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Heat Transfer in Liquid Helium II by Internal Convection†

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The heat transfer in liquid helium II by internal convection is calculated on the basis of Tisza's two-fluid model. One arrives at the following differential equation for the heat current density, **q**:

$$\text{curl curl}\mathbf{q} = -\Lambda \,\text{grad}T.$$

Here T = temperature, $\Lambda = (\rho s)^2 T/\eta_n$ is a numerical constant depending on temperature and pressure only (ρs = entropy density and η_n = the viscosity of the "normal" fluid). Comparison with the measurements of the heat transfer in very fine slits by Keesom and co-workers shows only partial agreement. It appears that the heat conductivity of helium II in narrow slits, even in the limit of very small flow, cannot be described by the usual laminary flow solutions of hydrodynamics within the two fluid model, even if one introduces further assumptions such as viscous slip and an Ohm's law type resistance.

I. INTRODUCTION

THE measurements of heat transfer in liquid helium II published recently by Keesom and Duyckaerts,[1] Mellink,[2] and Meyer and Mellink,[3] have shown that in extremely fine capillaries or slits (diameter 0.15–19μ) and for sufficiently small heat input *the transported heat becomes proportional to the temperature difference.* According to previous measurements it appeared that the heat current is proportional to $(\text{grad}T)^{\frac{1}{3}}$ (Keesom[4]). It now appears that this cube root dependence on the temperature gradient is restricted to heat flow in wider capillaries and to larger currents. The new results are of interest insofar as for the first time they seem to give qualitative support to general ideas promoted by Tisza[5] and H. London[6] interpreting the enormous heat conductivity in liquid helium II by a mechanism of "internal convection." This mechanism implies a heat current proportional to grad$T$. Since this result seemed to be in contradiction to every available experimental evidence so far the theory had never been developed in detail. It might now appear desirable to compare the theoretical expectations with the new experimental data.

According to the view in question, liquid helium II is described as consisting of two mutually interpenetrating fluids having densities ρ_n and ρ_s, respectively. The "normal" fluid (n) carries almost all the entropy of helium II and has ordinary viscosity, whereas the "superfluid" (s) has no viscosity and negligible entropy ($\eta_s = 0$, $s_s \simeq 0$). In first approximation the two fluids are considered as uncoupled, the macroscopic motion of each being described by its own hydrodynamic velocity field $\mathbf{v}_n$ and $\mathbf{v}_s$, respectively.

The "two-fluid" model was originally derived from the idea[7] that the condensation mechanism shown by an ideal Bose-Einstein gas might be responsible for the transition from helium I to helium II at the λ-point. The Bose-Einstein condensation exhibits the unique example of a system of two mutually interpenetrating phases one of which has zero entropy. However, the model in question may, of course, be employed without going into the details of any molecular theory. As a *macroscopic model* it has proved capable of accounting for a number of the peculiar properties of liquid helium II.

It leads to a very simple solution of the apparent paradox that viscosity experiments performed with superfluid helium flowing through

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[1] W. H. Keesom and G. Duyckaerts, Physica **13**, 153 (1947), (referred to in text as K and D).

[2] J. H. Mellink, Physica **13**, 180 (1947).

[3] L. Meyer and J. H. Mellink, Physica **13**, 197 (1947), (referred to in text as M and M).

[4] W. H. Keesom, *Helium* (Elsevier Publishing Company, Inc., Amsterdam, 1942).

[5] L. Tisza, Nature **141**, 913 (1938); Comptes Rendus **207**, 1186 (1938); J. de phys. et rad. **1**, 165, 350 (1940); Phys. Rev. **72**, 838 (1948).

[6] H. London, Proc. Roy. Soc. **A171**, 484 (1939).

[7] F. London, Nature **141**, 643 (1938); Phys. Rev. **54**, 947 (1938).

thin capillaries gave no measurable viscosity at all (Kapitza,[8] Allen and Misener[9]) while experiments using a rotating disk immersed in superfluid helium showed a quite normal viscosity of the same order as that of ordinary liquid helium I (Keesom and MacWood[10]).

It was further possible on the basis of this model to give an interpretation of the so-called fountain (or thermo-hydrodynamical) effect as caused by an essentially reversible mechanism of the type of the thermo-electric effects.[5,6] More precisely, the model of Tisza implies that two containers filled with liquid helium II and connected with each other by extremely fine capillaries will be in equilibrium, if a temperature difference ΔT maintained between the containers is accompanied by a pressure difference,

$$\Delta p = \rho s \Delta T, \tag{1}$$

where s = entropy per gram, and ρ = density of liquid helium II. This "thermo-hydrodynamical pressure difference" has subsequently been quantitatively confirmed by the measurements of Kapitza[11] and of Keesom and co-workers.[1,2,3]

Going beyond this result, Tisza inferred from the same model the possibility of a peculiar form of wave motion in liquid helium II in which temperature and entropy density perform periodic fluctuations, quite analogous to the pressure and density fluctuations in ordinary sound. These "temperature waves" represent density waves of ρ_n and ρ_s separately with a phase difference of 180° to each other so that $\rho_n+\rho_s$ is approximately constant. The same mechanism was again derived by Landau[12] who called it "second sound." The phenomenon was subsequently observed by Peshkov[13] and later by Lane and co-workers.[14] These authors found very close agreement between the measured and the theoretically predicted dependence on temperature of the propagation velocity of these temperature waves. Especially these experiments furnish a very sharp test of the validity of the general assumptions which are at the basis of the model under discussion.

However, the second sound waves are not able to transfer much entropy since they propagate nothing but the mechanical energy of their excitation which is extremely small, while the entropy density undergoes only periodic local fluctuations. Thus the second sound is not directly responsible for the "heat superconductivity" in liquid helium.

If one has a capillary or a fine slit connecting two vessels of liquid helium and maintains a temperature difference between them and thus also has a pressure difference according to (1), the normal fluid will not be entirely immobilized by its viscosity. A stationary state will be reached when a certain circulation of the two fluids is established: Superfluid carried by the thermo-hydrodynamic force will go from the cooler to the warmer container. There it will be excited into the normal fluid state, in which state the excess helium will return with viscous flow to the cooler container. Here its excess heat content will be carried away by the thermostat and the excited normal fluid will fall back again into the superfluid state. As only the normal fluid is supposed to carry entropy, a stationary heat flow will be established carrying heat from the warmer to the cooler container. This is the mechanism of heat transfer[5,6] which we shall call "internal convection" and which we are going to discuss in greater detail in the present paper.

II. THE DIFFERENTIAL EQUATIONS OF INTERNAL CONVECTION

The differential equations by which the model of the two mutually interpenetrating fluids has been described are still somewhat uncertain. In particular, the experimentally observed existence of something like an upper limit for the superflow has not yet found a satisfactory mathematical description within the theory. However, in the limit of small deviations from equilibrium, that is, if one neglects all except linear terms in the velocities $\mathbf{v}_n$ and $\mathbf{v}_s$ and in the derivatives of the other quantities entering into the equations, Tisza's assumptions definitely lead to the following set of equations:[12]

[8] P. Kapitza, Nature **141**, 74 (1938).
[9] J. F. Allen and A. D. Misener, Nature **141**, 75 (1938).
[10] W. H. Keesom and G. E. MacWood, Physica **5**, 737 (1939); **8**, 65 (1941).
[11] P. Kapitza, J. Phys. USSR **5**, 59 (1941).
[12] L. Landau, J. Phys. USSR **5**, 71 (1941).
[13] V. Peshkov, J. Phys. USSR **8**, 381 (1944); **10**, 389 (1946).
[14] C. T. Lane, H. Fairbank, H. Schultz, and W. Fairbank, Phys. Rev. **70**, 431 (1946); **71**, 600 (1947).

Conservation of mass:

$$\partial(\rho_n+\rho_s)\partial t = -\mathrm{div}(\rho_n \mathbf{v}_n+\rho_s \mathbf{v}_s); \qquad (2)$$

conservation of momentum:

$$\partial(\rho_s \mathbf{v}_s+\rho_n \mathbf{v}_n)/\partial t = -\mathrm{grad}p - \eta_n(\mathrm{curl}\ \mathrm{curl}\mathbf{v}_n - 4/3\ \mathrm{grad}\ \mathrm{div}\mathbf{v}_n), \qquad (3)$$

where η_n is the coefficient of viscosity of the normal component and p is the pressure.

Conservation of entropy:

$$\partial(\rho s)/\partial t = -\mathrm{div}(\rho s \mathbf{v}_n). \qquad (4)$$

It should be noted that in Eq. (4) terms have been omitted which would express an irreversible increase in entropy such as that due to the dissipation of kinetic energy caused by the viscosity of the normal component, since such terms are quadratic in the velocities.

Finally:

$$\partial \mathbf{v}_s/\partial t = -\rho^{-1}\ \mathrm{grad}p + s\ \mathrm{grad}T. \qquad (5)$$

The derivation of Eq. (5) implies some assumptions which might still appear very much open to argument. We will not enter here into such a discussion but rather take Eqs. (2), (3), (4), and (5) as the presumably simplest expression of the thermo-hydrodynamics of liquid helium II for small velocities. In particular Eqs. (3) and (4) express the assumption that viscosity and entropy are only attributed to the normal component. Equation (5) generalizes the equilibrium condition (1), making a very far-reaching step by attributing a thermo-hydrodynamical force to every volume element of the liquid in which a temperature gradient is being maintained. It is this Eq. (5) which is responsible for the appearance of temperature waves in the theory.

By subtracting ρ times (5) from (3) one obtains:

$$\rho_n \partial(\mathbf{v}_s-\mathbf{v}_n)/\partial t = \rho s\ \mathrm{grad}T + (\eta_n/\rho s T)(\mathrm{curl}\ \mathrm{curl}\mathbf{q} - 4/3\ \mathrm{grad}\ \mathrm{div}\mathbf{q}), \qquad (6)$$

similarly, Eq. (2) minus s^{-1} times Eq. (4) gives

$$\rho_s\ \mathrm{div}(\mathbf{v}_s-\mathbf{v}_n) = \rho\dot{s}/s = \rho/s\left[\left(\frac{\partial s}{\partial T}\right)_p \dot{T} + \left(\frac{\partial s}{\partial p}\right)_T \dot{p}\right]. \qquad (7)$$

In (6) we have written $\mathbf{q}$ for the heat current density vector

$$\mathbf{q} = \rho s T \mathbf{v}_n. \qquad (8)$$

Eliminating $\mathbf{v}_s-\mathbf{v}_n$ from (6) and (7) we obtain the wave equation of the temperature waves of Tisza and Landau plus an attenuation term:

$$-\frac{\rho_n}{\rho_s}\frac{\partial(1/s)}{\partial T}\ddot{T} = \mathrm{div}\ \mathrm{grad}\left[T - \frac{4}{3}\frac{\eta_n}{\rho}\frac{\partial(1/s)}{\partial T}\dot{T}\right]. \qquad (9)$$

Here we have omitted the term $(\partial s/\partial p)_T \dot{p}$ which entails a coupling between the temperature waves and the pressure (ordinary sound) waves. This is very small since $(\partial s/\partial p)_T = -(\partial V/\partial T)_p$ is very small for liquid helium II.

In case that *conditions of stationary heat transfer* are reached, we have to go back to Eq. (6). Since under those conditions $\mathrm{div}\mathbf{q}=0$, we obtain:

$$\Lambda\ \mathrm{grad}T = -\mathrm{curl}\ \mathrm{curl}\mathbf{q}, \qquad (10)$$

where

$$\Lambda = (\rho s)^2 T/\eta_n \qquad (11)$$

is a new characteristic coefficient.

Equation (10) replaces the usual equation

$$K\ \mathrm{grad}T = -\mathbf{q}$$

of ordinary heat conductivity. One might call (10) the differential equation of "heat superconductivity" or, more appropriately, of "internal heat convection." It expresses quantitatively the idea of H. London and Tisza that entropy is being transferred by a movement of the two phases relative to each other.

Whereas in ordinary heat conductivity

$$\mathrm{curl}\mathbf{q} = 0,$$

as follows directly from the usual heat conduction equation, we see that according to (10) a temperature gradient can be maintained in helium II only if $\mathrm{curl}\mathbf{q} \neq 0$. This is a consequence of the fact that in the present form of the theory the only resistance to internal convection at small velocities is the viscosity of the normal phase.

The *boundary conditions* derive from those of the normal phase.[15]

$$v_{n\parallel} = 0; \quad \rho s T v_{n\perp} = -K_a(\mathrm{grad}T)_{\perp\ \mathrm{solid}},$$

where K_a is the heat conductivity of the adja-

[15] L. Landau, J. Phys. USSR **8**, 1 (1944).

cent medium. This means

$$q_{\parallel}=0; \quad q_{\perp}=-K_a(\mathrm{grad}T)_{\perp\ \mathrm{solid}}.$$

For a *slit of width d* one obtains as solution the parabolic current distribution of laminary viscous flow over the cross section $(-\frac{1}{2}d\leqslant x\leqslant\frac{1}{2}d)$ of the slit in the form

$$\mathbf{q}=\tfrac{3}{2}\bar{\mathbf{q}}(1-4x^2/d^2),$$

where $\bar{\mathbf{q}}$ is the mean value of q taken over the cross section of the slit. Hence

$$\Lambda\ \mathrm{grad}T=-12\bar{\mathbf{q}}/d^2 \tag{12}$$

and accordingly the mean "heat conductivity" is given by

$$\bar{K}=\bar{q}/|\mathrm{grad}T|=\Lambda d^2/12=(\rho s)^2Td^2/12\eta_n. \tag{13}$$

Similarly for a circular capillary of radius R one calculates as mean heat conductivity

$$\bar{q}/|\mathrm{grad}T|=\Lambda R^2/8.$$

The entropy density of helium between 1°K and the λ-point can be approximately represented by the empirical formula

$$\rho s=0.72\times10^{-3}T^{5\cdot6}\ \mathrm{cal\ deg.^{-1}\ cm^{-3}}$$
$$=3\times10^4T^{5\cdot6}\ \mathrm{erg\ deg.^{-1}\ cm.^{-3}}$$

The viscosity measurements of Keesom and MacWood by the oscillating disk method actually determine the product $\rho_n\eta_n$; if one assumes that $\rho_n\sim s$, as is suggested by general arguments, it follows that η_n is roughly constant between 1°K and the λ-point and about equal to the viscosity of liquid helium I, that is, of the order $\eta_n\simeq2\times10^{-5}$[g cm^{-1} sec.$^{-1}$]. With these values one obtains for the mean heat conductivity (13) of the slit of width d:

$$\bar{K}\simeq0.9\times10^5T^{12\cdot2}d^2[\mathrm{cal\ deg^{-1}\ cm^{-1}\ sec^{-1}}]. \tag{13'}$$

For comparison we give the heat conductivity of ordinary helium I at 3.3°K:

$$K_{\mathrm{HeI}}=6\times10^{-5}[\mathrm{cal\ deg.^{-1}\ cm^{-1}\ sec.^{-1}}].$$

Accordingly even in a slit of only 10μ diameter the "mean heat conductivity" of liquid helium II at 2°K would be 10^7 times larger than that of helium I.

III. COMPARISON WITH EXPERIMENT AND DISCUSSION

a. Critical Velocity

In comparing the "mean heat conductivity" given by Eq. (13′) with the experimental data[1,2,3] one must take care to confine oneself to the region of presumable validity of the linear theory, i.e., to the region of sufficiently small heat currents where $\bar{\mathbf{q}}$ is proportional to grad$T$. In the narrowest slits ($d<.75\mu$) no deviations from the linear law are observed within the range of the given data (grad$T\simeq10^{-2}$ deg. cm^{-1}). As the width d of the slit increases, deviations appear first at temperatures close to the λ-point until, for $d>10\mu$, no linear dependence of $\bar{\mathbf{q}}$ on grad$T$ is observed even at the lowest temperatures ($T\simeq1.1$°K) and for the smallest measured temperature gradients. The values of $\bar{q}$ at which the deviations begin in the various experiments can be determined from the given data only very approximately. But it appears that the limiting feature is a critical velocity of the superflow $\mathbf{v}_s$ and not perhaps turbulence of the normal flow $\mathbf{v}_n$, as one might think. From Eq. (8) and the condition of no net flow

$$\rho_n\mathbf{v}_n+\rho_s\mathbf{v}_s=0,$$

one has

$$\bar{\mathbf{q}}=\rho sT\bar{\mathbf{v}}_n=-\rho sT\frac{\rho_s}{\rho_n}\bar{\mathbf{v}}_s\sim-T(T_\lambda^{5\cdot6}-T^{5\cdot6})\bar{\mathbf{v}}_s \tag{14}$$

where $T_\lambda=2.19$°K is the temperature at the λ-point.

Within the limits of accuracy of their determination from the experimental data the values of v_s (critical) obtained with the aid of (14) are *independent of the temperature*. They depend on the width d of the slit approximately as d^{-1}.

Typical examples of the determination of $\bar{v}_sd$ (critical) from the experimental data are collected in Table I. The column headed $\bar{q}\times d$ gives the mean heat current density multiplied by the width of the slit. In a slit of given width d and at a given temperature T the region of validity of a linear law is determined by the proportionality of $\bar{q}\times d$ to the temperature difference ΔT, that is by $\bar{q}\times d/\Delta T=$constant. The braces in Table II indicate the experimental limits within which in each case the proportionality of q to ΔT comes to an end. When $\bar{q}\times d/\Delta T$

TABLE I. Limits of the linear region; determination of critical velocities.

$d\times10^4$ cm	T °K	$\bar{q}d\times10^5$ cal/cm sec	$(\bar{q}d/\Delta T)\times10^3$ cal/cm sec. deg.	$\bar{v}_sd\times10^3$ cm²/sec.	R_n
0.75	2.097	4.03	8.5	1.45	2.3
	2.097	4.16	8.5	1.5	2.4
	2.097	6.65	7.7	2.4	3.9
1.15	1.989	2.2	8.2	0.5	1.4
	1.989	6.5	8.2	1.3	4.0
	1.989	8.9	7.9	1.8	5.4
	2.170	0.57	11.4	0.9	0.3
	2.170	1.04	10.4	1.7	0.6
	2.170	1.64	10.2	2.6	1.0
1.75	1.960	0.58	11.0	0.1	0.15
	1.960	3.34	11.0	0.63	0.8
	1.960	6.6	10.5	1.3	1.7
	1.960	9.4	10.3	1.8	2.4
	1.960	16.4	9.9	3.0	4.1
9.3	1.586	3.9	15.8	0.5	1.2
	1.586	12.6	15.8	1.6	3.9
	1.586	30.4	9.5	3.9	9.4
	1.807	5.8	72	0.8	1.8
	1.807	10.1	72	1.4	2.7
	1.807	25.1	67	3.6	6.8
	1.807	35.7	38	5.1	9.7
	1.962	6.1	190	1.2	1.5
	1.962	14.9	177	2.8	3.7
	1.962	31.1	133	5.8	7.8

begins to decrease the critical velocity has been exceeded. $R_n=\rho_n\bar{v}_nd/\eta_n$ is the Reynolds number for the "normal" flow. One sees from the table that in every case R_n is smaller by orders of magnitude than the value $\sim10^3$ at which turbulence may be expected to set in; thus the observed deviations from linear dependence of $\bar{q}$ on grad$T$ cannot be ascribed to turbulent flow of the viscous normal phase.

The product $d\times\bar{v}_s$ (critical) is seen to be of the order of magnitude

$$d\times\bar{v}_s(\text{critical})\simeq10^{-3}\ \text{cm}^2\ \text{sec.}^{-1}.$$

This is in agreement with the values obtained from Kapitza's experiments;[11] it is about 10 times larger than the values found by Daunt and Mendelssohn[16] for the transfer velocity in the supra-surface film.*

The d^{-1} dependence of $\bar{v}_s$ (critical) accounts for the fact that no region of linear dependence of $\bar{q}$ on grad$T$ has been observed in capillaries wider than 10μ. By virtue of the factor multiplying $\bar{\mathbf{v}}_s$ in (14) the critical value of $\bar{q}$ for a slit of given d approaches zero as the temperature approaches the λ-point. Thus for $T\rightarrow T_\lambda$ a linear formula can give no more than the slope $\partial\bar{q}/\partial|\text{grad}T|$ at grad$T=0$. This explains the fact that in capillaries of intermediate width $(1-10\mu)$ a region of linear dependence of $\bar{q}$ on grad$T$ is observed at low temperatures, whereas at temperatures close to the λ-point deviations from the linear law tend to occur even for the smallest measured values of grad$T$.

For the same reason, experimental curves of $\bar{q}$ *versus* T for given values of grad$T$—curves which show a maximum in the conductivity between 1.9° and 2.0° and then a rapid decrease as $T\rightarrow T_\lambda$—are misleading, since for any finite value of $|\text{grad}T|$ the critical velocity is inevitably exceeded if one goes sufficiently near the λ-point. This situation is exemplified in Fig. 1 which shows a typical experimental curve of $\bar{q}$ *versus* T for constant $|\text{grad}T|$ (heavy curve). The broken line is the corresponding curve of $\bar{q}$ (critical) as given by Eq. (14) with $d\times\bar{v}_s(\text{critical})=10^{-3}$ cm²/sec. One sees that the critical velocity is exceeded already at a temperature below the maximum of $\bar{q}$. This is quite generally the case.

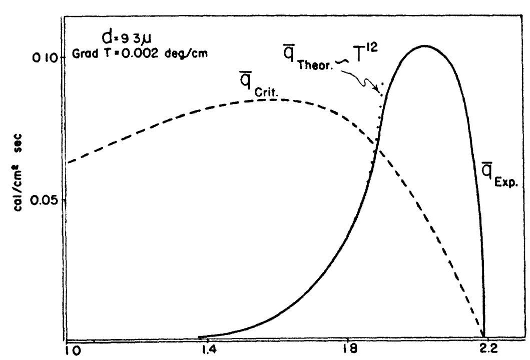

FIG. 1. Experimental heat current density $\bar{q}_{\text{exp}}$ as a function of the temperature T for constant temperature gradient (heavy curve); the curve plotted refers to a slit of width $d=9.3\times10^{-4}$ cm and to grad$T=0.002$ deg./cm. The broken curve gives the critical value of $\bar{q}$ as a function of T for a slit of the same width. The maximum in the experimental curve, as well as its decrease as the temperature approaches the λ-point, occurs in the region where $\bar{q}_{\text{exp}}>\bar{q}_{\text{critical}}$, i.e., outside the region of validity of the linear theory.

[16] J. G. Daunt and K. Mendelssohn, Proc. Roy. Soc. **A170**, 423 (1939); also: A. Bijl, J. de Boer, and A. Michels, Physica **8**, 655 (1941).

* *Note added in proof:* According to K. R. Atkins, Nature **85**, 925 (1948) the supra-surface film also has a considerably larger critical velocity in the presence of a temperature difference.

Hence one cannot expect to explain the decrease of $\bar{q}$ towards the λ-point by the present theory.

b. Linear Region: Experimental Values of $\bar{K}$

Representative values of the mean heat conductivity $\bar{K}_{\text{exp.}}=\bar{q}/|\text{grad}T|$ in the linear region are given in Table II. They were calculated from the experimental data of Meyer and Mellink (M and M) and Keesom and Duyckaerts (K and D). The corresponding theoretical values, $\bar{K}_{\text{theor.}}$, are given by Eq. (13′). In Table II L is the length of each slit; $\bar{K}_{\text{exp.}}/\bar{K}_{\text{theor.}}$ gives the ratio of the experimental to the theoretical value of the mean conductivity. The last column roughly gives the power of variation of $\bar{K}_{\text{exp}}$ with T.

The accuracy of some of the quantities entering into the tables might be somewhat doubtful, the greatest uncertainty probably coming from the determination of the width d of the slit in the various experiments. The widths were determined from optical interference measurements and from measurements of gas flow through the slits. Even granting the accuracy of those determinations (Meyer and Mellink estimate the probable error at 20 percent) they can only give the *average* width over the slit; the actual width may vary considerably in different parts of the slit. Since the mean heat conductivity depends rather strongly on the width, this is a source of considerable error. Another, though probably minor, source of error is the corrections applied to the observed heat flows to take account of losses of various kinds. From Table II one sees that the values of $\bar{K}_{\text{exp}}$ from Meyer and Mellink are consistently higher than those from Keesom and Duickaerts. M and M indicate that they had considerable difficulty in attaining a stationary state in their experiments, so that one might suspect that their measurements were not really made under stationary conditions and that this is the cause of the discrepancy between the two sets of data. The only other explanation one could think of is that this strange discrepancy has something to do with the length of the slits, those of M and M being less than half as long as those of K and D. Such an explanation does not seem very likely, however: "end effects" should be expected to be due to Bernoulli terms which have been neglected in the linear hydrodynamic equations. Since such terms are quadratic in the velocities of flow one would expect that if "end effects" are important they would destroy the proportionality between $\bar{q}$ and $|\text{grad}T|$.

Table II. Comparison of experimental and theoretical values of the mean heat conductivity.

	T(°K)	$\bar{K}_{\text{exp}}$ (cal/deg. cm sec.)	$\bar{K}_{\text{theor}}$ (cal/deg. cm sec.)	$\bar{K}_{\text{exp}}/\bar{K}_{\text{theor}}$	$\Delta\ln(\bar{K}_{\text{exp}})/\Delta\ln(T)$
$L=0.248$ cm	1.586	4.45	22	0.20	
$d=9.3\times10^{-4}$ cm					11.0
	1.807	18.7	103	0.18	
					10.3
(K and D)	1.962	43.5	295	0.15	
$L=0.275$ cm	1.223	0.066	0.032	2.05	
					11.9
$d=1.75\times10^{-4}$ cm	1.476	0.61	0.32	1.92	
					12.1
	1.705	3.5	1.86	1.88	
(K and D)					11.5
	1.960	17.2	10.2	1.69	
$L=0.275$ cm	1.799	5.35	1.55	3.45	
					11.6
$d=1.15\times10^{-4}$ cm	1.989	17.1	5.28	3.25	
					3.9
(K and D)	2.170	24	15.5	1.55	
$L=0.1$ cm	1.411	1.05	0.060	17.5	
					10.1
$d=1.0\times10^{-4}$ cm	1.802	12.4	1.19	10.4	
					7.3
	1.948	21.4	3.06	7.0	
(M and M)					3.6
	2.159	31	10.8	2.9	
$L=0.275$ cm	1.403	0.39	0.031	12.5	
					11.8
$d=0.75\times10^{-4}$ cm	1.600	1.85	0.156	11.8	
					10.3
(K and D)	2.097	30.2	4.3	7.0	
$L=0.1$ cm	1.086	0.124	0.0006	202	
					5.0
$d=0.5\times10^{-4}$ cm	1.274	0.277	0.0043	64	
					8.3
	1.315	0.354	0.0064	55	
(M and M)					8.3
	1.659	2.48	0.108	23	
$L=0.1$ cm	1.226	0.25	0.00097	260	
					6.4
$d=0.3\times10^{-4}$ cm	1.358	0.48	0.0034	140	
					7.5
	1.558	1.35	0.018	75	
(M and M)					6.0
	1.652	1.92	0.0365	54	

Altogether it appears that the listed values of $\bar{K}_{\text{exp.}}/\bar{K}_{\text{theor.}}$ may be expected to be uncertain by as much as a factor of 2 one way or the other. In spite of this the data indicate quite clearly the general dependence of $\bar{K}$ on the temperature and the width of the slit.

c. Temperature Dependence

Except in the two narrowest slits ($d=0.3$ and 0.5μ)** *the theoretical $T^{12\cdot2}$ dependence agrees rather*

** M and M have also made measurements in still narrower slits (0.15 and 0.2μ), but those measurements were made by a non-stationary method and therefore have not been used in the present analysis.

well with the experimental data up to about 1.9°; the fit is somewhat better with $d(\log\bar{K})/d(\log T) \simeq 11.5$. This may be accounted for theoretically if one assumes $\eta_n \sim T^{0.5}$, as is the case in an ideal gas and in liquid helium I,*** instead of taking $\eta_n = \text{constant}$ as was done in deriving Eq. (13′). When T approaches the λ-point $\bar{K}$ as a function of T levels off very definitely, but in no case has an actual maximum been observed. (Note that K is, of course, understood as function of T always in the limit $|\text{grad}T| \rightarrow 0$.)

In the narrowest slits the dependence on temperature is rather as T^7 or T^8 up to the highest temperatures at which measurements have been made ($T \simeq 1.7°$). No data are available on whether there is a leveling off at higher temperatures in this case also.

d. Absolute Value of $\bar{K}$ and Dependence on Width of Slit

A glance at the last column in Table II shows that the ratio of the observed to the theoretically predicted mean heat conductivity varies in order of magnitude from 10^2 in the narrowest slits to 10^{-1} in the widest. This looks rather discouraging; but it should be pointed out that except for its proportionality to d^2, which obviously does not fit the facts, the formula (13), based on the idea of heat transfer by internal convection, comes much closer to the experimental data than would a formula based on the usual type of heat conduction by "diffusion of energy." The latter mechanism would give a heat conductivity approximately independent of the temperature and of the order of that in helium I, that is too small by a factor of about 10^{-5} to 10^{-7}.

Actually there is an increase of the mean conductivity with d in the wider slits ($d \geqslant 10^{-4}$ cm), but it is not nearly as strong as d^2. In the narrowest slits the conductivity seems to be approximately independent of d.

*** As has been pointed out by Tisza (see reference 5), the small density of liquid helium and the temperature dependence of the viscosity of helium I make it appear likely that in contrast to all other liquids the viscosity of liquid helium is of the "kinetic" type usually found in gases. In this case one would expect

$$\eta_n \simeq \rho_n l_n \bar{u}_n,$$

where l_n is the mean free path and $\bar{u}_n$ the mean thermal velocity of the "normal" particles. The product $\rho_n l_n$ should be approximately temperature-independent and $\bar{u}_n \simeq T^{\frac{1}{2}}$

IV. ATTEMPTS TO ACCOUNT FOR THE DISCREPANCIES

The discrepancies between the observed values of $\bar{K}$ and the predictions of the theory as given by Eq. (13′) are essentially of two types: (*a*) *In the narrowest slits the observed values of the heat current density are some hundred times larger than those predicted by the theory.* (*b*) *In the widest slits in which a linear dependence of $\bar{q}$ on* grad$T$ *has been observed the heat currents are actually less than expected from the theory; furthermore at temperatures close to the λ-point the conductivity increases with temperature much more slowly than the theory predicts.*

To account for these discrepancies, one may attempt to generalize in two respects the equations by which internal convection has been described in Sec. 2 of this paper.

Item (*a*) points to a larger conductivity for the narrowest slits than is given by the theory. Since it appears impossible to get rid of the normal viscosity by manipulating the differential equations, one may conclude that the boundary condition $v_{n\parallel} = 0$ is too stringent. One may introduce a slip of the normal flow velocity v_n along the wall of the slit by replacing the boundary condition $v_{n\parallel} = 0$ by

$$v_{n\parallel} = -a\partial v_{n\parallel}/\partial N, \qquad (15)$$

where N is the outward normal to the boundary of the slit. The quantity a in (15) is of the dimension of a length and should be of the order of the mean free path of the particles in the normal phase. From the experimental value of the viscosity one estimates for the mean free path at the λ-point

$$l_n \simeq 10^{-7}\ \text{cm}(T = T_\lambda),$$

so that one should expect

$$a \simeq l_n \simeq 10^{-7}(\rho/\rho_n) \simeq 10^{-7}(T_\lambda/T)^{5.6}\ \text{cm}.$$

Item (*b*) indicates the presence of a resistance to internal convection in addition to the viscosity of the normal fluid. This additional resistance becomes significant for wider slits. One may try empirically to account for this discrepancy by formally introducing an Ohm's law type of resistance into the Eq. (6) describing the relative motion of the two phases. This means that we replace Eq. (10) by

$$-\mathrm{grad}T=\Lambda^{-1}(\mathrm{curl\ curl}\mathbf{q}+\beta^2\mathbf{q}) \qquad (16)$$

the function $\beta(T)$ to be determined empirically.

Band and Meyer[17] have recently fitted the heat conductivity in a wider capillary by assuming a resistance to the relative motion of the two phases equivalent to a term like the supplementary one in Eq. (16). However, since they were using data from the region of cube root dependence of q on $|\mathrm{grad}T|$, their resistance is a function not only of T but also of q itself (proportional to q^2). Furthermore, they completely neglect the viscous resistance of the normal phase, using *only* the Ohm's law type of resistance; thus in their formula the heat conductivity is independent of the width of the capillary. Because of its proportionality to q^2 the resistance proposed by Band and Meyer cannot be applicable to the linear region. At any rate it appears hopeless to try to understand the laws of motion of the two-phase model of helium II in the region of supra-critical velocities as long as one cannot account satisfactorily for the small heat currents.

Using (16) and the boundary condition (15) the stationary solution becomes

$$\mathbf{q}=\Lambda\beta^{-2}\,\mathrm{grad}T\left[\frac{\cosh\beta x}{\cosh(\beta d/2)+a\beta\sinh(\beta d/2)}-1\right], \qquad (17)$$

and one obtains for the mean heat current density:

$$\bar{q}=\Lambda\beta^{-2}|\mathrm{grad}T|\left[\frac{\xi^{-1}\sinh\xi}{\cosh\xi+a\beta\sinh\xi}-1\right], \qquad (18)$$

where we have put $\beta d/2=\xi$. Expanding this expression into powers of ξ we obtain as heat conductivity for slits of small width d:

$$\bar{q}/|\mathrm{grad}T|=\Lambda\beta^{-2}[a\beta\xi+(\tfrac{1}{3}-a^2\beta^2)\xi^2+\cdots]$$
$$=\frac{\Lambda d^2}{12}[1+6a/d-3a^2\beta^2+\cdots].$$

One sees that the new assumptions amount to a correction by a factor

$$[1+6a/d-3a^2\beta^2+\cdots] \qquad (19)$$

in the formula (13′) for the mean heat conductivity. If one takes a and $\beta(T)$ independent of d, as they ought to be if they are to have any reasonable physical meaning, one sees that a correction factor of the type (19) will not suffice to bring (13′) into agreement with the experimental data: The expression (18) for the mean heat conductivity still increases too fast with d; in particular for very narrow slits it is proportional to d rather than independent of d as the experimental data (of Table II) seem to indicate.

It is interesting to note that quite formally Eq. (18) is able to account for the T^7 dependence of $\bar{K}$ in the narrowest slits where one can presumably neglect the term proportional to d^2 compared to $6ad$. One can fit the data fairly well by setting

$$a\simeq 3\cdot 10^{-3}T^{-5\cdot 5}\ \mathrm{cm},$$

which has the right temperature dependence for the mean free path of the normal phase particles. However for two reasons this is not likely to have physical significance: First, the mean free path would be larger than the width of the slit; in that case, however, the whole hydrodynamic description of the flow would break down. Second, the mean free path implied by a slip of this magnitude is some hundred times larger than the mean free path corresponding to the experimental value of the viscosity as determined in the rotating disk experiments of Keesom and MacWood.[10] On the other hand it should be pointed out that there is no *direct* experimental evidence to exclude a slip of this magnitude: It is true that Keesom and MacWood failed to detect any measurable slip in their experiments. However, within the experimental accuracy of their measurements a slip of the magnitude in question here could not have been observed.

V. CONCLUSIONS

From the above discussion it appears that the present data on the heat flow in helium II in narrow slits cannot satisfactorily be described by the usual laminary flow solutions of the two-fluids hydrodynamics even with the additional assumptions of viscous slip and an Ohm's law type resistance. On the other hand, since our discussion was aimed at the limiting case of small

[17] W. Band and L. Meyer, Phys. Rev. **73**, 226 (1948).

velocities, when $\mathbf{v}_n$ is proportional to grad$T$, one cannot have recourse to assuming complications by turbulent flow either.

It is, of course, possible that the whole concept of the two-fluid hydrodynamics loses its validity when dissipative processes are considered. However, in view of the achievements of the theory in general, it appears unwarranted to go so far, and one will expect that, at least in the limit of very small velocities, the theory should be competent to account also for dissipative processes.

It might be pointed out that the critical values of $v_s d$ obtained from the present data, as well as those from Kapitza's flow experiments, are some ten times larger than those observed in connection with the supra-surface film (Daunt and Mendelssohn), the latter being of the order of $h/4\pi m \simeq 10^{-4}$. Hence it might be argued that the applicability of the theory is restricted to still smaller velocities, i.e., still smaller heat currents than the ones used in the experiments under discussion. This does not appear to be a valid argument since for most slits the observed heat conductivity is *larger* than that predicted by the theory. One cannot think of any perturbation mechanism which would lead to a *decrease* of the resistance with *increasing* flow velocity. At any rate it would be desirable to extend the measurements to still smaller heat currents.

It seems that in order to account for the present experimental data in terms of the general two-fluid theory some features of the theory must be revised. There appear to be two possibilities for doing this:

(1) One may have to abandon the idea of attributing an ordinary hydrodynamic viscosity to the normal phase of helium II. This would not be very satisfactory, since it is precisely the assumption that the normal phase has viscosity while the superfluid phase has none which in Tisza's theory explains some of the most striking properties of helium II (rotating disk experiments *versus* superfluid flow in narrow capillaries).

Or (2) one may have to assume that in the case of stationary flow through narrow slits the normal phase of the liquid helium II does not touch the wall of the slit at all, but that there is a film of pure superfluid next to the wall. This alternative is at any rate preferable to the first one. Allen and Misener[18] were led to a similar picture from their investigations of viscous flow of helium II in narrow capillaries ("sub-surface film"). This assumption would mean that there are large variations in the densities of the two phases over the width of the slit when stationary flow is being maintained.

This possibility is rather difficult to investigate quantitatively at the present time, since one does not know the correct hydrodynamic equations for the two-phase model if one cannot consider ρ_n and ρ_s as approximately constant.

One cannot entirely exclude the possibility that the phenomena may be describable in terms of end effects: Terms of the type grad(v_s^2), which have been neglected in the linear equations, may become large at the ends of the slit. It would appear that since such terms are quadratic in the velocities they cannot be important in the limit of small velocities, that is, in the region of proportionality of $\mathbf{v}_s$ and $\mathbf{v}_n$ to grad$T$. But it could be that the distance within which $\mathbf{v}_s$ and $\mathbf{v}_n$ reach the values they assume within the slit, is itself proportional to $\mathbf{v}_s$, so that grad(v_s^2) would actually be linear in $\mathbf{v}_s$. However, also this possibility cannot be investigated quantitatively without further refinement of the hydrodynamics of the model.

[18] J. F. Allen and A. D. Misener, Proc. Roy. Soc. **A172**, 467 (1939).