THE
PHYSICAL REVIEW

A journal of experimental and theoretical physics established by E. L. Nichols in 1893

SECOND SERIES, VOL. 74, NO. 2 JULY 15, 1948

On the Application of Heisenberg's Theory of *S*-Matrix to the Problems of Resonance Scattering and Reactions in Nuclear Physics

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(Received April 15, 1948)

It is shown in the present paper that the Heisenberg *S*-matrix can have only simple poles and zeros in the complex plane of momentum and energy variables when the range of force is short. This fact is sufficient to determine the dispersion formulae in nuclear physics.

1. INTRODUCTION

THE nuclear dispersion formulae as derived by various authors[1] account quite satisfactorily for the resonance phenomena in nuclear physics. We may classify all the derivations into two groups. The first group follows the perturbation method and the treatment is closely analogous to Dirac's theory of dispersion of light by charged particles. The perturbation treatment can hardly be considered as justified, as the interaction is usually very strong in the nuclear reactions. The second group makes use of complex energy states introduced by Gamow. A simplified model is usually needed in this method, and as a consequence the results vary with the different assumptions about the model that have to be introduced. Furthermore these derivations give no account of the potential scattering.

It has been pointed out recently by Heisenberg[2] that the divergence difficulty of the present quantum-field theory is perhaps due to the fact that the phenomena at small distances between the particles have not been described correctly. He proposed that in the future the collision matrix,[3] or the *S*-matrix, which gives only the asymptotic behavior of the wave functions, should be considered as the only fundamental quantity. Heisenberg and Møller[4] have shown that the knowledge of the *S*-matrix is sufficient to predict all the observable quantities if the analytic and unitary properties of the *S*-matrix and the completeness condition are assumed. In the present paper we shall show that the nuclear dispersion formulae also follow from these general properties of *S*-matrix. This means that these formulae really rest on a much

[1] G. Breit and E. P. Wigner, Phys. Rev. **49**, 519, 642 (1936). N. Bohr, Nature **137**, 344 (1936). H. A. Bethe and Placzek, Phys. Rev. **51**, 450 (1937). F. Kalcker, J. R. Oppenheimer, and R. Serber, Phys. Rev. **52**, 273 (1937). H. A. Bethe, Rev. Mod. Phys. **9**, 71 (1937). P. L. Kapur and R. Peierls, Proc. Roy. Soc. **166**, 277 (1938). A. J. F Siegert, Phys. Rev. **56**, 750 (1939). G. Breit, Phys. Rev. **58**, 606 (1940). E. P. Wigner, Phys. Rev. **70**, 15 (1946); Proc. Nat. Acad. Sci **32**, 302 (1946). H. Feshbach, D. C. Pearlee and V. F. Weisskoff, Phys. Rev. **71**, 145 (1947).

[2] W. Heisenberg, Zeits. f. Physik **120**, 513, 673 (1943); Zeits. f. Naturforsch., 11/12, 607 (1946), and several as yet unpublished papers, a complete account of which is given by C. Møller, Kgl. Danske Vid. Sels. Math-Fys. Medd. **23**, No. 1 (1945).

[3] This matrix was first introduced by Wheeler, see J. A. Wheeler, Phys. Rev. **52**, 1107 (1937).

[4] C. Møller, Kgl. Danske Vid. Sels. Math-Fys. Medd. **23**, No. 1 (1945); **22**, No. 19 (1946); Nature **158**, 403 (1946).

more solid basis than other theoretical derivations hitherto used.

2. GENERAL INVESTIGATION ON THE S-MATRIX

We shall first give a brief account of Heisenberg's theory. It is sufficient for this purpose to consider the simplest case of a non-relativistic particle in a central field of force. The Schrödinger equation is given by

$$[(1/r)(d^2/dr^2)](r\psi)+k^2\psi - [l(l+1)/r^2]\psi+V(r)\psi=0, \quad (1)$$

where l is the angular momentum. The asymptotic solution for ψ at large distance r in the continuous spectrum is given by

$$\psi\sim(1/r)\sin[kr+\eta_l(k)], \quad (2)$$

where $\eta(k)$ is the phase shift due to the interaction potential $V(r)$. Equation (2) can also be written in the following form, apart from an irrelevant factor depending on k:

$$\psi\sim(1/r)(e^{-ikr}-S_l(k)e^{ikr}), \quad (3)$$

where

$$S_l(k)=\exp[-2i\eta_l(k)] \quad (4)$$

is Heisenberg's S which is a matrix in the general case when the angular momentum and the number of particles is not fixed. Since $\eta_l(k)$ must be a real and odd function of k (this can be seen from the fact that if we reverse the sign of k in (2), no new solution of (1) should be obtained since (1) is even in k), it follows from (4) that

$$S^*(k)S(k)=1 \quad (5)$$

and

$$S(k)S(-k)=1. \quad (6)$$

It is expected that when $V(r)$ is a regular function of r, $S(k)$ should be an analytic function of k. According to the suggestion of Kramers and Heisenberg, the bounded states of the system is given by the negative imaginary values k_n of k, which satisfy $S(k_n)=0$ (or positive imaginary values k_n^* or $-k_n$ of k, which satisfy $S(k_n^*)=\infty$). Then ψ is given by

$$\psi_n\sim(1/r)e^{-ik_nr}=(1/r)e^{-|k_n|r}, \quad (7)$$

which falls off exponentially with r, and therefore represents a closed state. It has also been shown that the zeros of $S(k)$ for complex values of k represent quasi-stationary states of the system. For real values of k, $S(k)$ gives directly the scattering cross section of the particle by the central field of force. We see then that $S(k)$ determines all the observable quantities that can be obtained by solving (1). The above results hold also when the relativistic wave function is used.

In the following we shall show that with a possible exception at infinity the only singularities that $S(k)$ can have in the upper half of the complex k plane are poles of the first order. This follows directly from the general properties of $S(k)$. At least for the case when the range of force is very short, we can show that $S(k)$ can only have poles of the first order in the lower half of the complex k plane also. To show these we need a relation for $S(k)$ derived by Heisenberg from the completeness condition of the asymptotic wave functions. As probably most readers have not yet read Heisenberg's papers, we shall derive this relation once again as follows: The asymptotic wave functions for the continuous and discrete states are, respectively, given by

$$\psi_k(r)\sim A_k[\sin[kr+\eta(k)]/r] \quad (8)$$

and

$$\psi_n(r)\sim C_n[e^{ik_nr}/2\pi\sqrt{2}r], \quad (9)$$

where A_k and C_n are normalization constant and k_n is now given by $S(k_n)=\infty$. A_k is determined by the normalization condition for the continuous spectra:

$$\int 4\pi r^2\psi_k^*(r)\psi_{k'}(r)dr=\delta(k-k') \quad (10)$$

to be $A_k=1/\sqrt{2}\pi$. C_n can also be determined from the normalization condition by a method due to Kramers,[5] but it is determined more conveniently by the completeness condition

$$\int_0^\infty dk\psi_k(r)\psi_k^*(r') + \sum_n\psi_n(r)\psi_n^*(r')=(1/4\pi r^2)\delta(r-r'). \quad (11)$$

Inserting (8) and (9) into (11) and using the relation

$$\eta(k)=-\eta(-k), \quad (12)$$

[5] Kramers, Hand- und Jahrbuch d. Chem. Physik **1**, 312 (1938).

we easily obtain

$$\int_{-\infty}^{+\infty} dkS(k)e^{ik(r+r')} = \sum_n |C_n^2| e^{ik_n(r+r')}. \quad (13)$$

Now $S(k)$ has a pole at every point $k=k_n$ above the real axis in the complex k plane. By changing the path of integration towards increasing positive imaginary values of k, this path finally reduces to circles around the poles at k_n and an infinite semicircle above the real k axis. Therefore, one gets

$$\sum_n \oint_{k=k_n} dkS(k)e^{ik(r+r')} + \int_\Gamma dkS(k)e^{ik(r+r')} = \sum_n |C_n^2| e^{ik_n(r+r')}, \quad (14)$$

where $\int_\Gamma$ is the integration along the infinite semicircle. Since (14) holds for any value of $r+r'$, the coefficient of $\exp[ik_n(r+r')]$ of both sides must be equal. So we finally have

$$\oint_{k=k_n} dkS(k) = |C_n^2|, \quad (15)$$

$$\int_\Gamma dkS(k)e^{ik(r+r')} = 0. \quad (16)$$

Equation (15) is the relation for $S(k)$ obtained by Heisenberg in his papers in 1946. Equation (16) determines the nature of singularity, if it exists, at infinity. This point will be discussed later. From the above derivation we see immediately that if there is a quasi-stationary state determined by the poles of $S(k)$ above the real k axis, it should be included in the summation of (11) since other wise both sides of (14) cannot be equal. In fact, the wave function for this quasi-stationary state

$$\psi_n \sim \exp[iK_1 r - K_2 r] (K_2 > 0) \quad (17)$$

satisfies the same boundary conditions as the closed state. We have shown[6] that (17) represents a quasi-stationary state of the K-capturing type. It should be noted that this result is not contradictory to the conclusion from wave mechanics that the quasi-stationary states should not be included in the completeness condition (i.e., they cannot be considered as members of the orthonormal system determined by the wave equation), as the complex poles above the real k axis are only possible in the theories where particles can be created and annihilated; in that case the Hamiltonians are certainly not Hermitian so that complex eigenvalues cannot be excluded.

The complex poles of $S(k)$ below the real k axis give rise to the usual quasi-stationary radioactive states.[4] These states should not be included in the sum of the left-hand side of (11) since these poles make no contribution to the residue of the integral of (13).

For the case of relativistic particles, (15) and (16) are also obtained by the same procedure, except that, perhaps, some remarks should be added here. From the relation

$$E = (k^2+\mu^2)^{\frac{1}{2}}, \quad (18)$$

E being the energy and μ the rest mass of the particle, we see that E is a double-valued function of k. Consequently, the wave function ψ, which is a function of k and E, is also double valued. We may imagine the complex k plane as consisting of two Riemann sheets jointed along two cuts on the imaginary axis from $k=i\mu$ to $k=i\infty$ and from $k=-i\mu$ to $k=-i\infty$. The upper sheet represents only states with positive energy and the lower sheet only those with negative energy. The integral in (13) will then be along the real axes of both Riemann sheets and the summation in (13) contains the states determined by the poles in the upper half of the plane of both sheets. The same conditions (15) and (16) are obtained except that $\int_\Gamma$ is now taken along the two infinite semicircles in the upper half of the plane of both Riemann sheets.

We shall now extend Heisenberg's investigation to find the nature of the singularities of $S(k)$ in the upper half of the complex k plane. The general expression of $S(k)$ around the singular point k_n can be written

$$S(k) = f(k) + \sum_{s=1}^{\infty} [A_s/(k-k_n)^s], \quad (19)$$

where $f(k)$ is regular at the point $k=k_n$ and $A_s (s=1, 2, \cdots)$ are constants. Inserting this

[6] W. Heitler and N. Hu, Proc. Roy. Irish Acad. **51A**, No. 9 (1947), Nature **159**, 776 (1947).

into (14) and using (15), we have

$$\oint_{k=k_n} dkS(k)e^{ik(r+r')} = \oint_{k=k_n} \sum_{s=1}^{\infty} [A_s e^{ik(r+r')}/(k-k_n)^s] = \sum_{s=1}^{\infty} [2\pi i/(s-1)!] \times A_s[i(r+r')]^{s-1} e^{ik_n(r+r')}. \quad (20)$$

Since the above equation must hold for any value of $r+r'$, the coefficients of $(r+r')^s$ of both sides of (14) must be equal. Thus we have

$$A_1 = (1/2\pi i)|C_n^2|, \quad A_2 = A_3 = \cdots = 0 \quad (21)$$

and (19) becomes

$$S(k) = f(k) + (1/2\pi i)[|C_n^2|/(k-k_n)]. \quad (22)$$

Therefore, $S(k)$ has only poles of the first order in the upper half of the complex k plane. It should be noted that the same result can be obtained by Kramer's method of determining the normalization constant of the closed state.[5] The conclusion for the closed states was obtained also by Møller.

The relation (14) does not give any information about the nature of singularities in the lower half of the complex k-plane since these singularities are not included in the summation of (11). However, at least in the case of nuclear reaction where the range of force is much smaller than $1/K_1$, $k=-K=-K_1-iK_2(K_2>0)$ being the position of the pole, we can show that $S(k)$ has only poles of the first order also below the real k axis. From (6) we see that $k=K$ must be a zero of $S(k)$. The wave function becomes at this point

$$\psi_K = \varphi_K/r, \quad \varphi_K \sim e^{-iKr} = \exp[-iK_1 r + K_2 r], \quad (23)$$

which represents a radioactive decaying (capturing) state when K_1 is negative (positive). We have therefore at $k=K$ and, when r is very large,

$$\frac{d}{dk}\left[\varphi\left(\frac{d\varphi}{dr}+ik\varphi\right)\right]_{k=K} = \varphi_K\left(\frac{d^2\varphi}{dkdr}\right)_{k=K} + i\varphi_K^2 + iK\varphi_K\left(\frac{d\varphi}{dk}\right)_{k=K} = \varphi_K\left(\frac{d^2\varphi}{dkdr}\right)_{k=K} - \frac{d\varphi_K}{dr}\left(\frac{d\varphi}{dk}\right)_{k=K} + i\varphi_K^2. \quad (24)$$

For very large r, the left-hand side becomes simply $-2iK(dS/dk)_{k=K}$ if small terms of higher orders are neglected. Now inserting $\psi=\varphi/r$ into (1) we obtain

$$[(d^2/dr^2)+k^2-[l(l+1)/r^2]+V(r)]\varphi = 0. \quad (25)$$

Differentiating with respect to k, we have

$$[(d^2/dr^2)+k^2-[l(l+1)/r^2]+V(r)](d\varphi/dk)+2k\varphi = 0. \quad (26)$$

Multiplying (25) by $d\varphi/dk$ and (26) by φ and subtracting and integrating the result with respect to r, we obtain

$$\varphi(d^2\varphi/dkdr) - (d\varphi/dr)(d\varphi/dk) = -2k\int_0^r \varphi^2 dr. \quad (27)$$

Therefore, (24) becomes

$$-2iK(dS/dk)_{k=K} = -2K\int_0^r \varphi_K^2 dr + i\varphi_K^2. \quad (28)$$

We shall now introduce the assumption that the range of force is very small so that for $r>a$, where $a \ll 1/K_1$, φ_K represents a free wave:

$$\varphi_K(r) = (-i)^{l+1}(\pi Kr/2)^{\frac{1}{2}} H^{(2)}_{l+\frac{1}{2}}(Kr). \quad (29)$$

The integral in (28) may be broken into two parts

$$\int_0^r \varphi_K^2 dr = \int_0^a \varphi_K^2 dr + \int_a^r \varphi_K^2 dr. \quad (30)$$

The last integral can be evaluated easily and gives

$$\int_a^r \varphi_K^2 dr = (-1)^{l+1}(K\pi/2)\cdot(r^2/2) \times \{[H^{(2)}_{l+\frac{1}{2}}(Kr)]^2 - [H^{(2)}_{l+\frac{1}{2}}(Kr)] \times [H^{(2)}_{l+\frac{3}{2}}(Kr)]\}|_a^r. \quad (31)$$

The upper limit cancels with the last term of (28) when r is very large. For the lower limit we use the following approximate expression for small r:

$$H^{(2)}_{l+\frac{1}{2}}(Kr) \approx (2/\pi Kr)^{\frac{1}{2}}[(2l)!i/(l!(2Kr)^l)]. \quad (32)$$

We have, for $l>0$,

$$\int_a^r \varphi_K^2 dr - i(1/2K)\varphi_K^2 \approx (-1)^l[1/(2K)^{2l}] \quad (1/a^{2l-1})[(2l)!/l!]^2(1/2l-1). \quad (33)$$

To evaluate the first integral on the right-hand side of (30), we assume that the wave function for $r<a$ is nearly stationary so that the density of outgoing current is much smaller than the charge density. In other words, we assume that the wave function for $r<a$ is nearly real or nearly pure imaginary. From (29) and (33) we see that when l is even, φ_K is nearly real just outside $r=a$. Thus, by continuation, the wave function should also be nearly real inside $r=a$. When l is odd, φ_K is nearly pure imaginary just outside $r=a$ and, consequently, the wave function should be nearly pure imaginary also inside $r=a$. We have, therefore,

$$\int_0^a \varphi_K^2 dr \approx (-1)^l \int_0^a \varphi_K^* \varphi_K dr. \quad (34)$$

Adding (33) to (34), we have

$$i\left(\frac{dS}{dk}\right)_{k=K} = (-1)^l\left\{\int_0^a \varphi_K^* \varphi_K dr + \frac{1}{(2K)^{2l}} \frac{1}{a^{2l-1}}\left[\frac{(2l)!}{l!}\right]^2 \frac{1}{2l-1}\right\}. \quad (35)$$

It is seen that for $l>0$, (35) cannot vanish since both terms in the bracket are positive. For the case $l=0$, we have

$$\int_0^a \varphi_K^* \varphi_K dr \approx (1/2K_2)\varphi^*(a)\varphi(a) = 1/2K_2, \quad (36)$$

and (28) becomes

$$i(dS/dk)_{k=K} = (1/2K_2) - (1/2K_1 + 2iK_2). \quad (37)$$

Equations (35) and (37) furnish the proof that $(dS/dk)_{k=K} \neq 0$ and, consequently, the zero of $S(k)$ at $k=K$ is of the first order. From (6) we see that the pole of $S(k)$ at $k=-K$ below the real axis must also be of the first order.

In the above investigation we have only considered the isolated singularities of $S(k)$. It can easily be seen that $S(k)$ cannot have unisolated singularities such as those along a cut connecting different Riemann sheets other than the original two sheets in the relativistic case introduced by the fact that energy is a double-valued function of momentum. For if there is a cut between any two points a and b introduced by the factor like $[(k-a)(k-b)]^{\frac{1}{2}}$ in the expression for $S(k)$, from the condition $S(k)=1/S(-k)$, there must be another cut between $-a$ and $-b$. None of these two cuts can pass across the real axis, because if they do, additional degeneracy would be introduced to the continuous spectrum. Therefore at least one of these two cuts must lie above the real axis. Thus when we change the path of integration towards increasing positive imaginary values of k, in passing from (13) to (14), we would get an additional closed integral around this cut. Following the same argument leading to (16) we see that this closed integral must vanish separately. This however means that the cut cannot be there at all. Similar argument shows that $S(k)$ cannot have other factors which are irrational functions of E and k such as $(E-a)^{\frac{1}{2}}$, $(k-c)^{\frac{1}{2}}$, etc., because they would inevitably introduce cuts in the complex k plane. Therefore $S(k)$ can only be a rational function of E and k.

There is, however, a possibility that $S(k)$ may have an isolated singularity on the real k axis of the form

$$e^{ic(k-k_0)} \quad (38)$$

with c and k real and c also negative. The unitary condition $S^*(k)S(k)=1$ for real k will not be violated by this singularity. In the Appendix we shall show that (38) is the only possible essential singularity that $S(k)$ can have. Physically, this singularity amounts to the phenomena that the scattering cross section given by

$$\sin^2\eta = \sin^2[c/(k-k_0)] \quad (39)$$

will oscillate very rapidly as k approaches k_0. We shall dismiss the possibility of this singularity for finite value of k_0, on the physical ground that the cross section should be a smooth function of k. When k_0 goes to infinity, (38) becomes

$$e^{ic_\infty k}, \quad (40)$$

where c_∞ is a real negative constant. It will be of interest to note that $S(k)$ with this factor has been found in some known examples already. The simplest example is the case of scattering by a potential well of constant depth discussed by Møller.[4]

The general form of $S(k)$ for the non-relativistic case when the range of force is very small,

is therefore given by

$$S_l(k)=\pm e^{ick}\pi_n[(k-k_n^*)/(k-k_n)], \quad (41)$$

where $k_n(n=1, 2, \cdots)$ are the positions of the poles and are all distinct since all the poles are of the first order. The form of the factor $(k-k_n^*)/(k-k_n)$ follows from the unitary condition for real k: $S(k)S^*(k)=1$. From the further condition that $S(k)S(-k)=1$, we see that for any pole k_n of $S(k)$ not on the imaginary axis, there must be another pole $k_s=-k_n^*$. On the imaginary axis the last condition gives no other poles and zeros than those already demanded by the unitary condition. Thus if we denote by k_λ all the poles of $S(k)$ on the imaginary axis and by k_s all the poles on the right side of the imaginary axis, the final expression for $S(k)$ for the non-relativistic case becomes

$$S(k)=\pm e^{ick}\prod_\lambda\frac{(k-k_\lambda^*)}{(k-k_\lambda)}\prod_s\frac{(k-k_s^*)(k+k_s)}{(k-k_s)(k+k_s^*)}. \quad (42)$$

For the relativistic case a singularity of $S(k)$ is not only specified by its position k_s in the complex k plane, but also by the sign of $E_s=\pm(k_s^2+1)^{\frac{1}{2}}$ (in the following we shall put $\mu=1$) which shows whether the singularity is in the upper or the lower Riemann sheet. If $S(k)$ has a pole at $k=k_s$, $E=E_s(E_s^2=k_s^2+1)$, then from (5) and (6) we see that S has also a singularity at $k=-k_s^*$, $E=E_s^*$ and two zeros at $k=k_s^*$, $E=E_s^*$ and at $k=-k_s$, $E=E_s$. Therefore S must have a factor of the following form

$$S_s=\frac{(k_s+k_s^*)(E-E_s^*)-(E_s^*-E_s)(k-k_s^*)}{(k_s+k_s^*)(E-E_s)-(E_s-E_s^*)(k-k_s)}. \quad (43)$$

We can easily verify that the poles and zeros of (43) are simple. If S has also a pole at $k=k_s$, $E=-E_s$, then, following the same argument as before, it will have another factor given by (43) with E_s replaced by $-E_s$. We can easily verify that these two factors multiplied together are equal to the same factor

$$\text{Const.}\frac{(k-k_s^*)(k+k_s)}{(k-k_s)(k+k_s^*)} \quad (44)$$

of (42) as we would have expected. Next, if S has a pole at $k=k_\lambda$, $E=E_\lambda$ on the imaginary k axis, then it has also a zero at $k=k_\lambda^*=-k_\lambda$, $E=E_\lambda^*$. The expression for S has then the following factor

$$S_\lambda=(E-1+ib_\lambda k)/(E-1-ib_\lambda k) \quad b_\lambda=\text{real.} \quad 45$$

It can easily be verified that the pole and zero are simple and the position of the pole is given by

$$k_\lambda=2ib_\lambda/(1+b_\lambda^2), \quad E_\lambda=(1-b_\lambda^2)/(1+b_\lambda^2). \quad (46)$$

We see from (46) that $|k_\lambda|\leqslant 1$ and $|E_\lambda|\leqslant 1$, as should be the case for any closed state. Lastly, if S has an essential singularity at infinity, then it must contain a factor of the form

$$e^{ick}, \quad c<0, \quad (47)$$

which means that the same singularities are present in both Riemann sheets as (47) does not depend on the sign of E. That this must be so follows from the fact that $E=\pm\infty$, and $k=\pm\infty$ should be counted as only one point in the complex k- or E-plane. The final expression for $S(k)$ for the relativistic case is therefore:

$$S(k)=\pm e^{ick}\prod_\lambda\frac{E-1-ib_\lambda k}{E-1+ib_\lambda k}\prod_s\frac{(k_s+k_s^*)(E-E_s^*)-(E_s^*-E_s)(k-k_s^*)}{(k_s+k_s^*)(E-E_s)-(E_s-E_s^*)(k-k_s)}. \quad (48)$$

The sign $\pm$ in (42) and (48) remains so far undetermined. It should be noted that for any pole k_n of $S(k)$ above the real axis, we have from (15) the following additional condition to be satisfied by $S(k)$:

$$\oint_{k=k_n} dkS(k)>0. \quad (15')$$

Similarly, for any zero k_s of $S(k)$ above the real axis, we have, from (35) and (37) when the imaginary part of k_s is very small,

$$Rl\{(-1)^l i(dS/dk)_{k=k_s}\}>0 \quad (35')$$

(15') and (35') will determine among other things the sign of $S(k)$. It is rather difficult to write down explicitly the general form of $S(k)$ which also satisfies (15') and (35'). Therefore we shall consider (42) and (48) as our final form of $S(k)$ and (15') and (35') as additional conditions.

3. THE SCATTERING FORMULA

The wave function (3) can also be written in the following form:

$$\psi(r)\sim(1/r)(e^{-ikr}+(-1)^{l+1}e^{ikr}+[(-1)^l-S_l]e^{ikr}).$$

The first two terms represent the incident wave when there is no scattering. The last term represents the scattered wave. The scattering cross section is given by

$$Q_l(k)=(\pi/k^2)\,|\,1-(-1)^lS_l(k)\,|^2. \tag{49}$$

We shall first consider a special non-relativistic case where $S(k)$ has only two poles, k_s and $-k_s^*$, below the real axis. The sign can be fixed easily by (35′) to be $(-1)^l$. We put further $c=0$. This gives the so-called "one-level" formula:

$$Q(k)=\frac{\pi}{k^2}\left|1-\frac{(k-k_s^*)(k+k_s)}{(k-k_s)(k+k_s^*)}\right|^2=\frac{\pi}{k^2}\left|\frac{2k(k_s-k_s^*)}{k^2-k_s^2}\right|^2. \tag{50}$$

Now, since $E=k^2/2\mu$, where μ is the rest mass of the particle, we have

$$Q(k)=(4\pi\mu^2/k^2)\,|\,[2k(k_s-k_s^*)/(E-E_s)]\,|^2, \tag{51}$$

where $E_s=k_s^2/2\mu$. Since E_s is a complex quantity, we may write $E_s=W_s-\frac{1}{2}i\gamma_s$, thus

$$1/[\,|E-E_s|^2]=1/[(E-W_s)^2+\tfrac{1}{4}\gamma_s^2].$$

This gives the essential feature of the usual dispersion formulae. Using the relation

$$(1/4\mu)(k_s^{*2}-k_s^2)=\tfrac{1}{2}\gamma_s,$$

we have

$$Q(k)=[2\pi/(k_s^2+k_s^{*2})]\times\gamma_s^2/[(E-W_s)^2+\tfrac{1}{4}\gamma_s^2]. \tag{52}$$

Equation (52) becomes the well-known one-level formula when $(k_s^2+k_s^{*2})$ in the denominator is replaced by $2k^2$. From the above derivation we see that (52) holds only when $c=0$ and is therefore not the most general form. In the general case, where we have

$$Q(k)=\frac{2\pi}{(k_s^2+k_s^{*2})}\,\frac{4(E-W_s)^2\sin^2\tfrac{1}{2}ck+2(E-W_s)\gamma_s\sin ck+\gamma_s^2\cos^2\tfrac{1}{2}ck}{(E-W_s)^2+\tfrac{1}{4}\gamma_s^2}, \tag{53}$$

we shall have antiresonance scattering if $\cos^2\frac{1}{2}ck=0$. In that case $Q(k)$ vanishes when $E=W_s$. It seems, however, that c can only be a length of the order of the range of force a. We have, therefore, for nuclear scattering, $e^{ick}\approx 1$, and hence (52) will always be a good approximation except when $ka\approx 1$.

The corresponding one-level formula for the relativistic case when $c=0$ is given by

$$Q(k)=\frac{\pi}{k^2}\left|1-\frac{k_1(E-E_s^*)-\frac{1}{2}i\gamma_s(k-k_s^*)}{k_1(E-E_s)+\frac{1}{2}i\gamma_s(k-k_s)}\right|^2=\frac{\pi\gamma_s^2}{[k_1(E-W_s)-\frac{1}{2}\gamma_sk_2]^2+\frac{1}{4}\gamma_s^2(E^2-1)}, \tag{54}$$

where $E_s=W_s-i\frac{1}{2}\gamma_s$, $k_s=k_1+ik_2$, $E_s^2=k_s^2+1$. If we consider $E-W_s$, k_2, and γ_s as small quantities of higher order in comparison with W_s and k_1 in the neighborhood of resonance scattering, and retain only terms of the lowest order in them, we shall see after some easy calculation that (54) reduces to the same formula (52).

The many-level formulae can be easily evaluated from (42) and (48), but we shall not write them down here. From the above derivation we see that the scattering cross section is entirely determined by the value of c and the positions of all the poles of $S(k)$. These values depend on the particular form of potential function $V(r)$. The resonance scattering takes place around those levels for which $W_s\gg\gamma_s$, i.e., when E_s lies very near to the positive real E axis. Those poles far from the positive real E axis shall only contribute to the potential scattering or the background scattering. In the former theories no account of the potential scattering can be given. According to the present theory, the potential scattering is really the resonance scattering due to quasi-stationary states determined by poles far away from the positive real k axis and closed states of the system. Our formula differs in detail from other derivations given before. For instance, (52) differs from the usual one-level formula by a factor. The discrepancy must be due to the simplified assumptions about the model introduced in those derivations.

Equation (43) gives only the scattering when the incident wave is a spherical wave of angular momentum l. When the incident wave is a plane wave, the total cross section is given by

$$Q=\sum_l(2l+1)Q_l. \tag{55}$$

4. THE RESONANCE REACTIONS

We can easily apply our foregoing result to the nuclear reactions of the type

$$A+B\rightleftarrows C+D. \tag{56}$$

When the incident plane wave contains only A and B, the wave function for the system are given by

$$\begin{aligned}\psi(\mathbf{k}_A\mathbf{k}_B)&=\delta(\mathbf{k}_A-\mathbf{k}_A{}^0)\delta(\mathbf{k}_B-\mathbf{k}_B{}^0)\\&\quad+p(E_A+E_B-E_A{}^0-E_B{}^0)\\&\quad\times(\mathbf{k}_A\mathbf{k}_B|\,U|\,\mathbf{k}_A{}^0\mathbf{k}_B{}^0),\\\psi(\mathbf{k}_C\mathbf{k}_D)&=p(E_C+E_D-E_A{}^0-E_B{}^0)\\&\quad\times(\mathbf{k}_C\mathbf{k}_D|\,U|\,\mathbf{k}_A{}^0\mathbf{k}_B{}^0),\end{aligned} \tag{57}$$

where

$$p(E)=(1/E)-\tfrac{1}{2}i\pi\delta(E). \tag{58}$$

In the other case when the incident wave contains only C and D, we have

$$\begin{aligned}\psi(\mathbf{k}_A\mathbf{k}_B)&=p(E_A+E_B-E_C{}^0-E_D{}^0)\\&\quad\times(\mathbf{k}_A\mathbf{k}_B|\,U|\,\mathbf{k}_C{}^0\mathbf{k}_D{}^0),\\\psi(\mathbf{k}_C\mathbf{k}_D)&=\delta(\mathbf{k}_C-\mathbf{k}_C{}^0)\delta(\mathbf{k}_D-\mathbf{k}_D{}^0)\\&\quad+p(E_C+E_D-E_C{}^0-E_D{}^0)\\&\quad\times(\mathbf{k}_C\mathbf{k}_D|\,U|\,\mathbf{k}_C{}^0\mathbf{k}_D{}^0).\end{aligned} \tag{59}$$

The S-matrix for the system is given by

$$S=1+2i\pi U, \tag{60}$$

where

$$U^0=\begin{pmatrix}(\mathbf{k}_A\mathbf{k}_B|\,U^0|\,\mathbf{k}_A{}^0\mathbf{k}_B{}^0) & (\mathbf{k}_A\mathbf{k}_B|\,U^0|\,\mathbf{k}_C{}^0\mathbf{k}_D{}^0)\\(\mathbf{k}_C\mathbf{k}_D|\,U^0|\,\mathbf{k}_A{}^0\mathbf{k}_B{}^0) & (\mathbf{k}_C\mathbf{k}_D|\,U^0|\,\mathbf{k}_C{}^0\mathbf{k}_D{}^0)\end{pmatrix}. \tag{61}$$

$\mathbf{k}_A$, $\mathbf{k}_B$, $\cdots$ are the momenta of the particles A, B, $\cdots$, respectively. U^0 is the submatrix of U with the initial and final states having the same total energy and momentum. Since the S-matrix is unitary, we may apply an unitary transformation to transform it into the following normal form

$$\bar{S}=u^+Su, \tag{62}$$

where

$$\bar{S}=\begin{pmatrix}(\mathbf{k}_A\mathbf{k}_B|\,\bar{S}_1|\,\mathbf{k}_A{}^0\mathbf{k}_B{}^0) & 0\\0 & (\mathbf{k}_C\mathbf{k}_D|\,\bar{S}_2|\,\mathbf{k}_C{}^0\mathbf{k}_D{}^0)\end{pmatrix}, \tag{63}$$

where $(\mathbf{k}_A,\mathbf{k}_B|\,\bar{S}_1|\,\mathbf{k}_A{}^0,\mathbf{k}_B{}^0)$ and $(\mathbf{k}_C,\mathbf{k}_D|\,\bar{S}_2|\,\mathbf{k}_C{}^0,\mathbf{k}_D{}^0)$ are two unitary matrices, and u is the unitary matrix given by

$$u=\begin{pmatrix}(\mathbf{k}_A\mathbf{k}_B|\,u_{11}|\,\mathbf{k}_A{}^0\mathbf{k}_B{}^0) & (\mathbf{k}_A\mathbf{k}_B|\,u_{12}|\,\mathbf{k}_C{}^0\mathbf{k}_D{}^0)\\(\mathbf{k}_C\mathbf{k}_D|\,u_{21}|\,\mathbf{k}_A{}^0\mathbf{k}_B{}^0) & (\mathbf{k}_C\mathbf{k}_D|\,u_{22}|\,\mathbf{k}_C{}^0\mathbf{k}_D{}^0)\end{pmatrix}, \tag{64}$$

u^+ is the adjoint of u. If we use the center of mass coordinate system, we have $-\mathbf{k}_A=\mathbf{k}_B$, $-\mathbf{k}_C=\mathbf{k}_D$. Therefore all the states can be labeled by a single variable $\mathbf{k}_A$ or $\mathbf{k}_C$. $\bar{S}_1$ and $\bar{S}_2$ can further be transformed into diagonal form by using the representation of spherical harmonics. Let u_l, S_l, and $\bar{S}_l$ be the submatrices of u, S, and $\bar{S}$ when the angular momentum has a definite value l. The general form of u and S can then be written

$$u_l=\begin{pmatrix}\cos\alpha_l e^{i\beta_l} & \sin\alpha_l e^{i\beta_l}\\-\sin\alpha_l e^{-i\beta_l} & \cos\alpha_l e^{-i\beta_l}\end{pmatrix}\quad \bar{S}_l=\begin{pmatrix}\bar{S}_{l1} & 0\\0 & \bar{S}_{l2}\end{pmatrix}, \tag{65}$$

where α_l and β_l are two arbitrary parameters and $\bar{S}_{l1}$ and $\bar{S}_{l2}$ are pure numbers. S_l is thus given by

$$S=\begin{pmatrix}\cos^2\alpha\bar{S}_1+\sin^2\alpha\bar{S}_2 & \sin\alpha\cos\alpha e^{2i\beta}(-\bar{S}_1+\bar{S}_2)\\\sin\alpha\cos\alpha e^{-2i\beta}(-\bar{S}_1+\bar{S}_2) & \sin^2\alpha\bar{S}_1+\cos^2\alpha\bar{S}_2\end{pmatrix}. \tag{66}$$

$\bar{S}_1$ and $\bar{S}_2$ may be considered as representing two normal modes of the system in analogy to the system of coupling oscillators. As is seen from (63), each normal mode represents a pure scattering system. We can, therefore, determine $\bar{S}_1$ and $\bar{S}_2$ by applying the theory developed in the last two sections. We obtain

$$\begin{aligned}\bar{S}_1&=e^{ic_1k_A}\prod_\lambda\frac{(k_A-k_{A\lambda}{}^*)}{(k_A-k_{A\lambda})}\\&\qquad\prod_s\frac{(k_A-k_{As}{}^*)(k_A+k_{As}{}^*)}{(k_A-k_{As})(k_A+k_{As})},\\\bar{S}_2&=e^{ic_2k_C}\prod_\lambda\frac{(k_c-k_{c\lambda}{}^*)}{(k_c-k_{c\lambda})}\prod_s\frac{(k_c-k_{cs}{}^*)(k_c-k_{cs}{}^*)}{(k_c-k_{c\lambda})(k_c+k_{cs})},\end{aligned} \tag{67}$$

with $-k_A=k_B$, and $-k_C=k_D$ in the center of mass system. The reaction cross section is given by

$$\begin{aligned}
Q_l^{AB\to AB}&=(\pi/k_A^2)\,|\,1-(k_A\,|\,S_l\,|\,k_A)\,|^2\\
&=(\pi/k_A^2)\,|\,1-\cos^2\alpha\bar{S}_1-\sin^2\alpha\bar{S}_2\,|^2,\\
Q_l^{AB\to CD}&=(\pi/k_A^2)\,|\,1-(k_c\,|\,S_l\,|\,k_A)\,|^2\\
&=(\pi/k_A^2)\,|\,1-\sin\alpha\cos\alpha e^{2i\beta}\\
&\qquad\times(-\bar{S}_1+\bar{S}_2)\,|^2,\\
Q_l^{CD\to CD}&=(\pi/k_c^2)\,|\,1-(k_c\,|\,S_l\,|\,k_c)\,|^2\\
&=(\pi/k_c^2)\,|\,1-\sin^2\alpha\bar{S}_1-\cos^2\alpha\bar{S}_2\,|^2,\\
Q_l^{CD\to AB}&=(\pi/k_c^2)\,|\,1-(k_A\,|\,S_l\,|\,k_c)\,|^2\\
&=(\pi/k_c^2)\,|\,1-\sin\alpha\cos\alpha e^{-2i\beta}\\
&\qquad\times(-\bar{S}_1+\bar{S}_2)\,|^2. \qquad (68)
\end{aligned}$$

The cross sections are completely determined by the constant parameters α and β, the energy levels, the line breadth, and the value of c_1 and c_2 of the two equivalent systems. The values of these constants cannot be given in a general theory of resonance without knowledge of the interaction potentials of the particular system. Their theoretical determination will be very tedious or even impossible for a complicated system. It is usually more practical to determine these constants empirically from experiments. Equation (68) gives only the cross sections when the incident wave has a definite total angular momentum l. The total cross section when the incident wave is a plane wave is given by

$$Q=\sum_l(2l+1)Q_l. \qquad (69)$$

Finally, the author wishes to express his thanks to Professor C. Møller for his kind interest.

APPENDIX

We shall now show that $S(k)$ can only have singularities of the form

$$e^{ic/(k-k_0)} \qquad (70)$$

on the real k axis, where c is a negative real constant. The general form of the function $F(k)$, which has a singularity at $k=k_0$, k_0 being real, and satisfies the unitary condition $F^*(k)F(k)=1$ for real k, is

$$F(k)=\exp\left\{i\sum_{n=1}^{\infty}[C_n/(k-k_0)^n]\right\}, \qquad (71)$$

where c_n are real constants. To identify $F(k)$ with $S(k)$ we have further to satisfy the condition (13) of Section 2, which requires that the integral

$$\oint_c dkS(k)e^{ik(r+r')}, \qquad (72)$$

which appears on the left-hand side of (14), obtained by deforming the path of integration towards increasing imaginary value of k in passing from (13) to (14), should vanish separately. $\oint_c$ means the integration along a closed path starting from $k=k_0-a$ (a being real and positive) to $k=k_0+a$ along the real k axis, and then returning to $k=k_0-a$ along a half-circle above the real axis of radius a with the center at $k=k_0$. For a given value of a we can find an integer N such that the value of the series

$$\sum_{n=1}^{\infty}C_n/(k-k_0)^n$$

on this circle may be replaced by

$$\sum_{n=1}^{N}C_n/(k-k_0)^n$$

with a negligible error. The integration along the real k axis gives a contribution of the order of a since $|S(k)|=1$ on the real axis. The condition for the vanishing of (72) becomes therefore, on neglecting terms of order a when a is very small,

$$\oint_{|k-k_0|=a}dk\exp\left\{i\sum_{n=1}^{N}[C_n/(k-k_0)^n]\right\}\cdot e^{ik(r+r')}\approx 0, \qquad (73)$$

where the integration is taken along the half-circle of radius a. We may deform this path to a new path starting from $k=k_0-a$ to k_0-b (b being positive and real and smaller than a), then along a half-circle above the real axis with center $k=k_0$ to the point $k=k_0+b$, and finally along the real axis from $k=k_0+b$ to $k=k_0+a$. The contribution to the integral from the path along the real axis is again negligible since it is of the order $a-b$. We may choose b so small that the summation

$$\sum_{n=1}^{N}[C_n/(k-k_0)^n]$$

can be replaced by a single term $c_N/(k-k_0)^N$ on the smaller half-circle. Equation (73) then

becomes

$$\int_{|k-k_0|=b} dk \exp\{i[C_N/(k-k_0)^N]\} \times e^{ik(r+r')} \approx 0. \quad (74)$$

The integral (74) for very small values of $|k-k_0|=b$ can be evaluated easily by the method of variational phase. We have

$$\int_{|k-k_0|=b} dk e^{ic_N/(k-k_0)N} e^{ik(r+r')} = e^{ik_0(r+r')} \times \int_0^{\pi} \exp[ic_N(1/b^N)(\cos N\theta - i \sin N\theta)] b e^{i\theta} d\theta$$
$$= 2\pi i e^{ik_0(r+r')} b \sum_l \exp[c_N(1/b^N)(-1)^l] \times \exp[i(2l+1/2N)\pi]. \quad (75)$$

Where the summation is over the range $0<(2l+1)\leqslant 2N$, i.e., from $l=0$ to $l\leqq(2N-1)/2$. Equation (75) will be of the order $\exp(1/b^N)$, which is a very large quantity when b is very small, except when $N=1$ and $c_N<0$. In the latter case (75) will be a very small quantity and thus only in this case can (74) be satisfied. This furnishes the proof that the singularity of $S(k)$ on the real k axis can at most be of the form

$$\exp[ic/(k-k_0)] \qquad (c<0). \quad (76)$$

When k_0 goes to infinity, (73) goes just to (16) of Section 2, and (76) becomes, on putting $c_\infty = c/k_0^2$,

$$e^{ic_\infty k}. \quad (77)$$

PHYSICAL REVIEW VOLUME 74, NUMBER 2 JULY 15, 1948

Scattering of Particles by the Gas in a Synchrotron*

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(Received April 5, 1948)

Successive elastic encounters of a proton with atoms of air in a proton synchrotron may build up a betatron oscillation whose amplitude exceeds the aperture of the vacuum chamber; in this case the particle is lost. The Rutherford scattering formula, suitably cut off at large and small scattering angles, is used to determine the r.m.s. amplitude of this oscillation as a function of the increasing kinetic energy of the particle. It reaches a maximum, given by Eq. (11), when the latter is four times its value at injection. The differential equation is found, which the distribution of amplitudes as a function of path length must obey, and it is solved, by use of appropriate boundary conditions, for the case in which damping of the oscillations is ignored. The solution makes possible an estimate of the fraction of the original particles which is scattered to the wall (Fig. 1). These results are applied to two proposed synchrotrons to determine the air pressures which gives a ten percent loss of particles.

I. INTRODUCTION

THE protons in a proton synchrotron will move through a path of length nearly ct_0 if the acceleration takes place over a time t_0. For $t_0=1$ sec. this is 3×10^{10} cm. Even at a pressure of 10^{-6} mm Hg this is equivalent to about 40 cm of air at atmospheric pressure, and appreciable scattering and loss may occur. Since cross sections for inelastic scattering, which includes all nuclear disintegration processes, are generally smaller by factors of about 10^{-7} than those for elastic or Rutherford scattering, it is necessary only to be sure that the elastic scattering is not serious.

The elastic scattering gives rise to "betatron"

* Work done under the auspices of the Atomic Energy Commission.