger, Daudin, Fréon, and Maze, Comptes Rendus 226, 169, 569 (1948).

spread of electrons in a cascade and of the fluctuations in the lateral density distribution of electrons would enable one to decide by an analysis of experiment whether intervening links other than electrons and gamma-rays play any part in the generation of large air showers. Perhaps the accurate numerical answers to some of these problems, particularly those connected with the lateral spread of showers, will be given only with the help of the new types of calculating machines which are now being developed in different parts of the world. Nevertheless, a good deal can be done by analytical methods, and in some problems answers can be given which are at least as accurate as the experimental data available at present. For this reason we have though it desirable to push the analytical solutions as far as possible, and we believe that the results given in this paper are more accurate than any that have been given till now.

There is one approximation which underlies all the treatments that have been given hitherto, namely, neglect of the lateral spread of a shower and its treatment as a one-dimensional phenomenon. Only very qualitative estimates of the lateral spread of showers have been made in some cases. With this universal underlying simplification, the only complete and rigorous solution of the cascade problem has been given by K. S. K. Iyengar.[2] He has treated the collision loss of electrons as a constant independent of the energy, but he has taken the exact cross sections for radiation loss and pair creation as given by Bethe and Heitler for all energies and not only their asymptotic form for high energies. However, it is not easy to derive numerical values from Iyengar's solution, and it has therefore not been of use in interpreting and analyzing experimental data. Taking only the asymptotic forms of the cross sections for radiation and pair creation at very high energies, Snyder,[3] and later Serber,[4] have given solutions which if correct*

would enable one to calculate the total number of particles in a shower with fair accuracy provided the depth of material is not too small. Their solutions, however, do not allow the number of electrons to be calculated for small thicknesses because of the circumstance that their boundary condition corresponds to a spectrum of lower energy electrons incident on the material in addition to the high energy electron whose effect we are investigating. They are also not suitable for calculating the energy spectrum of electrons in a cascade.

With the same basic assumptions we gave a solution of the cascade problem in two previous papers[5, 6] which allowed the spectrum of electrons in a shower to be calculated with considerable accuracy. Although our solution was valid for all thicknesses, numerical data could be extracted from it previously only for thicknesses which were not too small. We have now found a method of overcoming this limitation, so that figures are given in this paper for the growth of a shower in thin layers of material. We can now trace with considerable confidence the course of a shower from its very beginning to thicknesses three to four times that at which it reaches its maximum, and calculate the energy spectrum throughout this range. Secondly, our solution was in the form of an infinite series and it was shown that for most of the energies and thicknesses concerned, the first term alone gave almost the entire contribution. We have now extended these figures to higher energies and greater thicknesses, and, in addition, evaluated the next term of the series. This has thrown further light on the regions in which the first term is insufficient and increased our confidence in the figures which have been given in the rest of the range. For

[2] K. S. K. Iyengar, Proc. Ind. Acad. Sci. A15, 195 (1942).
[3] H. Snyder, Phys. Rev. 53, 960 (1938).
[4] R. Serber, Phys. Rev. 54, 317 (1938).
* The difficulty is essentially the following. The solution depends on finding the value at the point $r = -s$ of a function $k(s, r)$ which satisfies a difference equation (see, for example, Rossi and Greisen, Rev. Mod. Phys. 13, 240 (1941), Eq. (A12)). The values of the function $k(s, r)$ can be given explicitly as functions of s at all positive and negative integral values of r, and from this the value of $k(s, -s)$ is arrived at by graphical interpolation on the

assumption that $k(s, r)$ is a smooth function of r. However, it can be shown from the difference equation that provided $k(s, -s) \neq 0$ then $k(s, -s-1)$ must be infinite although $k(s, r)$ is finite for the integral values of r on each side of the point $r = -s-1$. This alone casts some doubt on the validity of the assumption that $k(s, r)$ is a smooth function of r in a finite range round the point $r = -s$. In any case, this assumption remains to be justified mathematically, and an estimate of the error in arriving at the value of $k(s, -s)$ by the empirical method used needs to be given. Till this is done, one does not know what faith to place in the numerical figures obtained.

[5] H. J. Bhabha and S. K. Chakrabarty, Proc. Roy. Soc. A181, 267 (1943), referred to in this paper as A.
[6] H. J. Bhabha and S. K. Chakrabarty, Proc. Ind. Acad. Sci. A15, 464 (1942).

example, if the total number of particles is considered, the second term of our series only becomes comparable with the first at depths which are three to four times that at which the cascade reaches its maximum.

It is shown that the third and higher terms of our series are of importance only at large thicknesses where the shower is being absorbed, and then too their contribution is mainly to the number of electrons whose energy is much smaller than the critical energy. But this is precisely the range where radiation loss and pair creation are badly represented by their asymptotic forms for high energies. Thus it is a satisfactory feature of our solution that the neglect of the higher terms becomes serious precisely in the conditions in which the basic physical assumptions are no longer sufficiently accurate.

We have given a formula (37) which, with the help of the table we have given, allows one to calculate the low energy spectrum of electrons with considerable accuracy. As far as possible we have given our numerical results in the form of tables and graphs which would make them easily accessible for the experimenter who wishes to compare theory and experiment, or to use theory for an analysis of his experiments.

It must be emphasized that there is a considerable range of the energy spectrum below the critical energy for which the use of the radiation and pair-creation cross sections for complete screening is fully justified, especially in substances of low atomic number like air or water where the critical energy is of the order of a hundred million volts. In this range full confidence can be placed in our figures. Even in this range our figures differ considerably from those given by Arley,[7] which must now be regarded as definitely erroneous.

II. THE MATHEMATICAL SOLUTION

In our original paper we gave only a formal proof of the correctness of our solution, and its convergence was not demonstrated. This lacuna was filled by Iyengar, who established the convergence of our solution rigorously in the paper already quoted. Since Iyengar's paper does not appear to be as well known as it deserves and

some misapprehension exists about the rigorousness of our solution, we think it desirable to take this opportunity to sketch the steps by which it can be rigorously established.

Denoting by $P(E, t)dE$ the mean number of electrons, positive and negative, in the energy range E, $E+dE$ to be found in a cascade at a depth t, and by $Q(E, t)$ the corresponding expression for the number of quanta, the fundamental equations of the cascade theory are[5]

$$\frac{\partial P(E, t)}{\partial t} - \beta\frac{\partial P(E, t)}{\partial E}$$

$$= \int_E^\infty P(E', t)R(E', E'-E)\frac{E'-E}{E'^2}dE'$$

$$-P(E, t)\int_0^E R(E, E')\frac{E'dE'}{E^2}$$

$$+2\int_E^\infty Q(E', t)R(E, E')\frac{dE'}{E'}, \quad (1a)$$

$$\frac{\partial Q(E, t)}{\partial t} = \int_E^\infty P(E', t)R(E', E)\frac{EdE'}{E'^2}$$

$$-Q(E, t)\int_0^E R(E', E)\frac{dE'}{E}. \quad (1b)$$

The thickness of the substance t is measured in radiation units of length, namely,

$$\left[4\frac{Z^2N}{137}\left(\frac{e^2}{mc^2}\right)^2 \log 183Z^{-\frac{1}{3}}\right]^{-1},$$

Z being the atomic number of the substance, N the number of atoms per cubic centimeter, and the other symbols having their usual meaning. The collision loss per radiation unit is denoted by β. The probabilities for radiation loss and pair creation are expressed in terms of the function $R(E, U)$ defined by

$$R(E, U) = \left(1 - \frac{4}{3}\frac{E}{U} + \frac{4}{3}\frac{E^2}{U^2}\right)(x_1+x_2) - x_2, \quad (2)$$

x_1 and x_2 being functions of a variable

$$\varsigma = 100\frac{mc^2}{Z^{\frac{1}{3}}}\frac{U}{E|U-E|}. \quad (3)$$

[7] N. Arley, Proc. Roy. Soc. A168, 519 (1938).

Integrating (1b), remembering that $Q(E, t) = 0$ at $t = 0$ is the required boundary condition, and substituting the resulting expression for Q in (1a), we get

$$\frac{\partial P}{\partial t} - \beta \frac{\partial P}{\partial E} = \left[\int_E^\infty R(U, U-E) \frac{U-E}{U^2} P(U, t) dU \right.$$

$$- P(E, t) \int_0^E R(E, U) \frac{U}{E^2} dE \right]$$

$$+ 2 \int_0^t \int_E^\infty \int_E^U R(E, V) R(U, V)$$

$$\times \exp(D(V)(t'-t)) dV \cdot \frac{1}{U^2} P(U, t') dU dt', \quad (4)$$

where

$$D(E) = \int_0^E \frac{R(U, E)}{E} dU.$$

In the limit of very high energies $\zeta \to 0$ and then

$$x_1 \to 1, \quad x_2 \to \tfrac{3}{4}\alpha = (12 \log 183 Z^{-\frac{1}{3}})^{-1}. \quad (5)$$

(5) is the approximation which underlies most of the treatments. In the limit (5)

$$D = (7/9) - (1/6)\alpha.$$

Let A_S, B_S, and C_S be those functions of the complex variable s defined by

$$A_S = \left(\frac{4}{3} + \alpha \right) \left\{ \frac{d}{ds} \log\Gamma(s) + \gamma - 1 + \frac{1}{s} \right\}$$

$$+ \frac{1}{2} - \frac{1}{s(s+1)}, \quad (6a)$$

$$B_S = 2 \left\{ \frac{1}{s} - \left(\frac{4}{3} + \alpha \right) \frac{1}{(s+1)(s+2)} \right\}, \quad (6b)$$

$$C_S = \frac{1}{s+1} + \left(\frac{4}{3} + \alpha \right) \frac{1}{s(s-1)}. \quad (6c)$$

γ is the Euler-Mascheroni constant. Then the differential equation

$$\left\{ \frac{\partial^2}{\partial t^2} + (A_S + D) \frac{\partial}{\partial t} + (A_S D - B_S C_S) \right\} \psi_0(s, t) = 0 \quad (7)$$

has the solution

$$\psi_0(s, t) = \frac{D - \lambda_S}{\mu_S - \lambda_S} e^{-\lambda_S t} + \frac{\mu_S - D}{\mu_S - \lambda_S} e^{-\mu_S t}, \quad (8)$$

$$\lambda_S = \tfrac{1}{2}(A_S + D) - \tfrac{1}{2}\{(A_S - D)^2 + 4 B_S C_S\}^{\frac{1}{2}}, \quad (9a)$$

$$\mu_S = \tfrac{1}{2}(A_S + D) + \tfrac{1}{2}\{(A_S - D)^2 + 4 B_S C_S\}^{\frac{1}{2}} \quad (9b)$$

for any fixed value of s. The solution (8) satisfies the boundary conditions

$$\psi(s, 0) = 1, \quad \left(\frac{\partial \psi(s, t)}{\partial t} \right)_{t=0} = -A_S \text{ at } t = 0. \quad (10)$$

Let $\psi_1(s, t)$, $\psi_2(s, t)$, $\cdots \psi_n(s, t)$, $\cdots$ be a set of functions defined by the recurrence formula

$$\psi_n(s, t) = \int_0^t \psi_0(s+n, t-t') \psi_{n-1}(s, t') dt'. \quad (11)$$

Then it can be shown without difficulty, as has already been done in the appendix of our previous paper A, that ψ_n and ψ_{n-1} satisfy the equation

$$\left\{ \frac{\partial^2}{\partial t^2} + (A_{S+n} + D) \frac{\partial}{\partial t} \right.$$

$$+ (A_{S+n} D - B_{S+n} C_{S+n}) \left. \right\} \psi_n(s, t)$$

$$= \left(\frac{\partial}{\partial t} + D \right) \psi_{n-1}(s, t). \quad (12)$$

This follows simply by substituting (11) in (12) and using (10). We now prove by induction that (11) is equivalent to a formula evolved by Iyengar, namely,

$$\psi_n(s, t) = \int_0^t \psi_0(s, t_1) \psi_{n-1}(s+1, t-t_1) dt_1. \quad (13)$$

Let (13) hold for all integral n from 1 to $n-1$. Then submitting for ψ_{n-1} by (13) in (11) we get

$$\psi_n(s, t) = \int_0^t \psi_0(s+n, t') \psi_{n-1}(s, t-t') dt'$$

$$= \int_0^t \psi_0(s+n, t') \left\{ \int_0^{t-t'} \psi_0(s, t_1) \right.$$

$$\times \psi_{n-2}(s+1, t-t'-t_1) dt_1 \left. \right\} dt',$$

and on changing the order of the t' and t_1 integrations

$$= \int_0^t \psi_0(s, t_1) \left\{ \int_0^{t-t_1} \psi_0(s+n, t') \right. $$
$$\left. \times \psi_{n-2}(s+1, t-t_1-t')dt' \right\} dt_1$$

$$= \int_0^t \psi_0(s, t_1)\psi_{n-1}(s+1, t-t_1)dt_1$$

by (11).

Now consider the expression

$$P(E, t) = \frac{1}{2\pi i E_0} \int_C \left(\frac{E_0}{E}\right)^s$$
$$\times \left\{ \sum_{n=0}^{\infty} \left(-\frac{\beta}{E}\right)^n \frac{\Gamma(s+n)}{\Gamma(s)} \psi_n(s, t) \right\} ds \quad (14)$$

for $E \leqslant E_0$, where the contour C of integration in the complex plane runs along the straight lines from $-\infty - i\tau$ to $\sigma - i\tau$, then to $\sigma + i\tau$ and then to $-\infty + i\tau$, τ being a large real number and $\sigma > 1$. It has been proved by Iyengar that

$$\psi_n(s, t) < (1+\epsilon)^{n+1} \frac{t^n}{n!} \quad (15)$$

for every value of s on C. Indeed (15) is simply a consequence of (13) and the inequality $|\psi_0(s, t)|$ $< 1 + \epsilon$. It follows that the series in curly brackets in (14) is uniformly and absolutely convergent if $E > \beta t$. Moreover, it follows from the inequalities for $\psi_n(s, t)$ given by Iyengar that the integral on the contour C is convergent. $P(E, t)$ is therefore a well defined function of E and t, and the absolute and uniform convergence of the series in curly brackets allows one to interchange the orders of summation and integration. We introduce (14) into the right-hand side of (4), interchange the orders of the E and s integrations, and carry out the E integration first. Using the approximation for complete screening, namely, (5), and remembering the definitions (6), we find that it is equal to the left-hand side of (4) provided $\psi_n(s, t)$ and $\psi_{n-1}(s, t)$ satisfy Eq. (12) for all integral $n \geqslant 0$. It follows that (14) must be an exact solution of (4) for $E > \beta t$. This restriction does not interfere with the proof since in (4) $P(E, t)$ on the left is connected only

through the integral on the right with value of $P(E', t)$ for which $E' \geqslant E$.

Now introduce the functions

$$g(s, t) = \frac{\psi_1(s, t)}{\psi_0(s, t)}, \quad (16)$$

$$f_n(s, t) = \left\{ \psi_0(s, t)\frac{g^n}{n!} - \psi_1(s, t)\frac{g^{n-1}}{(n-1)!} + \cdots \right\}. \quad (17)$$

Write the integrand of (14) in the form

$$E_0^s \sum_{n=0}^{\infty} (-\beta)^n \frac{\Gamma(s+n)}{\Gamma(s)} \frac{\psi_n(s, t)}{\{(E+\beta g) - \beta g\}^{n+s}}$$

$$= E_0^s \sum_{n=0}^{\infty} (-\beta)^n \frac{\Gamma(s+n)}{\Gamma(s)} \psi_n(s, t) \frac{1}{(E+\beta g)^{n+s}}$$

$$\times \sum_{m=0}^{\infty} \frac{\Gamma(n+s+m)}{\Gamma(m+1)\Gamma(n+s)} \left(\frac{\beta g}{E+\beta g}\right)^m \quad (18)$$

by expanding $\{(E+\beta g) - \beta g\}^{n+s}$ in powers of $\beta g/(E+\beta g)$.

The double series is absolutely convergent because of (15) and its terms can therefore be rearranged. We therefore get

$$P(E, t) = \frac{1}{2\pi i E_0} \int_C \left(\frac{E_0}{E+\beta g}\right)^s$$
$$\times \left\{ \sum_{n=0}^{\infty} \left(\frac{\beta}{E+\beta g}\right)^n \frac{\Gamma(s+n)}{\Gamma(s)} f_n(s, t) \right\} ds. \quad (19)$$

The restriction $E > \beta t$ can be dropped now, since (19) exists for $E \geqslant 0$. It follows from the principle of analytic continuation that (19) is the exact solution of (4) for all $E \geqslant 0$. It can be shown that the contribution to (19) from the parts of the contour C from $-\infty - i\tau$ to $\sigma - i\tau$ and from $\sigma + i\tau$ to $-\infty + i\tau$ tends to zero as $\tau \to \infty$. The contour C can therefore be replaced by a line running from $\sigma - i\infty$ to $\sigma + i\infty$ with $\sigma > 1$.

To get the total number of particles whose energy is greater than E we integrate (19) from E to E_0 and get

$$N(E, t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{1}{s-1} \left(\frac{E_0}{E+\beta g}\right)^{s-1}$$
$$\times \left\{ \sum_{n=0}^{\infty} \left(\frac{\beta}{E+\beta g}\right)^n \frac{\Gamma(s+n)}{\Gamma(s)} f_n(s, t) \right\} ds, \quad (20)$$

TABLE I. Values of $g(s, t)$ as function of s and t.

$t\backslash s$	1.2	1.3	1.4	1.5	1.8	2.0	2.3	2.5	2.8	3.0	3.5	4.0	4.5	5.0	5.5	6.0
0.25		0.2095		0.2205	0.2280	0.2312	0.2338	0.2352	0.2367	0.2376	0.2391					
0.5		0.3195		0.3673	0.4027	0.4168	0.4301	0.4365	0.4441	0.4481	0.4555					
1.0	0.3100	0.3790	0.4392	0.4892	0.5965	0.6309	0.6985	0.7264	0.7565	0.7738	0.8054	0.8317	0.8475	0.8630	0.8712	0.8846
2.0	0.3099	0.3904	0.4622	0.5269	0.6868	0.7705	0.8765	0.9339	1.006	1.048	1.129	1.189	1.239	1.287	1.313	1.347
4.0	0.3099	0.3921	0.4667	0.5353	0.7086	0.8009	0.9048	0.9564	1.010	1.037	1.066	1.064	1.047	1.027	1.000	0.9693
6.0		0.3922		0.5360	0.7149	0.8119	0.9259	0.9735	1.023	1.041	1.047	1.029	0.9978	0.9621		
8.0		0.3922		0.5361	0.7166	0.8164	0.9331	0.9891	1.043	1.063	1.069	1.048	1.020	0.9803		
10.0	0.3099	0.3922	0.4673	0.5364	0.7167	0.8186	0.9387	0.9993	1.059	1.083	1.094	1.067	1.027	0.9876	0.9526	0.9156
15.0			0.4673	0.5364	0.7167	0.8186			1.009	1.112	1.136	1.120	1.080	1.036		
20.0									1.011	1.126	1.164	1.158	1.120	1.074·		
∞	0.3099	0.3922	0.4673	0.5364	0.7167	0.8186	0.9450	1.013	1.095	1.141	1.226	1.292	1.350	1.386		

the expression obtained by putting $E = E_0$, which has to be subtracted from (20), being exactly zero, as can be seen by deforming the contour C into an infinite semicircle to the right of the imaginary axis.

It remains to determine the boundary conditions satisfied by $P(E, t)$ at $t = 0$. For this purpose it is more convenient to consider $N(E, t)$. It follows from (10) that for small t

$$\psi_0(s, t) = 1 - A_s t + \cdots, \tag{21}$$

and from (13) that

$$\psi_n(s, t) = \frac{t^n}{n!} + 0(t^{n+1}). \tag{22}$$

Hence, for small t,

$$g(s, t) = t + 0(t^2) \tag{23}$$

as already proved in our previous paper.[5] Thus,

$$N(E, 0) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{1}{s-1} \left(\frac{E_0}{E}\right)^{s-1} ds = 1$$

for all $E \leqslant E_0$. Since we are then left with only the residue of the integrand at the point $s = 1$. Our solution therefore corresponds to exactly one electron of energy E_0 incident on the surface $t = 0$. It similarly corresponds to no quanta incident on the surface $t = 0$. This is automatically ensured by the substitution we made for $Q(E, t)$ in deriving (4) from (1a) and (1b).

Expression (20) can be written

$$N(E, t) = \sum_{n=0}^{\infty} N_n(E, t), \tag{25a}$$

where

$$N_n(E, t) = \frac{1}{2\pi i} \int \frac{1}{s+n-1} \left(\frac{E_0}{\beta}\right)^{s-1}$$

$$\times \left(\frac{\beta}{E+\beta g}\right)^{s+n-1} \frac{\Gamma(s+n)}{\Gamma(s)} f_n(s, t) ds. \tag{25b}$$

It should be noted that by definition $f_0 = \psi_0$ and $f_1(s, t) = 0$, so that $N_1(E, t) = 0$ always. The second term of the series (25) is $N_2(E, t)$, and it is already proportional to β^2.

Multiplying Eqs. (1a) and (1b) by E, integrating with respect to E from 0 to ∞, and adding the two equations, we get

$$\frac{\partial}{\partial t} \left\{ \int_0^\infty EP(E, t)dE + \int_0^\infty EQ(E, t)dE \right\}$$

$$= -\beta \int_0^\infty P(E, t)dE = -\beta N(0, t), \tag{26}$$

which simply states that the change in the total energy of the shower is entirely due to collision loss. Integrating (26), with respect to t, from 0 to ∞ we get, for a shower initiated by an electron of energy E_0

$$\int_0^t N(0, t)dt = E_0/\beta. \tag{27}$$

Since the solution (25) satisfies Eqs. (1) exactly, it must satisfy (27). This need not, of course, be true of the first term alone. Some misapprehension appears to have existed on this point.[8] We shall return to this point in the next section, where we show that $N_0 + N_2$ together contribute between 70 and 85 percent of the whole primary energy. From the physical point of view, however, Eq. (27) must be taken with caution, especially in substances of high atomic number where the critical energy is low. It is not true that all the energy of a cascade is dissipated by the collision loss of *cascade* electrons alone. A good deal of energy is lost in the form of quanta of energy less than $2mc^2$ which are incapable of further pair creation. Thus the complete series

[8] I. E. Tamm and S. Belenky, Phys. Rev. **70**, 660 (1946).

(25a) for $N(0, t)$ must give too many cascade electrons of low energy at large thicknesses, and the first two or three terms of the series may well give a truer picture of the physical process in substances of high atomic number.

III. NUMERICAL RESULTS; ENERGY SPECTRUM OF SHOWER ELECTRONS

It is next necessary to evaluate the first term of the series (25a), namely, $N_0(E, t)$. Now $f_0(s, t)$, which occurs in the integrand, is given by (8) and consists of a sum of two parts. As is well known, the coefficient λ_S is much smaller than μ_S for real values of s, so that for all but small t the second part containing $\exp(-\mu_S t)$ makes a negligible contribution compared with the first. The first part can be evaluated with considerable accuracy by the saddle point method. The second part expressed the result of transition effects, and it is therefore necessary to evaluate it if one wishes to know what happens at small thicknesses. However, it cannot be evaluated in the same way as the first part since the integrand has no saddle point. We have now found a method of overcoming this difficulty. Our method is equally applicable to the higher terms $N_n(E, t)$, so that it will be developed in general. Let

$$\phi_n(s, r) = \int_0^\infty e^{-rt} f_n(s, t) dt. \qquad (28a)$$

Then

$$f_n(s, t) = \frac{1}{2\pi i} \int_{\rho-i\infty}^{\rho+i\infty} e^{rt}\phi_n(s, r) dr, \qquad (28b)$$

where ρ is a large real number such that all the singularities of ϕ_n lie to the left of the path of integration. Introducing this expression into (25b) we get

$$N_n(E, t) = \frac{1}{(2\pi i)^2} \int_{\sigma-i\infty}^{\sigma+i\infty} ds \int_{\rho-i\infty}^{\rho+i\infty} dr \frac{1}{s+n-1}$$

$$\times \left(\frac{E_0}{\beta}\right)^{s-1} \left(\frac{\beta}{E+\beta g}\right)^{s+n-1} \frac{\Gamma(s+n)}{\Gamma(s)}$$

$$\times \exp(rt)\phi_n(s, r). \qquad (29)$$

(29) is an exact expression and it has the advantage that the double integral can be evaluated by the saddle point method. Writing the integral in the form $\exp\omega_n(s, r)$, so that

$$\omega_n(s, r) = (s-1)y_0 + (s+n-1) \log\frac{\beta}{E+\beta g}$$

$$+rt - \log(s+n-1) + \log\Gamma(s+n)$$

$$-\log\Gamma(s) + \log\phi_n(s, r), \qquad (30)$$
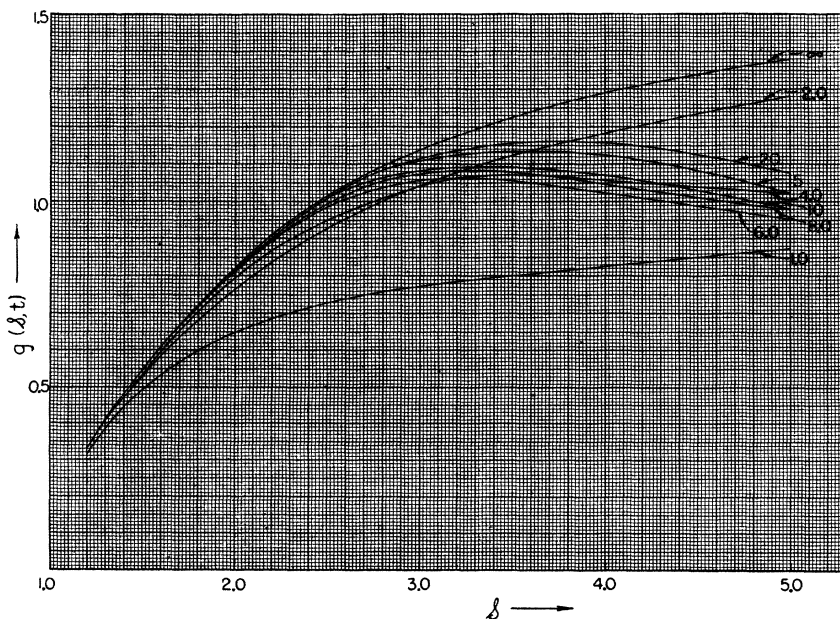
where $y_0 = \log E_0/\beta$. We determine the double

FIG. 1. $g(s, t)$ as a function of s.

TABLE II. y_0 as a function of t and s as determined by Eq. (31a). The figures in line with each t give the values of y_0 for the principal term $n=0$. Those below give y_0 for the next term $n=2$.

$t\backslash s$	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0	2.2	2.3	2.4	2.5	2.6	3.0	3.5
0.25		2.54		0.877												
0.5		5.77		3.23			1.93		1.53							
1.0	14.13	8.56	6.05	4.64					2.11				1.36		1.01	
2.0	23.69	14.06	9.79	7.42		4.93			3.22				1.99		1.40	1.05
	18.02	10.12	6.74	4.94			2.64		2.03				1.39		1.13	
4.0		24.95	17.16	12.88	10.20		7.07		5.31				3.09		2.04	1.45
		20.83	14.04	10.32	8.03		5.42		4.01		2.83		2.35		1.63	
6.0			24.50	18.29		11.78	9.88		7.34		5.10		4.13		2.62	
			20.82	15.70	11.17		8.20		5.99				3.32		2.13	
8.0			23.69	18.62			12.69		9.35		6.41		5.14		3.18	2.12
			21.08	15.36			10.97		7.97	6.08			4.28		2.63	
10.0			29.08	22.81			15.48		11.35		7.70		6.14		3.71	2.42
				20.58			13.75		9.95				5.25		3.13	
12.0				27.00		21.93	18.27		13.35		9.00				4.24	
				24.76			16.52		11.93				6.21		3.63	
15.0						26.99	22.44	19.01	16.33		10.92				5.02	
						25.02	20.69		14.91				7.66		4.38	
20.0								24.85	21.30	16.09		12.46		9.81	6.31	
							27.63	23.26	19.86		13.00		10.07		5.63	

saddle point s_n, r_n through the equations

$$\frac{\partial \omega_n}{\partial r}=0, \quad \frac{\partial \omega_n}{\partial s}=0,$$

namely,

$$t+\frac{\partial}{\partial r}\log\phi_n(s, r)=0, \tag{31a}$$

$$y_0+\left[\log\frac{\beta}{E+\beta g}-(s+n-1)\frac{\beta}{E+\beta g}\frac{\partial g}{\partial s}\right]$$

$$-\frac{1}{s+n-1}+\frac{\partial}{\partial s}\{\log\Gamma(s+n)-\log\Gamma(s)\}$$

$$+\frac{\partial}{\partial s}\log\phi_n(s, r)=0. \tag{31b}$$

We then get as usual

$$N_n(E, t)=\frac{\exp(\omega_n(s_n, r_n))}{2\pi\left\{\frac{\partial^2\omega_n}{\partial r_n^2}\frac{\partial^2\omega_n}{\partial s_n^2}-\left(\frac{\partial^2\omega_n}{\partial r_n\partial s_n}\right)^2\right\}^{\frac{1}{2}}}. \tag{32}$$

The values of $g(s, t)$ for different values of s and t are given in Table I. For this purpose we have used the exact expression for $g(s, t)$ as defined by (16) and given explicitly by formula (57) of our previous paper A, and not the approximate expression A(58). Table I is, in fact, an extension of Table 4B of A to cover a much greater range of values of t and a larger number of values of s. We note that $g(s, t)$ is a fairly slowly varying function of s of the order 1. For $E \gg \beta$ the second term in square brackets in (31b) is negligible compared with the first. Its neglect means that the variation of $g(s, t)$ with s plays no part in determing the saddle point. (See Fig. 1.) Thus as far as the high energy part of the spectrum, $E \gg \beta$, is concerned, the effect of collision loss is simply to shift it to lower energies by a constant amount $\beta g(s_n, t)$. The spectrum is, however, not represented by a simple power law for high energies, since the value of s_n changes with the value of E for a given E_0. Only if s_n were fixed, would the spectrum be of the form $\{E_0/E+\beta g\}^{s_n}$, and this is not so. Indeed, for fixed E_0 and t the saddle point s_n shifts to large values as E increases, so

that the number of shower particles of high energy is greater than would be obtained by a simple power law.

For low energies E less than or of the order β, the second term in square brackets cannot be neglected. However, the saddle point s_n changes very slowly with E, and one can therefore take for s_n the value obtained by putting $E=0$. For small E (32) can therefore be written approximately in the form

$$N_n(E, t) = \frac{\exp(\omega_n'(s_n, r_n))}{2\pi \left\{ \frac{\partial^2 \omega_n'}{\partial r_n^2} \frac{\partial^2 \omega_n'}{\partial s_n^2} - \left(\frac{\partial^2 \omega_n'}{\partial r_n \partial s_n} \right)^2 \right\}^{\frac{1}{2}}}$$

$$\times \left(\frac{\beta}{E + \beta g(s_n, t)} \right)^{s_n + n - 1}, \quad (33)$$

where ω_n' differs from ω_n defined by (30) only by the omission of the second term. This approximation is not permissible for primary energies E_0 of the order β, and then we have to turn to the more exact expression (32).

The contribution to the total number of electrons is obtained by putting $E=0$. We write $N_n(t) = N_n(0, t)$ for brevity. (33) shows that for

electrons below the critical energy

$$N_n(E, t) \approx \left\{ \frac{\beta g(s_n, t)}{E + \beta g(s_n, t)} \right\}^{s_n + n - 1} N_n(t), \quad (34)$$

and the contribution of the corresponding term to the low energy spectrum is

$$P_n(E, t) = -\frac{\partial N_n(E, t)}{\partial E} \approx \frac{s_n + n - 1}{\beta g(s_n, t)}$$

$$\times \left\{ \frac{\beta g(s_n, t)}{E + \beta g(s_n, t)} \right\}^{s_n + n} N_n(t), \quad (35a)$$

the total energy spectrum being

$$P(E, t) = \sum_{n=0}^{\infty} P_n(E, t). \quad (35b)$$

This double saddle point method is particularly easy to apply to the main term $N_0(E, t)$. Inserting f_0, given by (8) (28a) yields

$$\phi_0(s, r) = \frac{D + r}{(r + \lambda_S)(r + \mu_S)}. \quad (36)$$

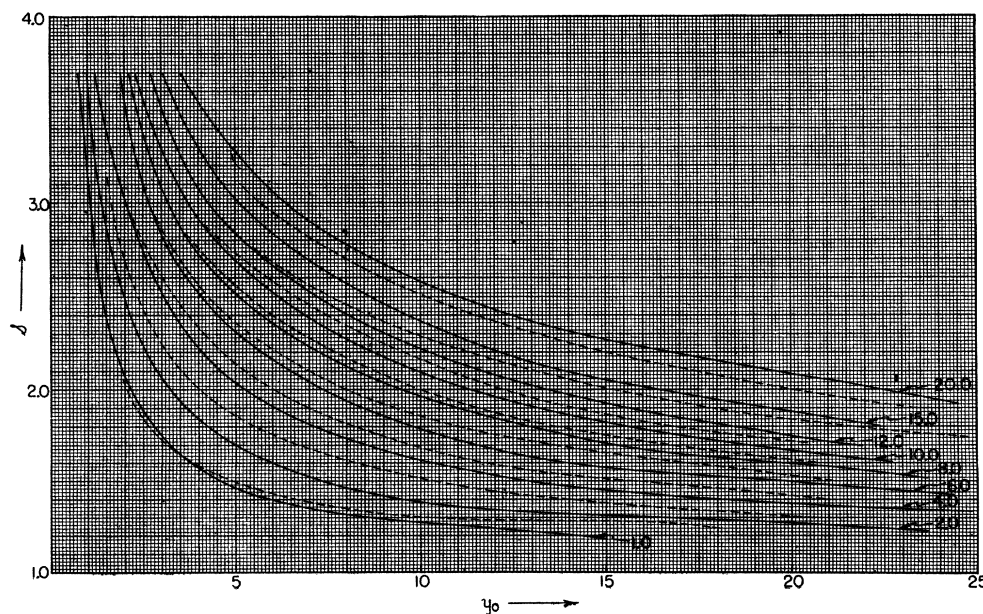An insertion of (36) into the right-hand side of (28b) obviously leads to (28a), since we are left

FIG. 2. s as a function of y_0 for different values of t. Numbers indicate values of t. ———relation between s and y_0 for the first term $N_0(t)$; ———— relation between s and y_0 for the second term $N_2(t)$.

with only the residues of the integrand at the points $r = -\lambda_S$ and $r = -\mu_S$.

From the point of view of numerical calculation it is easiest to proceed as follows. We assume a value of t and s_n. We then determine r_n by Eq. (31a), and y_0 by Eq. (31b). One can then determine $N_n(t)$ by (32). In this way we obtain y_0 and $N_n(t)$ as functions of s_n for different values of t. From these figures it is easy to obtain $N_n(t)$ as a function of y_0. The upper figures of Table II give y_0 for different values of t and s_0, that is, the corresponding value of y_0 for the principal term. The lower figures give y_0 as determined by (31b) for the second term N_2, of the series (25). The values of s at the head of the table are then to be understood to denote s_2. The same results have been given graphically in Fig. 2. Here s_0 and s_2 have been plotted as functions of y_0 for different values of t.

The values of N_0 and N_2 calculated from (25) are given in Table III for different values of y_0 and t, the upper figure in each space giving N_0 and the lower figure N_2. The space for N_2 has been left a blank, whenever the number is quite negligible compared with N_0. In Fig. 3 both N_0 and N_2 are plotted logarithmically as functions of t for four typical values of y_0. Table III as well as Fig. 3 show that N_2 becomes equal to N_0 only at thicknesses which are three to four times those at which the maximum number of particles occurs in the shower. One may be sure that as long as N_2 is less than N_0, N_3 and the higher terms will be still smaller. Their contribution is significant only at the tail end of a shower, and that only to electrons of very low energy.

The spectrum of electrons whose energy is of the order of the critical energy β, or less, is given by (35). Whenever $N_2 \ll N_0$ the spectrum is determined almost entirely by the first term of (35b) and is of the form of a modified power law $\{\beta g/(E+\beta g)\}^{s_0}$. Here g is a quantity approximately equal to 1, the exact value of which can be read from Table I, for any given value of t and s_0. The value of s_0 can be obtained from Fig. 2 for any t and y_0. At the maximum of the shower $s_0 \approx 2$, while before the maximum s_0 moves from about 1.3 to 2 with increasing t. This means that the power of the modified power law gradually increases with thickness, corresponding to an increasing concentration of

TABLE III. The total number of particles $N = N_0 + N_2 + \cdots$ as a function of thickness t and primary energy, $y_0 = \log E_0/\beta$. The figures in line with each t give N_0, the figures below give N_2.

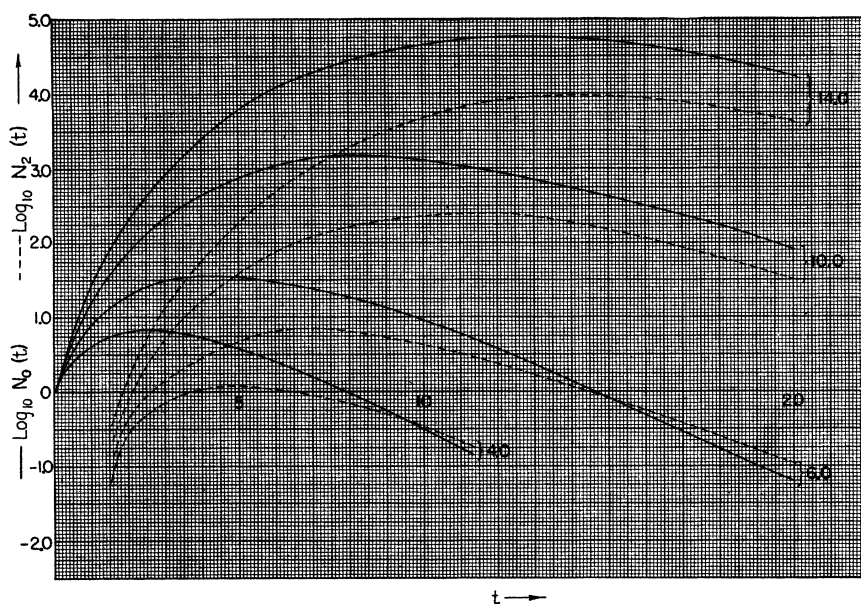| $t\backslash y_0$	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	12.0	14.0	16.0	18.0	20.0	22.0	24.0
0.25	1.140	1.231	1.310	1.390	1.472	1.561	1.650	1.720									
0.5	1.151	1.501	1.838	2.184	2.510	2.851	3.195	3.520									
1.0	0.8730	1.897	2.824	3.788	5.449	7.002	8.698	10.57	12.64	14.86	19.95	26.54					
2.0	0.1272	0.9734	2.809	5.957	11.33	18.10	26.93	36.34	54.03	75.35	130.5	210.6	318.6	463.0	661.8	951.0	
	0.09660	0.1765	0.2025	0.2439	0.3066	0.3818	0.4666	0.5651	0.6643	0.7507	0.9660	1.236	1.605	2.180			
4.0	0.2596	1.384	5.006	13.88	33.90	72.23	142.1	261.7	452.8	1211	2792	5991	1.223×10^{4}	2.322×10^{4}	4.201×10^{4}	7.323×10^{4}	
	0.2565	0.5293	0.9906	1.729	2.904	4.678	6.486	10.95	15.47	29.38	56.92	102.3	173.7	278.4			
6.0	0.4762	2.382	8.923	28.86	82.64	209.4	485.0	1062		4152	1.337×10^{4}	3.751×10^{4}	9.690×10^{4}	2.279×10^{5}	5.076×10^{5}	1.114×10^{6}	
	0.4496	1.159	2.742	6.005	12.30	23.61	43.06	75.12		205.2	511.3	1223	2677	5283			
8.0	0.1448	4.433	17.88	61.50	187.2	525.4	1365	7364		3.106×10^{4}	1.133×10^{5}	3.826×10^{5}	1.162×10^{6}	3.197×10^{6}			
	0.9031	2.164	6.084	14.95	34.83	74.93	155.7	615.6		2038	6000	1.597×10^{4}	3.962×10^{4}				
10.0	0.04263	0.3076	1.764	8.538	35.62	130.8	435.2	1311		8950	5.051×10^{4}	2.452×10^{5}	1.054×10^{6}	4.004×10^{6}	1.293×10^{7}		
	0.07312	0.3222	1.191	3.913	11.80	33.11	87.55	214.7		1079	4.599×10^{4}	1.685×10^{5}	5.631×10^{5}	1.799×10^{6}			
12.0	0.2563	3.763	11.61	60.38	268.0	935.2	8925	6.412×10^{4}		3.664×10^{5}	1.936×10^{6}	7.920×10^{6}	3.209×10^{7}	1.166×10^{8}			
	0.6014	2.349	8.110	25.86	77.19	216.5	1434	7.574×10^{4}		3.520×10^{5}	1.429×10^{6}	5.158×10^{6}	1.733×10^{7}	5.596×10^{7}			
15.0	0.1261	0.8107	4.625	23.27	104.6	425.9	5399	5.285×10^{4}		4.215×10^{5}	2.787×10^{6}	1.613×10^{7}	8.163×10^{7}	3.711×10^{8}			
	0.1716	0.7862	3.295	12.47	42.76	138.7	1269	9.591×10^{4}		5.939×10^{5}	3.136×10^{6}	1.511×10^{7}	6.405×10^{7}	2.356×10^{8}			
20.0	0.05720	0.3970	2.484	13.87	69.97	1376	2.034×10^{4}	2.427×10^{5}		1.859×10^{6}	1.998×10^{7}	1.434×10^{8}	9.209×10^{8}				
	0.09140	0.5148	2.475	10.37	38.65	381.8	2.813×10^{4}	2.813×10^{5}		1.856×10^{6}	2.528×10^{7}	1.856×10^{8}	8.270×10^{8}				

FIG. 3. N_0 and N_2 as functions of t. Numbers on the curves indicate the value of y_0.

electrons at the low energy end. The spectrum is nearly a modified inverse square law at the maximum of the shower. The effect of N_2 and the higher terms is to still further accentuate this tendency. For as shown by Fig. 2, s_0 and s_2 do not differ very much in the region where N_2 is comparable with N_0. Hence according to (35), N_2 makes a contribution to the spectrum of the form of a modified power law with a power 2 higher than N_0. Indeed, keeping only the first two terms of (35), we get roughly

$$P(E, t) \approx \text{const.}\left\{ (s_0-1)\left(\frac{\beta g}{E+\beta g}\right)^{s_0} N_0 \right.$$

$$\left. + (s_0+1)\left(\frac{\beta g}{E+\beta g}\right)^{s_0+2} N_2 \right\}. \quad (37)$$

Thus, even if N_2 is a little less than N_0 the effect of the factors s_0-1 and s_0+1 is to make the contribution of the second term greater than the first for energies lower than those determined by

$$\left(\frac{E+\beta g}{\beta g}\right)^2 \sim \frac{s_0+1}{s_0-1}\frac{N_2}{N_0}.$$

As a concrete example, take $t=20$, $y=10$. Then $s_0=2.58$, $s_2=2.50$, $N_0=70.0$, $N_2=38.7$, $g=1.01$.

Hence

$$\frac{s_0+1}{s_0-1}\frac{N_2}{N_0}=1.25.$$

We therefore see that N_2 makes a larger contribution than N_0 to the spectrum of electrons of energy *less than a tenth* of the critical energy. For any given thickness and primary energy, the contribution of the first two terms to the low energy spectrum can be calculated easily by using the tables and figures of this paper. As we have already stated, the contribution of the higher terms is negligible, except for the very tail end of a shower and low energies.

Electrons below the critical energy arise by electrons of high energy emitting large quanta, and by pair creation, and not only by electrons of energy above the critical energy coming into regions of lower energy by collision loss, as has been assumed by Arley. Indeed, for any thickness before the shower reaches its maximum the former processes certainly predominate. Iyengar has proved that for thickness $t < 1/((4/3)+\alpha) \approx \frac{3}{4}$ there is still a trace left of the delta-function representing the original electron, but for $t > 1/((4/3)+\alpha)$ this completely disappears. For $t \ll 1$ the form of the spectrum was given by us in A. For $t > 1/((4/3)+\alpha)$ formula (37) shows that the spectrum increases monotonically as E decreases, contrary to the results of Arley.

Iyengar has shown that this result remains true even if the variation of the radiation and pair creation cross sections with energy is taken into account.

Table III brings out an interesting feature of cascades started by low energy electrons corresponding to, say, $y_0 \lesssim 2$. Take, for example, $y_0 = 1$, that is, $E_0 = 2.7\beta$. The first column of Table III shows that N_2 becomes comparable with N_0 only when $t \gtrsim 2$, and here N_0 has already fallen to about one-tenth of its value at the maximum. Thus the contribution of N_0 to the left-hand side of (27) can hardly be greater than 1.5. In order that (27) should be satisfied the cascade must have a long tail of height <0.2, say, extending to $t \sim 6$. Now it has already been remarked that the critical energy is in fact the collision loss in characteristic units, so that

because of collision loss alone a particle of energy $E_0 = 2.7\beta$ cannot travel more than a distance 2.7. The mean number of particles can therefore be other than zero at a distance greater than 2.7 only if no energy has been lost by collision loss for part of the distance. This is possible only if no particles have traveled along some of this distance. Thus, for showers produced by low energy primaries, at certain thicknesses there may only be quanta but no electrons present in the shower, and at a greater distance quanta may materialize into a pair of particles. For small showers, the whole path of the shower is not covered by ionizing particles, as is the case for larger showers. This result is simply a rather interesting case of the fluctuation phenomenon which may be of considerable importance in interpreting the experiment.
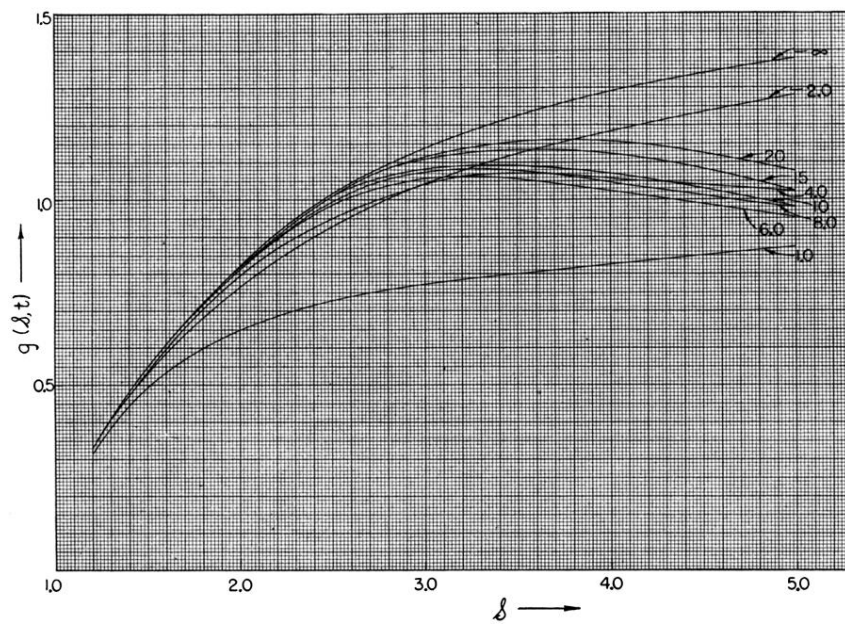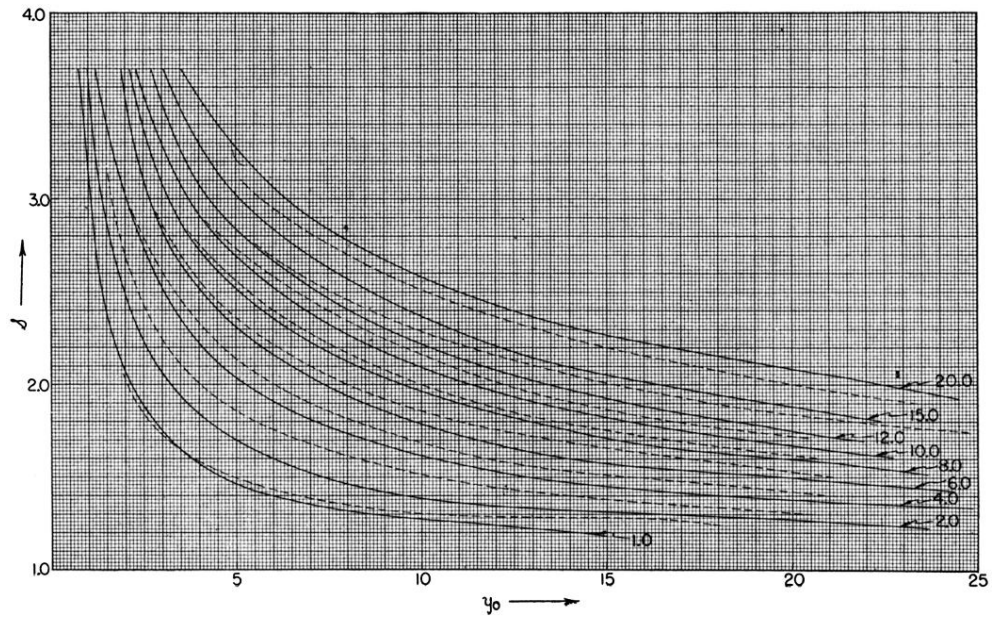
Fig. 1. $g(s, t)$ as a func-
tion of s.

FIG. 2. s as a function of y_0 for different values of t. Numbers indicate values of t. ——————relation between s and y_0 for the first term $N_0(t)$; ----relation between s and y_0 for the second term $N_2(t)$.
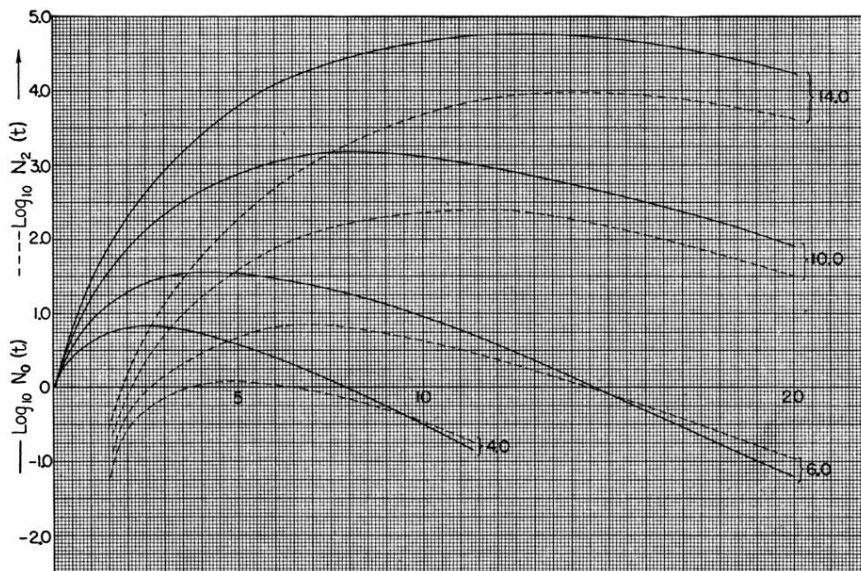
FIG. 3. N_0 and N_2 as functions of t. Numbers on the curves indicate the value of y_0.