Depolarization of Neutrons During Diffusion

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A theoretical analysis is given of the depolarization to be expected when a plane, monochromatic, polarized beam of thermal neutrons is incident normally on a slab of finite thickness in which the neutrons are scattered isotropically and without capture. The problem reduces to the solution of an integro-differential equation of the Wiener-Hopf type. The equation is solved approximately by a method of G. C. Wick. Experimental possibilities for measuring the scattering amplitudes by this method even in the case where neutron-nuclear forces are not very spin-dependent, are excellent. The case where the capture cross section is different from zero is also included.

I. INTRODUCTION

SPIN dependence of the interaction between a neutron and a nucleus manifests itself in the difference of the scattering amplitude when the neutron and nuclear spins are parallel and the amplitude when they are anti-parallel. Inasmuch as a consistent theory of the nucleus should be able to predict this spin dependence, experimental determination of this difference gives us a valuable datum for testing such theories.

Such an investigation has been carried out in the case of the neutron-proton interaction by observing[1–3] the differences of the scattering cross sections for slow neutrons in the ortho and para modifications of the hydrogen molecule, and by making a theoretical analysis of these differences.[4,5] In this case, the large spin dependence of the neutron-proton force results in a substantial difference between the ortho and para cross sections. Decisive information could therefore be deduced without requiring experimental results of extreme accuracy. A similar theoretical analysis[6] for ortho and para deuterium indicates that accuracy of but a few percent in the measurement of the ortho and para cross sections would be necessary, under the best circumstances to get comparable information about the neutron-deuteron force. There is also the possibility if this force is not very spin-dependent, that the best experiment of this type would still be indecisive. For nuclei heavier than deuterium, experiments of this sort would be extremely difficult to perform.

More recently,[7] a determination of parallel and anti-parallel scattering amplitudes has been attempted by comparing the magnitude of the diffuse scattering with the interference peaks obtained when slow neutrons are scattered from crystals such as sodium hydride and sodium deuteride. In this case too, if the scattering amplitudes do not differ by much, the experimental accuracy required is beyond the limits of available techniques.

Another possibility for the experimental determination of the spin dependence of the interaction is to study the depolarizing effect of scattering processes on a beam of polarized neutrons. Experimentally, techniques for obtaining polarized neutron beams and for the analysis of their polarizations are well known.[8,9] Theoretically, the probability that a slow neutron will change its spin orientation in a single collision, has been calculated[10] and is given by:

$$Q=\tfrac{2}{3}(2i+1)^{-1}[(a_1-a_0)^2 i(i+1)]/[(i+1)a_1^2+ia_0^2], \quad (1.1)$$

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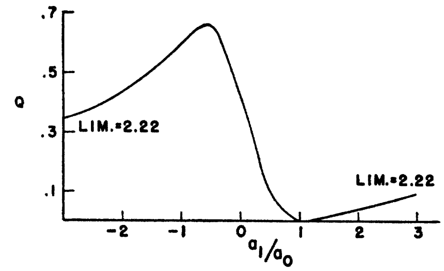

FIG. 1. Depolarizing factor Q as a function of the ratio of the scattering amplitudes when $i=1$.

where $i=$the spin of the interacting nucleus, $a_1=$the scattering amplitude for parallel spin, and $a_0=$the scattering amplitude for anti-parallel spin. $a_1=a_0$, implying no spin-dependence, has as a consequence $Q=0$ or no depolarization. This is clear since interactions which are not spin-dependent will produce no changes in spin orientation.

When several isotopes are present the depolarizing factor[10] is:

$$Q=\frac{2}{3}\sum_p |b_p|^2(2i_p+1)^{-1} \times\frac{(a_1{}^p-a_0{}^p)^2 i_p(i_p+1)}{(i_p+1)(a_1{}^p)^2+i_p(a_0{}^p)^2}, \quad (1.2)$$

where $|b_p|^2$ is determined by the relative abundance of the pth isotope ($\sum_p |b_p|^2=1$). The total scattering cross section[10] is:

$$\sigma_n=4\pi(2i+1)^{-1}[(i+1)a_1{}^2+ia_0{}^2], \quad (1.3)$$

when only one isotope is present, and is given[10] by:

$$\sigma_n=4\pi\sum_p |b_p|^2(2i_p+1)^{-1} \times[(i_p+1)(a_1{}^p)^2+i_p(a_0{}^p)^2], \quad (1.4)$$

when there is more than one isotope. A combination of two measurements, namely total scattering and Q is sufficient to determine the scattering amplitudes when only one isotope is present. This is no longer true when there are several isotopes.

The statement that the scattering amplitudes are determined by two measurements when only one isotope is present must be qualified. In Fig. 1, we have a plot of Q *vs.* a_1/a_0 when $i=1$. Except when a_1/a_0 is equal to 1 or $-\frac{1}{2}$, a determination of Q does not determine a_1/a_0 uniquely. In addition, the absolute sign of either amplitude cannot be found from these two measurements. In fact, no experiment in which the spin of the scattering nucleus is arbitrarily oriented in space will yield more information.[6]

If a_1-a_0 is small, the resultant depolarization for a single collision may not be measurable. It has been suggested,[6] therefore, that the cumulative effect of several depolarizing collisions be observed in order to determine the value of Q. This paper is concerned with the theoretical analysis of such an experiment. This technique will be most valuable when Q is in the neighborhood of zero, i.e., when a_1/a_0 is approximately 1. It is only in this region that our method is of interest. For a_1/a_0 considerably different from one, the depolarization in a single scattering process will be sufficiently large to enable determination of Q. In Fig. 2 we show the behavior of Q in this region for $i=1$.

II. FORMULATION OF THE PROBLEM

Specifically, we shall consider the following situation: a plane monochromatic beam of completely polarized (spins all pointing in some direction, p) slow neutrons, traveling in the positive z direction, is incident at $z=0$ on an infinite slab of amorphous scattering material perpendicular to the z axis and of thickness h. The neutrons are assumed to be scattered isotropically, in the laboratory system of coordinates. They are sufficiently slow so that no inelastic collisions are possible. We assume further, that no energy changes occur on collision, and for the moment we shall neglect capture. Physically this corresponds to a situation in which we are dealing with the interaction between a thermal neutron beam and a heavy nucleus, or with the interaction with a light nucleus such as deuterium which is combined in a heavy molecule whose other elements do not contribute to the scattering.

As the neutrons diffuse through the scatterer, the beam will become partially depolarized. In the material there will be two competing processes in the depolarization: Those neutrons whose spins point in the negative p direction will have a probability Q for spin reversal on collision, with a resultant increase in the polarization. Those neutrons whose spins are oppositely directed, will undergo depolarizing collisions with the same probability. At the outset, it should be made clear that if the scatterer were infinitely thick, the emergent beam would be completely depolarized no matter what the value of Q since the depolarization process tends to equalize the populations of the two spin states. Thus, it is essential to consider a scatterer of finite thickness. Let:

$N^+(z,\mu)$ = the number of neutrons per unit volume whose spins point in the positive p direction and are traveling in a direction given by $\mu=\cos\theta$, in the angular range $d\mu$ (θ is the angle made with the positive z axis).

$N^-(z,\mu)$ = the number of neutrons per unit volume whose spins point in the negative p direction and are traveling in the direction given by μ, in the angular range $d\mu$.

V = the neutron speed.

l = the scattering mean free path.

If the time of collision is small compared to the time between collisions, then the transport equation for this process yields the following:

$$\frac{dN^+}{dt}=-\frac{V}{l}N^+(z,\mu,t)+\frac{V}{2l}\int_{-1}^{+1}N^+(1-Q)d\mu'+\frac{V}{2l}\int_{-1}^{+1}N^-Qd\mu', \quad (2.1)$$

and

$$\frac{dN^-}{dt}=-\frac{V}{l}N^-(z,\mu,t)+\frac{V}{2l}\int_{-1}^{+1}N^-(1-Q)d\mu'+\frac{V}{2l}\int_{-1}^{+1}N^+Qd\mu'. \quad (2.2)$$

The factor $\frac{1}{2}$ which appears before the integral sign is a normalization factor for the isotropic scattering function $1/4\pi$ and is determined by the condition

$$1/4\pi\int_0^{2\pi}\int_{-1}^{+1}Nd\mu'd\phi=1.$$

We consider a steady state and for this case, we have:

$$\mu\frac{\partial N^+}{\partial z}+\frac{1}{l}N^+=\frac{1}{2l}\int_{-1}^{+1}N^+(1-Q)d\mu+\frac{1}{2l}\int_{-1}^{+1}N^-Qd\mu', \quad (2.3)$$

and

$$\mu\frac{\partial N^-}{\partial z}+\frac{1}{l}N^-=\frac{1}{2l}\int_{-1}^{+1}N^-(1-Q)d\mu'+\frac{1}{2l}\int_{-1}^{+1}N^+Qd\mu'. \quad (2.4)$$

The problem reduces to the solution of a coupled system of integro-differential equations with appropriate boundary conditions. Equations (2.3) and (2.4) may be simplified by adding and subtracting them to give:

$$\mu\frac{\partial\sigma}{\partial z}+\sigma=\frac{1}{2}\int_{-1}^{+1}\sigma d\mu', \quad (2.5)$$

$$\mu\frac{\partial\Delta}{\partial z}+\Delta=\frac{\alpha}{2}\int_{-1}^{+1}\Delta d\mu', \quad (2.6)$$

where we have introduced the mean free path 1, as the unit of length and set

$$\sigma=N^++N^-,$$
$$\Delta=N^+-N^-,$$
$$\alpha=1-2Q.$$

Equations (2.5) and (2.6) are integro-differential equations of the Wiener-Hopf type. They arise from problems dealing with the radiative equilibrium of stellar atmospheres.[11] In most prob-

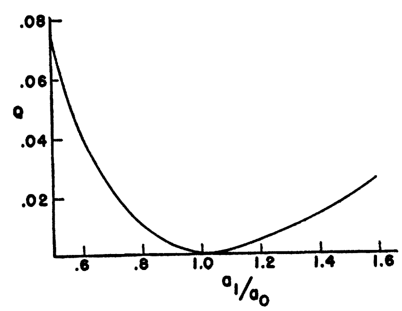

FIG. 2. Depolarizing factor Q as a function of the ratio of the scattering amplitudes when $i=1$, and the ratio $=1$.

[11] E. Hopf, "Mathematical problems of radiative equilibrium," *Cambridge Tracts*, No. 31 (1934); N. Wiener and E. Hopf, Berliner Ber. Math. Phys. Klasse, 696 (1931).

lems of interest to the astrophysicist the atmosphere is taken to have an infinite depth. As pointed out in I, we are obliged to deal with scatterers of finite thickness.

In order to facilitate the application of our special boundary conditions, we introduce into (2.5) and (2.6) the following transformations:

$$\sigma = J + \delta(\mu - 1)e^{-z/\mu}, \quad (2.7)$$

$$\Delta = I + \delta(\mu - 1)e^{-z/\mu}. \quad (2.8)$$

$\delta(\mu) =$ Dirac delta function; and

$$\int_{-1}^{+1} f(\mu')\delta(\mu' - \beta)d\mu' = f(\beta).$$

This gives for Eqs. (2.5) and (2.6)

$$\mu\frac{\partial J}{\partial z} + J = \frac{1}{2}\int_{-1}^{+1} J d\mu' + \tfrac{1}{2}e^{-z}, \quad (2.9)$$

and

$$\mu\frac{\partial I}{\partial z} + I = \frac{\alpha}{2}\int_{-1}^{+1} I d\mu' + \frac{\alpha}{2}e^{-z}. \quad (2.10)$$

Inasmuch as most detectors for slow neutrons measure neutron densities rather than flux, we will express our principal results in these terms. We will indicate later how these expressions are changed to represent the flux. Accordingly we define the polarization density as

$$P(z) = \int_{-1}^{+1} \Delta d\mu' \bigg/ \int_{-1}^{+1} \sigma d\mu', \quad (2.11)$$

and are looking for $P(d)$, where $d =$ thickness of the scatterer in units of mean free path $(= h/l)$.

Our boundary conditions are:

at $z = 0$;

$$\left.\begin{matrix} \sigma(0, \mu) = \delta(\mu - 1); & J(0, \mu) = 0 \\ \Delta(0, \mu) = \delta(\mu - 1); & I(0, \mu) = 0 \end{matrix}\right\} \mu > 0; \quad (2.12)$$

at $z = d$;

$$\left.\begin{matrix} \sigma(d, \mu) = 0; & J(d, \mu) = 0 \\ \Delta(d, \mu) = 0; & I(d, \mu) = 0 \end{matrix}\right\} \mu < 0. \quad (2.13)$$

The boundary condition (2.12) is an expression of the fact that a completely polarized plane beam is incident normally on the scatterer. Condition (2.13) states that after transmission, the vacuum returns no neutrons to the scatterer.

III. SOLUTION OF THE PROBLEM

The exact solution of the Wiener-Hopf equation for a finite thickness of scatterer has not yet been found. Inasmuch as our problem will not lead to significant results unless the slab is finite, we must resort to an approximate method of solving (2.9) and (2.10). The method we have used is one originally proposed by G. C. Wick[12] and which has been subsequently used and developed by S. Chandrasekhar.[13] The principle of the method is to replace the integral which appears on the right side of (2.9) and (2.10) by a polynomial, and thus reduce the integro-differential equation to a system of first-order differential equations. The special polynomial used is the one originally proposed by Gauss[14] in his quadrature method of approximating integrals:

$$\int_{-1}^{+1} I(z, \mu')d\mu' \cong \sum_{\substack{i=-n \\ n\neq 0}}^{n} a_i I(z, \mu_i) = \sum_{\substack{i=-n \\ n\neq 0}}^{n} a_i I_i. \quad (3.1)$$

The coefficients are the Christoffel[15] numbers which satisfy the conditions:

$$\sum_{i=1}^{n} a_i = 1; \quad a_i = a_{-i}. \quad (3.2)$$

The μ_i is determined in the nth approximation as solution of the equation $P_{2n}(\mu) = 0$, where $P_{2n}(\mu)$ is the spherical harmonic of order $2n$. The solution of Eq. (2.10) by this method is found by letting

$$I_i = g_i e^{-\lambda z}, \quad (3.3)$$

for the homogeneous part and adding the particular solution

$$I_i = (\alpha/2)\Gamma e^{-z}/(1 - \mu_i), \quad (3.4)$$

where

$$\Gamma = 1 \bigg/ \left(1 - \alpha \sum_{i=1}^{n} \frac{a_i}{1 - \mu_i^2}\right). \quad (3.5)$$

The complete approximate solution for I_i is

$$I_i = \sum_{p=1}^{n} \frac{L_p e^{-\lambda_p z}}{1 - \mu_i \lambda_p} + \sum_{p=1}^{n} \frac{L_{-p} e^{\lambda_p z}}{1 + \mu_i \lambda_p} + \frac{\alpha}{2}\frac{\Gamma e^{-z}}{1 - \mu_i}, \quad (3.6)$$

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[14] Riemann-Weber, *Diff. Gleichungen der Physik*, I (Dover Publications, New York), p. 394.

[15] A. N. Lowan, N. Davids, and A. Levenson, Bull. Am. Math. Soc. **48**, 739 (1942).

where λ_p is determined by the equation

$$\alpha\sum_{i=1}^{n}\frac{a_i}{1-\mu_i^2\lambda^2}=1. \tag{3.7}$$

Equation (2.9) is solved in much the same way, except that in this case, (when $\alpha=1$) $\lambda=0$ identically satisfies (3.7) because of (3.2). The solution in this case is

$$J_i=b\left\{\sum_{p=1}^{n-1}\frac{M_pe^{-\zeta_pz}}{1-\mu_i\zeta_p}+\sum_{p=1}^{n-1}\frac{M_{-p}e^{\zeta_pz}}{1+\mu_i\zeta_p}+z-\mu_i+R\right\}+\frac{1}{2}\frac{\gamma e^{-z}}{1-\mu_i}, \tag{3.8}$$

where

$$\gamma=1\bigg/\left(1-\sum_{i=1}^{n}\frac{a_i}{1-\mu_i^2}\right), \tag{3.9}$$

and ζ_p is a solution of the equation

$$\sum_{i=1}^{n}a_i/(1-\mu_i^2\zeta^2)=1, \tag{3.10}$$

which is different from zero.

The boundary conditions (2.12) and (2.13) now give the following system of linear equations for the determination of L_p, L_{-p}, M_p, M_{-p}, b and R.

$$\sum_{p=1}^{n}\frac{L_p}{1-\mu_i\lambda_p}+\sum_{p=1}^{n}\frac{L_{-p}}{1+\mu_i\lambda_p}+\frac{\alpha}{2}\frac{\Gamma}{1-\mu_i}=0 \quad (i=1,2\cdots n); \tag{3.11}$$

$$\sum_{p=1}^{n}\frac{L_pe^{-\lambda_pd}}{1+\mu_i\lambda_p}+\sum_{p=1}^{n}\frac{L_{-p}e^{\lambda_pd}}{1-\mu_i\lambda_p}+\frac{\alpha}{2}\frac{\Gamma e^{-d}}{1+\mu_i}=0 \quad (i=1,2\cdots n); \tag{3.12}$$

$$b\left\{\sum_{p=1}^{n-1}\frac{M_p}{1-\mu_i\zeta_p}+\sum_{p=1}^{n-1}\frac{M_{-p}}{1+\mu_i\zeta_p}-\mu_1+R\right\}+\frac{1}{2}\frac{\gamma}{1-\mu_i}=0 \quad (i=1,2\cdots n); \tag{3.13}$$

$$b\left\{\sum_{p=1}^{n-1}\frac{M_pe^{-\zeta_pd}}{1+\mu_i\zeta_p}+\sum_{p=1}^{n-1}\frac{M_{-p}e^{\zeta_pd}}{1-\mu_i\zeta_p}+d+\mu_i+R\right\}+\frac{1}{2}\frac{\gamma e^{-d}}{1+\mu_i}=0 \quad (i=1,2\cdots n). \tag{3.14}$$

IV. CALCULATIONS

The particular virtue of the method outlined above for approximating the solution of the Wiener-Hopf equation is that the convergence is so rapid. In most cases, in dealing with an infinite scattering atmosphere the second approximation gives results within 1 percent accuracy. In estimating the accuracy of our solutions in the various approximations, it was not possible to compare them with an exact solution. We have assumed that convergence to the exact solution was uniform, and if the difference between the nth and $n+1$th approximation is sufficiently small, we consider the $n+1$th result to be exact.

A. First Approximation

If one writes Eq. (3.3) in the first approximation, one has:[12]

$$\frac{1}{\sqrt{3}}(\partial I_1/\partial z)+I_1=\frac{\alpha}{2}I_1+\frac{\alpha}{2}I_{-1}, \tag{4.1}$$

$$-\frac{1}{\sqrt{3}}(\partial I_{-1}/\partial z)+I_{-1}=\frac{\alpha}{2}I_1+\frac{\alpha}{2}I_{-1}. \tag{4.2}$$

The total density is $I=I_1+I_{-1}$. I_1 may be considered to be the density moving to the right and I_{-1} may be considered that moving to the left. One can eliminate I_{-1} from (4.1) by differentiating with respect to z, then substituting for $\partial I_{-1}/\partial z$ from 4.2 and for I_{-1} from (4.1). In the same way one can eliminate I_1 and one gets for I, I_1, and I_{-1} the following:

$$(\tfrac{1}{3})\partial^2I/\partial z^2+(\alpha-1)I=0, \tag{4.3}$$

which is the diffusion equation.

To gain some insight into the applicability of the first approximation, one starts with an exact equation such as (2.6) and expands the unknown function in spherical harmonics.

$$I=\frac{1}{4\pi}\sum_n(2n+1)P_n(\mu)I_n(z), \tag{4.4}$$

(we call the variable I instead of Δ). Resubstituting (4.4) into (2.6) we get:

$$\sum_n(2n+1)\mu P_n(\mu)\frac{\partial I_n}{\partial z}+\sum_n(2n+1)P_n(\mu)I_n=\alpha I_0. \tag{4.5}$$

TABLE I. Christoffel numbers and zeros of the spherical harmonics used in the various approximations.

First approximation		Third approximation	
a_1	1	a_1	0.467914
μ_1	0.577350	a_2	0.360762
Second approximation		a_3	0.171324
a_1	0.652145	μ_1	0.238619
a_2	0.347855	μ_2	0.661209
μ_1	0.339981	μ_3	0.932470
μ_2	0.861136		

First we integrate (4.5) from -1 to $+1$, and get

$$\partial I_1/\partial z + I_0 = \alpha I_0, \tag{4.6}$$

then we multiply by μ and integrate from -1 to $+1$. Then this gives:

$$2\partial I_2/\partial z + \partial I_0/\partial z + 3\partial I_1/\partial z = 0, \tag{4.7}$$

since $\mu^2 = (2P_2 + P_0)/3$. Neglecting I_2 and substituting for I_1 from (4.7) into (4.6), we have:

$$\tfrac{1}{3}(\partial^2 I_0/\partial z^2) + (\alpha - 1)I_0 = 0, \tag{4.8}$$

which is the diffusion equation again. Thus we see that if the distribution function is spherically symmetric, the Wiener-Hopf equation corresponds to the diffusion equation. Attempting to solve (4.8) instead of (2.8) means that we are assuming that the contributions to the solution of the higher terms in the expansion in spherical harmonics are negligible. This assumption corresponds exactly to using the first approximation in our approximate method.

For the particular boundary conditions that we are considering here, the distribution at $z=0$ is far from spherically symmetric. After several collisions it will be closer to spherical symmetry. We may expect therefore that the true transmitted density will be given more closely by the first approximation, the thicker the scatterer.

B. Calculation of Experimental Quantities

Having solved Eqs. (3.10) through (3.13) it remains to be seen how the experimentally measured quantities, i.e.,

$$\Delta_0(d) = \int_0^1 \Delta(d, \mu')d\mu',$$

and

$$\sigma_0(d) = \int_0^{+1} \sigma(d, \mu')d\mu',$$

depend on the values of L_p, L_{-p}, M_p, M_{-p}, b and R. By Eqs. (2.8), (3.6), (3.7), (3.5), (3.1), and (3.2) we have:

$$\begin{aligned}
\Delta_0(d) &= \int_0^1 \Delta(d, \mu')d\mu' = \int_{-1}^{+1} I d\mu' \\
&\quad + \int_{-1}^{+1} \delta(\mu'-1)e^{-z/\mu'}d\mu' \\
&\cong \sum_{\substack{i=-n \\ n\neq 0}}^{n} a_i \sum_{p=1}^{n} \frac{L_p e^{-\lambda_p d}}{1-\mu_i\lambda_p} \\
&\quad + \sum_{\substack{i=-n \\ n\neq 0}}^{n} a_i \sum_{p=1}^{n} \frac{L_{-p} e^{\lambda_p d}}{1+\mu_i\lambda_p} + \frac{\alpha}{2} \sum_{\substack{i=-n \\ n\neq 0}}^{n} \frac{a_i \Gamma e^{-d}}{1-\mu_i} \\
&= 2\sum_{p=1}^{n} L_p e^{-\lambda_p d} \sum_{i=1}^{n} \frac{a_i}{1-\mu_i^2\lambda_p^2} \\
&\quad + 2\sum_{p=1}^{n} L_{-p} e^{\lambda_p d} \sum_{i=1}^{n} \frac{a_i}{1-\mu_i^2\lambda_p^2} \\
&\quad + \alpha\Gamma e^{-d} \sum_{i=1}^{n} \frac{a_i^2}{1-\mu_i^2}.
\end{aligned}$$

$$\Delta_0(d) = \frac{2}{\alpha}\Big\{\sum_{p=1}^{n} (L_p e^{-\lambda_p d} + L_{-p} e^{\lambda_p d})\Big\} + \Gamma e^{-d}. \tag{4.9}$$

In a similar way we find that:

$$\begin{aligned}
\sigma_0(d) &= \int_0^1 \sigma(d, \mu')d\mu' \\
&= 2b\Big\{\sum_{p=1}^{n-1} (M_p e^{-\zeta_p d} + M_{-p} e^{\zeta_p d}) \\
&\qquad + d + R\Big\} + \gamma e^{-d}.
\end{aligned} \tag{4.10}$$

For the sake of completeness, we include expres-

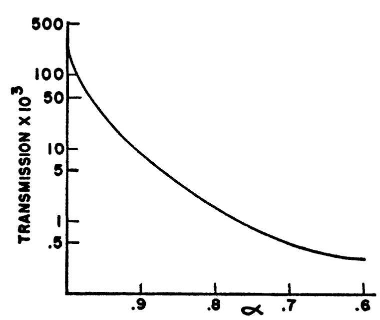

FIG. 3. Transmitted density as a function of $1-2Q$ in the second approximation when $d=10$ (mean free paths).

sions for the albedo (reflection coefficient) from a finite plate:

$$\Delta_0(0)=\int_{-1}^{0}\Delta(0,\,\mu')d\mu'$$

$$=\frac{2}{\alpha}\Big[\sum_{p=1}^{n}(L_p+L_{-p})\Big]+\Gamma-1, \quad (4.11)$$

$$\sigma_0(0)=\int_{-1}^{0}\sigma(0,\,\mu')d\mu'$$

$$=2b\Big[\sum_{p=1}^{n-1}(M_p+M_{-p})+R\Big]+\gamma-1. \quad (4.12)$$

If a deep detector is used one would measure currents rather than densities. We therefore include the currents corresponding to Eqs. (4.9) to (4.12).

$$\int_0^1\sigma(d,\,\mu')\mu'd\mu'=-\tfrac{1}{3}b; \quad (4.13)$$

$$\int_{-1}^{0}\sigma(0,\,\mu')\mu'd\mu'=-\tfrac{1}{3}b-1; \quad (4.14)$$

$$\int_0^1\Delta(d,\,\mu')\mu'd\mu'=\left(\frac{2}{\alpha}-2\right)\times\left\{\sum_{p=1}^{n}\left(\frac{L_pe^{-\lambda_pd}}{\lambda_p}-\frac{L_{-p}e^{\lambda_pd}}{\lambda_p}\right)\right\}+(1-\alpha)\Gamma e^{-d}. \quad (4.15)$$

$$\int_{-1}^{0}\Delta(0,\,\mu')\mu'd\mu'=\left(\frac{2}{\alpha}-2\right)\times\left\{\sum_{p=1}^{n}\left(\frac{L_p}{\lambda_p}-\frac{L_{-p}}{\lambda_p}\right)\right\}+(1-\alpha)\Gamma-1. \quad (4.16)$$

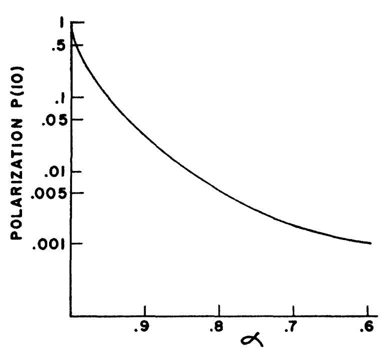

FIG. 4. Polarization as a function of $1-2Q$ in the second approximation for $d=10$ (mean free paths).

TABLE II. Solution of the equation $\alpha\sum_{i=1}^{n}a_i/1-\mu_i^2\lambda^2=1$ for various α's.

1st approx.		2nd approx.		3rd approx.		
α	λ_1	λ_1	λ_2	λ_1	λ_2	λ_3
1		1.972027	—	3.202945	—	1.225211
0.95	0.387298	2.01257	0.379497	3.24943	0.37948	1.23255
0.9	0.547723	2.05519	0.525560	3.29669	0.52500	1.24087
0.8	0.774597	2.14577	0.711876	3.39312	0.71045	1.26096
0.7	0.948686	2.24183	0.834508	3.49150	0.829085	1.28609
0.6	1.09545	2.34138	0.922637	3.591195	0.900563	1.31067

C. Results

Using the values tabulated in Tables I and II, the transmitted density was calculated in the second approximation for $d=10$ (mean free paths), from $\alpha=0.6$ to $\alpha=1.0$. Formulas (4.9) and 4.10 were used for this calculation after having found L_p, L_{-p}, M_p, and M_{-p}, by solving (3.10)–(3.13). The values of Γ and γ are found from Eqs. (3.7) and (3.9). The results are shown in Fig. 3. The polarization $P(10)$ as defined by (2.11), if found in the case of no capture, by dividing the value of $\Delta_0(10)$ for a particular α, by the value of $\sigma_0(10)[=299\times10^{-3}]$. Thus the value of $P(10)$ for $\alpha=0.9$, is found by dividing 9.33×10^{-3} by 299×10^{-3}. This depolarization curve is shown in Fig. 4.

To test the accuracy of the second approximation for this thickness, both the diffusion and third approximation were calculated for $\alpha=0.9$ and compared.

	1st approx.	2nd approx.	3rd approx.
$\Delta_0(10)$	9.21×10^{-3}	9.334×10^{-3}	9.337×10^{-3}

It is thus seen (assuming uniform convergence as indicated above), that the second approximation is excellent, and the diffusion approximation more than adequate. The error in the first approximation is approximately given by $\Delta\alpha/\alpha=0.005$ percent which corresponds to an error in Q of ~0.05 percent if we assume that $Q\sim0.05$.

These results point to excellent experimental possibilities since the beam is depolarized by a factor of 30 when Q changes from zero to 0.05. This involves, however, the measurement of a neutron density after the polarized beam intensity has been reduced by a factor of 10,000. If a strong neutron source is not available, it may be advisable to reduce the thickness of the scatter-

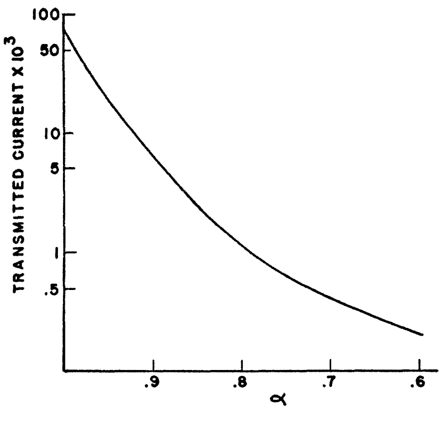

FIG. 5. Transmitted current as a function of $1-2Q$, second approximation, $d=10$ (mean free paths).

ing slab. This will increase the transmitted density, but at a sacrifice in the amount of depolarization.

To check the accuracy of the various approximations for the smaller thicknesses, we have calculated the transmitted densities in the first and third approximations for $d=5$. The results are:

	1st approx.	2nd approx.	3rd approx.
$\Delta_0(5)$	0.133	0.1249	0.1253

As is to be expected, the diffusion approximation is not as good as it was for $d=10$. In fact it is no longer adequate since the error in α is now approximately 0.25 percent corresponding to an error in Q of 2.5 percent again assuming that $Q\sim 0.05$. The second approximation still seems to be remarkably good. In Table III, we have included a summary of some typical results which can be evaluated on the basis of the calculation which we have made.

For the sake of completeness, we have included too the curves in the second approximation for

TABLE III. Summary of calculations for 5 and 10 mean free paths for two typical values of Q.

Q	Polarization	Calculation accuracy 1st approx. (%)	Calculation accuracy 2nd approx. (%)	% error in Q when error in measurement of P is 1%	Intensity down approx.
		5 mean free paths			
0.025	0.62	—	—	16.3	330
0.05	0.27	2.5	0.06	3.7	330
		10 mean free paths			
0.025	0.16	1.2	0.001	1.9	10,000
0.05	0.03	0.05	0.001	0.28	10,000

the transmitted current (Fig. 5), the reflected density (Fig. 6), and the reflected current (Fig. 7).

It may be remarked at this time that although we have neglected capture in this analysis, this restriction is by no means necessary. The effect of including capture is to change the fundamental Eqs. (2.7) and (2.8) as follows: in (2.7) the factor $\frac{1}{2}$ on the right is changed to $\beta/2$,[16] where β is the ratio of the total mean free path (capture $+$scattering) to the scattering mean free path; in Eq. (2.6) the factor α is changed to $\epsilon=(1-2Q)\beta$. Figure 3 can still be used to draw a new polarization curve. For example when $\beta=0.99$ and $Q=0.05$, the transmitted density for $\beta=0.99$ is read from Fig. 4 to be 160×10^{-3}. The transmitted density for $\epsilon=0.99(1-0.1)=0.891$ is again read from Fig. 3 and is equal to 8×10^{-3}. The polarization is therefore $8/160=0.05$ when $Q=0.05$. The rest of the curve is plotted the same way. It should be noted that distances are now measured in terms of total mean free paths, $1/l_{\text{tot.}}=1/l_{\text{cap.}}+1/l_{\text{scatt.}}$.

V. SUMMARY AND CONCLUSIONS

We have described and analyzed an experiment suitable for the measurement of the difference between the two scattering amplitudes in the neutron-nucleus interaction. The techniques required for the experiment are the production and analysis of polarized neutron beams. The accuracy of determination of a_1-a_0 (or a_1/a_0) will depend on the accuracy of measuring polari-

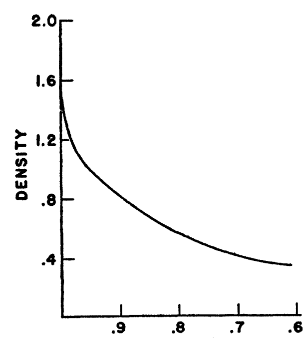

FIG. 6. Reflected density as a function of $1-2Q$, second approximation, $d=10$ (mean free paths).

[16] O. Halpern, R. Luneburg, and O. Clarke, Phys. Rev. **53**, 173 (1938).

zations of neutron beams of relatively low density, and on the magnitude of the depolarizing factor Q.

Referring to Table III, we can see that for each value of Q, which must be determined approximately by some previous experiment, there will be an optimum value of scattering thickness. On the one hand, the thickness must not be so large as to completely depolarize the beam. On the other hand, the thickness must not be so small that a small error in the measurement of the depolarization will result in a large error in Q.

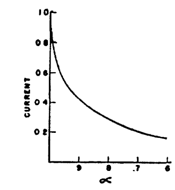

FIG. 7. Reflected current as a function of $1-2Q$, second approximation, $d=10$ (mean free paths).

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A Direct Determination of the Energy of the He³ Nucleus from the D−D Reaction

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The energy of the He^3 nuclei emitted at 90° to the incident deuteron beam in the D−D reaction has been measured directly by deflecting them through a 90°, 15-cm radius, cylindrical electrostatic analyzer. The He^3 ions were detected by allowing them to eject secondary electrons from the first plate of a 12-element electron multiplier tube. Thus the ions were not required to traverse any foil or window material between the heavy ice target and the point of their detection.

The corrections to be applied to the electrostatic analyzer were experimentally investigated and when applied to the kinetic energies deduced from the observed critical deflecting voltages, give a Q value of 3.30 ± 0.01 Mev for this reaction.

INTRODUCTION

THE D−D reaction yielding a neutron and a He^3 nucleus has been studied by observing the recoil He^3 nuclei obtained when a beam of deuterons impinges on a thick target of D_2O ice. The energy of the He^3 ions emitted at 90° to the incident deuteron beam was measured by deflecting them 90° with an electrostatic analyzer, and the ions were counted with an electron multiplier tube* of the type developed by J. S. Allen.[1] From the knowledge of the energy of the incident deuteron and the energy of the He^3 nucleus emitted at 90°, one has from conservation of energy and momentum the energy released in the reaction

$$D^2+D^2\rightarrow He^3+n+Q, \quad (1)$$

$$Q=E_D+4E_{He}, \quad (2)$$

where Q is the energy released, E_D is the kinetic energy of the incident deuteron, and E_{He} is the kinetic energy of the He^3 nucleus emitted at 90° in the laboratory system.

APPARATUS

The deuteron beam was accelerated by a Cockcroft-Walton voltage quadrupling circuit of conventional design employing a low voltage arc source developed by S. K. Allison.[2] Accelerating voltages up to 400 kv could be attained. This voltage was measured by a resistance stack of approximately 10^{10} ohms in series with a sensitive galvanometer. The current through the resistance stack was known to 0.1 percent accuracy, and the resistance to about 0.2 percent. The beam

* The construction of the electron multiplier tubes for this work was assisted by the Joint Program of the Office of Naval Research and the Atomic Energy Commission.

[1] J. S. Allen, Phys. Rev. **55**, 336 (1939); **55**, 966 (1939).

[2] S. K. Allison, Rev. Sci. Inst. **19**, 291 (1948).