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The Energy Distribution in Cosmic-Ray Showers*

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The track length distribution $f(E)dE$ in a shower is defined as the total distance traveled by all electrons while in the energy interval dE. It is representative for the average energy distribution in a shower. The present paper obtains by numerical methods the complete functions for both electrons as well as for photons in showers of high primary energy. Calculations have been made for air and for lead, down to energies of about 4 Mev. Close approximations to the actual cross sections have been used for all energies and all physically significant processes have been included—radiation, pair production, energy loss through collision, Compton effect, and knock-on electrons. As a special result of importance we mention the high number of low-energy photons that accompany a shower, a quantity that has here been evaluated for the first time.

1. INTRODUCTION

THE well-known cascade theory of cosmic-ray showers[1] has been successfully carried through to the evaluation of total numbers of electrons and photons and their energy distribution for not too low energies. For energies around and below the so-called ionization limit (cf. below) an analytical evaluation of the energy distribution is made difficult by the complicated formulae for the probabilities of elementary processes and the incoming of additional effects, such as the Compton effect and knock-on electrons, which are usually neglected.

The first attempt to determine the number of low energy electrons in a shower is due to Arley.[2] He neglects entirely ionization losses above the critical energy and radiative effects below it, and he takes into account only those electrons in the low energy range that originate from high energy electrons which are thrown into the low range by emission of quanta. Thus, the electrons produced with small energy by quanta are omitted. The result is a serious underestimate of the number of slow electrons. The ratio of slow-to-fast electrons near the shower maximum is, according to Arley, about 1, while the actual ratio as computed in this paper is of order 3 to 4.

A more systematic treatment of the electron distribution at low energies has been given by Tamm and Belenky[3] who use simplified cross sections even in the low energy range, and a somewhat questionable mathematical approximation and by Rossi and Klapman[4] who, by a numerical method, obtain the total track length in air above energies of 10 Mev as function of the energy of a generating electron or photon.

In the present paper a numerical method is developed to obtain the track-length distribution down to low energies ($\sim$4 Mev) in showers starting with high initial energies.

In the low-energy end the nature of the material is of importance and numerical results have been obtained for air and lead. No terms of significance have been neglected, and our results should be accurate to about ± 10 percent. Our results include also the distribution of photons of low energy which has not been evaluated before.

2. THE EQUATIONS FOR THE TRACK LENGTH

The track-length distribution of a shower $f(E)dE$ is defined as the total distance traveled by all shower electrons while their energy lies between E and $E+dE$. Similarly we can define a photon track-length distribution $g(K)dK$ where we denote the photon energy by K for easy distinction. This "normal" distribution is much easier to calculate than the energy distribution as function of distance, and it has almost the same physical significance as the latter, since it

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[1] For all older work on shower theory, reference is made to the excellent summary by B. Rossi and Kenneth Greisen, Rev. Mod. Phys. **13**, 240 (1941).

[2] N. Arley, Proc. Roy. Soc. **A168**, 519 (1938).

[3] I. Tamm and S. Belenky, J. Phys. U.S.S.R. **1**, 177 (1939); Phys. Rev. **70**, 660 (1946).

[4] B. Rossi and S. J. Klapman, Phys. Rev. **61**, 414 (1942).

will be the same as the distribution at the shower maximum, and it will predominate over most of the longitudinal extension of the shower with the exception of its very early and late stages.

The following general results are known about the track-length distribution. Since all the energy in a shower is dissipated finally by ionization through electrons, the integral track-length over all energies is

$$\int_0^{E_p} f(E)dE = E_p/\beta_i, \tag{1}$$

where E_p is the primary energy and β_i the energy loss through ionization which can be considered as constant over the interesting energy range of 2 Mev to 300 Mev.

The track-length distribution has been evaluated by Nordheim and Hebb[5] in the high-energy limit where ionization losses can be neglected relative to radiation losses, and by Rossi and Greisen[1] in the next approximation, where ionization losses can be considered as small. Their result can be expressed as follows

$$f(E) = 0.437\frac{E_p}{\beta_i}\left[\frac{E}{\beta_i} + 0.818 - \frac{0.392}{(E/\beta_i)} + \cdots\right]^{-2}, \tag{2}$$

$$g(K) = 0.437 \times \frac{9}{7}\frac{E_p}{\beta_i}\left[\frac{K^2}{\beta_i^2}\left(1 + \frac{0.769}{(K/\beta_i)} + \cdots\right)\right]^{-1}. \tag{3}$$

These expressions give a good approximation down to about E and $K \sim 3\beta_i$. For lower energies, as already mentioned, the usual mathematical treatments break down, and also the cross sections for occurring processes lose their universal character.

It is to be noted that for high energies $\beta_i \ll E \ll E_p$, the distributions are proportional to $1/E^2$. We will normalize the distribution so that

$$\left.\begin{aligned} f(E) &\rightarrow 1/E^2 \\ g(K) &\rightarrow \frac{9}{7}\frac{1}{E^2} \end{aligned}\right\} \quad \text{for} \quad E \gg \beta_i.$$

The absolute number of particles produced by a primary of energy E_p is then $0.437(E_p/\beta_i)f(E)$ and correspondingly for $g(K)$.

[5] L. W. Nordheim and M. H. Hebb, Phys. Rev. 56, 494 (1939).

In the following, we will introduce the natural units of shower theory, the critical energy, where radiation and collision losses are equal, and the unit length on which the energy loss through ionization is just this critical energy. In these units $\beta_i = 1$.

We consider the following processes: for photons, pair production and Compton effect, for electrons, radiation, ionization losses and knock-on electrons. The cross sections, as used in this paper, are summarized in the appendix. All these processes are of the type that a primary of energy E' or K' produces a secondary of energy E or K respectively.

We denote the cross sections for such a process by a function of two arguments. $\sigma(E', K)$, for instance, is the probability that an electron E' produces a quantum K and so on. Of course, $\sigma(E', K)$ is the same as $\sigma(E', E'-K)$, since the electron remains at an energy $E'-K$ after emission of the photon K. If necessary we add an index to the cross sections, for instance:

σ_r radiation by an electron,
σ_p pair production,
σ_c Compton effect,
σ_k knock-on electron.

The equations that describe the balance of all effects are then as follows

$$\int_E^\infty g(K')\sigma(K', E)dK' + \int_E^\infty f(E')\sigma(E', E)dE' - \int_0^E f(E)\sigma(E, E')dE' + \frac{d}{dE}(\beta_i f(E)) = 0, \tag{4}$$

$$\int_K^\infty f(E')\sigma(E', K)dE' + \int_K^\infty g(K')\sigma(K', K)dK' - \int_0^K g(K)\sigma(K, E')dE' - \int_0^K g(K)\sigma(K, K')dK' = 0. \tag{5}$$

The first terms describe the production of a "particle" of one kind (electron or photon) by a particle of the other kind. The next terms give the number of particles added or removed from an energy interval by processes involving energy changes. The last term in (4) takes account of the continuous energy loss by ionization for electrons.

The principal difficulty in the treatment of these equations comes from the second and third terms of Eq. (4).

$$D(E)=\int_E^\infty f(E')\sigma_r(E',E)dE' - \int_0^E f(E)\sigma_r(E,E')dE', \quad (6)$$

since it is well known that the radiation probabilities diverge for small K, i.e., $E'-E$, as $1/K$ (infra-red catastrophe). The expression (6) has, therefore to be understood as the finite limit

$$D(E)=\lim_{\epsilon=0}\left[\int_{E+\epsilon}^\infty f(E')\sigma_r(E',E)dE' - \int_0^{E-\epsilon} f(E)\sigma_r(E,E')dE'.\right. \quad (6a)$$

If the energy loss due to radiation could be considered as of continuous nature, D should be representable by an expression similar to the last term of Eq. (4), i.e.,

$$D(E)=\frac{d}{dE}[\beta_r(E)f(E)], \quad (7)$$

where $\beta_r(E)$ is the average energy loss to radiation as function of energy.

We shall now prove that Eq. (7) is in fact a sufficient approximation for our purpose.

The cross section for radiation at high energies is of the form

$$\sigma(E,E')dE'=\frac{dE'}{E-E'}S\left(\frac{E'}{E}\right), \quad (8)$$

where S is a homogeneous function of the fractional energy reduction E/E'. The total average energy loss is then

$$\beta_r=\int_0^E(E-E')\sigma(E,E')dE'=\int_0^E S\left(\frac{E'}{E}\right)dE' = E\int_0^x S(x)dx = E\int_1^\infty S\left(\frac{1}{x}\right)\frac{dx}{x^2}. \quad (9)$$

Introducing the new variable $E''=(E^2/E')$ into the second integral in Eq. (6) and renaming it afterwards again E', we obtain the transformation

$$D(E)=\int_E^\infty\left[f(E')-\frac{E}{E'}f(E)\right]S\left(\frac{E}{E'}\right)\frac{dE'}{E'-E}.$$

We know now from Eq. (2) that $f(E)$ behaves at high energies similar to $1/E^2$. Also, the discontinuity of the energy loss is most important for high energies. We can put, therefore,

$$f(E)=\frac{\varphi(E)}{E^2};\quad \varphi(E')=\varphi(E)+(E'-E)\frac{d\varphi}{dE},$$

where φ is a slowly varying function of E for the representation of which the first term in its Taylor development is sufficient. We obtain then

$$D(E)=\left(\frac{d\varphi}{dE}-\frac{\varphi}{E}\right)\int_E^\infty S\left(\frac{E}{E'}\right)\frac{dE'}{E'^2} = \left(\frac{d\varphi}{dE}-\frac{\varphi}{E}\right)\frac{1}{E}\int_1^\infty S\left(\frac{1}{x}\right)\frac{dx}{x^2} = \frac{d}{dE}\left(\frac{\varphi}{E}\int_0^1 S(x)dx\right)=\frac{d}{dE}(\beta_r f(E)).$$

We have proved thus the substitution Eq. (7) for the limit of high energies, where the ionization losses that produce the deviation of $\varphi(E)$ from a constant are not too important. In the opposite limit of low energies radiation losses become, in any case, of lesser importance, and we are thus justified to combine D and the last term in Eq. (4) to the simple expression

$$\frac{d}{dE}[(\beta_r+\beta_i)f(E)]=\frac{d}{dE}[\beta(E)f(E)]. \quad (10)$$

This substitution has been checked for the distribution as found later in this paper, which permits a numerical evaluation of Eq. (6). The error made by Eq. (10) turned out to be of order 8 percent for $E=1$.

It is of advantage for the further treatment to introduce the total number of new electrons produced by photons and by knock-on collisions as a separate function through the definition***

$$h(E)=\int_E^\infty g(K)\sigma(K,E)dK+\int_{2E}^\infty f(E')\sigma_k(E',E)dE'. \quad (11)$$

Equation (4) then takes the form

$$h(E)+\frac{d}{dE}[(\beta_r+\beta_i)f(E)]=0.$$

Integration of this relation between two energy values gives

$$f(E)=\frac{\beta(E_0)f(E_0)}{\beta(E)}+\frac{1}{\beta(E)}\int_E^{E_0} h(E')dE'. \quad (12)$$

We can bring Eq. (5) for the photons into a similar form

$$g(K)=\frac{1}{\sigma(K)}\left\{\int_K^\infty f(E)\sigma(E,K)dE+\int_K^\infty g(K')\sigma(K',E)dK'\right\}, \quad (13)$$

where

$$\sigma(K)=\int_0^\infty \sigma(K,E')dK', \quad (14)$$

is the total absorption coefficient for photons of energy K.

The functions $f(E)$, $g(K)$, and thus also $h(E)$ are known for sufficiently high energies, let us say above E_0, from Eqs. (2) and (3). The integral equations (11), (12), (13) permit us now to compute these functions for lower energies with the help of an iteration process.

For energies E that are not too much lower than E_0 a zero approximation for $f(E)$ will be given by

$$f_0(E)=\frac{\beta(E_0)f(E_0)}{\beta(E)}. \quad (15)$$

We find then a g_0 through

$$g_0(K)=\frac{1}{\sigma(K)}\left\{\int_{E_0}^\infty f(E')\sigma(E',K)dE'+\int_E^{E_0} f_0(E')\sigma(E',K)dE'\right\}, \quad (16)$$

and an h_0 through

$$h_0(E)=\int_{E_0}^\infty g(K)\sigma(K,E)dK+\int_E^{E_0} g_0(K)\sigma(K,E)dK. \quad (17)$$

These functions are then introduced into the right side of the full Eqs. (11) to (13) and new functions $f_0(E)+f_1(E)$, and so on, are calculated. If necessary the cycle is repeated. It has been found that this process converges rapidly. The iteration was stopped when it could be estimated that the next step would give a contribution of less than 5 percent. It was in no case necessary to go beyond an f_2.

The method is practicable since most integrations can be arranged with a variable lower limit, so that the process of integration from an E_0 to an E yields at the same time all intermediate values of the function in question.

The convergence of the method becomes less good for too low values of the final energy, since f_0 is in that case not a good approximation. The obvious remedy is, of course, to carry out the integration in steps, i.e., firstly to an energy E_1, and after evaluation of all functions to this value, to resume the process with E_1 in place of E_0.

Actual calculations have been carried out for air and lead. They will be described in the following section.

3. CALCULATIONS

a. Air

The analytical expressions for f and g (Eqs. (2), (3)) can be used for energies as low as 3 in

*** The second term in the expression for $h(E)$ represents new electrons produced in knock-on collisions. The minimum primary energy E' to produce a secondary of energy E is $2E$. The change of energy of the primary can be considered as already contained in the term for energy loss through ionization.

TABLE I. Track-length distribution in air.*

Primary Energy	$E_p \gg 3$					$E_p=1$			
E or K	$f_{TB}(E)$	$f(E)$	$F(E)$	$g(K)$	$G(K)$	$f(E)$	$F(E)$	$g(K)$	$G(K)$
0			2.3		∞				
0.05	5.9	5.5	1.9	45	6.6	1.7	0.63	16	1.6
0.10	3.8	3.7	1.6	23	5.0	1.3	0.56	7.8	1.0
0.15	2.8	2.8	1.5	15	4.1	1.0	0.50	4.9	0.73
0.20	2.2	2.25	1.4	11	3.5	0.89	0.45	3.4	0.53
0.30	1.5	1.61	1.2	6.0	2.7	0.74	0.37	1.7	0.31
0.40	1.1	1.27	1.04	3.9	2.2	0.64	0.31	0.91	0.19
0.50	0.89	0.97	0.92	2.8	1.8	0.57	0.25	0.58	0.12
0.75	0.53	0.62	0.73	1.4	1.3	0.48	0.12	0.20	0.02
1.0	0.36	0.42	0.61	0.79	1.05	0.42		0	
1.5	0.21	0.23	0.46	0.41	0.77				
2.0	0.14	0.145	0.37	0.25	0.60				
2.5	0.10	0.100	0.31	0.16	0.51				
3.0	0.073	0.073	0.27	0.12	0.44				

* E or K = energy in units 86 Mev. In order to obtain absolute values multiply the $E_p \gg 3$ columns by $0.437E_p$, the $E_p=1$ column by 1.2.
$f(E)$ = track-length distribution for electrons, $f_{TB}(E)$ = track-length distribution as calculated by Tamm and Belenky.

$F(E)=\int_E^\infty f(E')dE'$.

$g(K)$ = track-length distribution for photons.

$G(K)=\int_K^\infty g(K')dK'$.

For E and $K>3$, use Eqs. (2) and (3).

units of the shower theory. The asymptotic cross sections (A4), (A6) for complete screening are valid down to about $E=1$. A first-iteration cycle was carried through between these energies. For the evaluation of the terms containing radiation and pair production it is only necessary to compute terms of the form

$$\int_E^\infty f(E')dE'/E'^n,$$

with $n=0$ to 3 as functions of the lower limit, where the functions f are known from previous steps, and where high energy tails can be evaluated analytically. After carrying through the iteration for two cycles, the knock-on and Compton contributions were added. The former ones are completely negligible, while the number of Compton electrons remains small. Therefore, it was sufficient to use the integrated probability (A9) and to assume that the produced electrons have the average energy (A10).

After the functions $f(E)$ and $g(K)$ were determined down to $E=1$, the iteration process was resumed with this new upper limit, under use of the cross sections (A5) and (A7). The Compton effect is of considerable importance in this interval and has to be recycled, and also "post-Compton" photons have to be included at lower energies. The expression for the number h of electrons produced by a photon distribution $g(K)$ through the Compton effect (compare (A8))

$$h_c(E)=B\int_{E+(mc^2/2)}^\infty \frac{g(K)}{K(K-E)}\left\{1+\frac{(K-E)^2}{K^2}\right\}dK,$$

contains the lower limit also in the integrand. Thus, strictly speaking, a complete new integration would be necessary for every value of E. It was found, however, that a computation based on the assumption that all electrons produced by quanta of energy K have the same average energy, Eq. (A10) gave values of h that were in a fairly constant ratio to the true values. This was verified by carrying out the full process for the values $E=0.3$ and 0.05, where the ratios of the complete integrals to the approximate ones were 1.2 and 1.25. For other values of E, therefore, the approximate values were calculated and corrected by multiplication with 1.23.

The results of the calculations are shown in Tables I and II which give $f(E)$, $h(E)$, and $g(K)$, and the total number of electrons above energy E, i.e., the integral

$$F(E)=\int_E^\infty f(E')dE';$$

also the total number of photons

$$G(K)=\int_K^\infty g(K')dK'.$$

The function $f(E)$ can be approximated for low energies by

$$f(E)=2.64\log(1/E)-2.4, \quad (18)$$

showing, as is to be expected, a logarithmic in-

TABLE II. Electrons produced with low energy by pair production and Compton effect (air).*

E	h_{pair}	$h_{Compton}$	h_{total}
0.05	13.9	26.5	40.4
0.10	9.4	8.3	17.7
0.15	6.7	4.1	10.8
0.20	4.9	2.3	7.2
0.30	3.1	1.00	4.1
0.40	2.2	0.58	2.8
0.50	1.65	0.35	2.00
0.75	1.00	0.11	1.11
1.00	0.70	0.06	0.76

* $h(E)dE$ = number of electrons produced in energy interval dE. Same units and normalization as in Table I.

crease for very low energy values.† The relation (18) permits an approximate evaluation of the integrated number of electrons to $E=0$. We find the contribution

$$\int_0^{0.05} f(E)dE = 0.405.$$

The total track length of all electrons in air as evaluated by us is thus 2.26. The normalized value for a primary E_p is thus $0.437\times 2.26E_p = 0.989E_p$. The factor is so near to the required value unity (comp. Eq. (1)) that we can be sure that no physically important effect has been omitted, and it furnishes thus a good check on our calculations.

Since for air a new cycle was started at $E=1$, it was comparatively easy to evaluate the track-length distribution produced by single primary electrons with this energy. The results are also given in Table I. In order to normalize these functions, we observe that the track length for a single electron near its initial energy should be $1/(dE/dt)=1/\beta$ or 0.5 in our case. The figures of the table should, therefore, be multiplied by $0.5/0.42=1.2$ in order to correspond to one primary electron.

b. Lead

For lead the critical energy is only 6.7 Mev. The asymptotic formulae (4), (5), are applicable to energies for which the asymptotic forms of the cross sections are valid, i.e., to about $E=10$ or ~ 67 Mev. The iteration process was therefore started from this energy on. It was possible to carry it down to $E=0.5$ or 3.4 Mev, which is nearly the same in absolute value as our lower limit for air ($E=0.05$ or 4.3 Mev). No new complications arise. It may be stated only that knock-on electrons and post-Compton photons were entirely negligible since the energy as measured in radiation units remains high for the whole range. The results are collected in Table III.

4. DISCUSSION

In comparing air and lead we see that the electron distributions are essentially the same if expressed in their respective radiation units. The photon distributions are, however, quite different, the number of photons at half the critical energy in lead being 2.3 times the corresponding number for air. The reason for this difference lies in the smaller absorption coefficient for photons in lead for the same reduced units which increases their free path and therefore their track length. This effect does not react strongly on the electron distribution since finally all photons are reconverted into electrons.

One of the results of our calculation is the ratio of the number of quanta to the number of electrons in a shower as a function of energy and also, of course, the evaluation of the total number of photons. While at high energies the ratio of photons to electrons is 9/7, the photons are preponderant at low energies. If we consider all energies down to our lower limit of ~ 4 Mev, we find that for both air and lead there are about 3 times as many photons as electrons.

The calculations for both air and lead have about the same lower energy limit of ~ 4 Mev in absolute units. Since the total track length is given by Eq. (1) for all materials, it is possible to determine the fraction of the shower of energy less than 4 Mev. It is about $\frac{1}{6}$ for air and $\frac{1}{2}$ for lead. The fraction of the total shower above the critical energy is 27 percent for both air and lead. A further comparison can be made with the calculations of Rossi and Klapman.[4] They find for the fraction of the track length above 10 Mev in air a value $3.06/4.3=0.71$. This is in complete agreement with the value from our calculations (1.64/2.26).

TABLE III. Track-length distribution of electrons and photons in lead.*

	$E_p \gg 10$			
E or K	$f(E)$	$F(E)$	$g(K)$	$G(K)$
0.5	0.90	1.02	6.4	2.8
1.0	0.43	0.71	2.0	1.8
1.5	0.26	0.54	0.89	1.16
2.0	0.17	0.43	0.49	0.83
3.0	0.089	0.30	0.21	0.51
4.0	0.054	0.23	0.11	0.37
5.0	0.036	0.18	0.072	0.28
6.0	0.025	0.152	0.048	0.22
7.0	0.018	0.130	0.032	0.18
8.0	0.014	0.114	0.022	0.15
9.0	0.011	0.102	0.016	0.13
10.0	0.0086	0.092	0.012	0.12

* Unit of E or K = 6.7 Mev. Normalization and meaning of symbols same as in Table I.

† The photon distribution behaves like $(1/K)$, due to this factor in the radiation cross section, and since the probability to produce a photon with $K<E$ is approximately independent of E.

It is interesting to compare our results with the analytical formula of Tamm and Belenky.[3] This formula, normalized as ours to $1/E^2$ at high energies, can be written as follows:

$$f(E)_{\text{T.B.}} = 2.3^2\left\{(1+\epsilon)e^{\epsilon}\left[-Ei(-\epsilon)-1+\frac{1+\epsilon}{\epsilon_p}e^{-(\epsilon_p-\epsilon)}\right]-(1+\epsilon)e^{\epsilon}[-Ei(-\epsilon_p)]\right\}. \quad (19)$$

The symbols have the following meaning: $\epsilon=2.3E$; $\epsilon_p=2.3E_p$, where E_p is the primary energy of the shower-producing electron in radiation units. Ei is the exponential integral according to the definition in Jahnke-Emde.[6] The values of the above function for $E_p=\infty$ (i.e., infinitely high primary energy) are also given in Table I. It is surprising how close the agreement between our values and the Tamm-Belenky function is, in spite of the neglections made in the derivation of the latter. The explanation of this behavior lies probably in the fact that the Tamm-Belenky function is so constructed that it has the correct asymptotic behavior as $1/E^2$ at high energies and that it satisfies the energy principle (Eq. (1)). These requirements already determine to a considerable extent the character of the function. However, an attempt to calculate a photon distribution by the Tamm-Belenky method would probably give rather inaccurate results. Equation (19) includes the dependence of the track-length distribution on the primary energy of the shower-producing electron. According to the preceding remarks it will be safe to use this formula if not too high an accuracy is required.

APPENDIX

Units and Cross Sections

For a complete discussion of cross sections, compare W. Heitler[7] and B. Rossi and K. Greisen.[1]

For the unit of length we take the radiation length t_0

$$\frac{1}{t_0}=\frac{4}{137}\left(\frac{e^2}{mc^2}\right)^2 NZ\bar{Z}\,\lg 183Z^{-\frac{1}{3}}, \quad (A1)$$

where N is the number of atoms per cc and $\bar{Z}=Z+1$. $\bar{Z}$ is introduced in order to include approximately the effects of the atomic electrons. The value of t_0 is 3.03×10^4 cm for air and 0.519 cm for lead.

In order to find the unit of energy β_i we equate the total energy loss by radiation

$$-\left(\frac{dE}{dt}\right)_r = E\frac{NZ\bar{Z}}{mc^2}\left(\frac{e^2}{mc^2}\right)^2\times\left(4\lg 183Z^{-\frac{1}{3}}+\frac{2}{9}\right), \quad (A2)$$

to the energy loss by collision

$$-\left(\frac{dE}{dt}\right)_{\text{coll}} = \frac{2N\pi e^4 Z}{mc^2}\left[\lg\frac{E_1E_0^2}{2mc^2I^2Z}\right], \quad (A3)$$

where $I=13.5$ ev according to Bloch, and E_1, the maximum energy of secondary electrons, is taken as 10 mc^2. Secondary electrons of higher energy are introduced separately as knock-on electrons. The values for E_0 are 86 Mev for air and 6.7 Mev for lead. With these units, we obtain the following probabilities:

Pair Production

At high energies (complete screening)

$$\sigma(K,E)dE=\frac{dE}{K}2\left[1-\frac{4}{3}\frac{K}{E}+\frac{4}{3}\frac{K^2}{E^2}\right]. \quad (A4)$$

The factor 2 is introduced to include both electron and positron. At low energies, the approximation

$$\sigma(K,E)dE=\frac{dE}{K}\times 2\sigma_{\text{pair}} \quad (\text{for } E<E_0), \quad (A5)$$

is made, where σ_{pair} is the total cross section for pair production as evaluated by Heitler.[7] E_0 has been taken as 1.15 for air and 10 for lead.

Radiation by Electrons

At high energies the cross section is

$$\sigma(E,K)dK=\frac{dK}{K}\left[\frac{4}{3}-\frac{4}{3}\frac{K}{E}+\frac{K^2}{E^2}\right]. \quad (A6)$$

[6] E. Jahnke and F. Emde, *Tables of Functions* (Dover Publications, New York, 1943).

[7] W. Heitler, *Theory of Radiation* (Oxford University Press, London, 1945), 2nd ed.

At lower energies the simple interpolation formula

$$\sigma(E, K)dK = \frac{dK}{K}\left[\frac{4}{3} - \frac{4}{3}\frac{K}{E} + \frac{K^2}{EE_0}\right], \quad \text{(A7)}$$

gives an extremely satisfactory approximation as can be verified by comparison with the graphs given by Rossi and Greisen.[1]

Compton Effect

The probability for a Compton transition per unit radiation length is for $K' \gg mc^2$

$$\sigma(K, K')dK' = \frac{BdK'}{KK'}\left(1 + \frac{K'^2}{K^2}\right), \quad \text{(A8)}$$

with $B = 137\pi mc^2/4\bar{Z}\ \lg 183Z^{-\frac{1}{3}} = 0.017$ for air and 0.0265 for lead. The integrated Compton probability is

$$\sigma(K) = \frac{B}{K}\left(\lg\frac{2K}{mc^2} + \frac{1}{2}\right). \quad \text{(A9)}$$

The average energy $\bar{E}$ of an electron produced in a Compton collision with a quantum K is

$$\bar{E} = K\left[1 - \frac{4}{3}\left(\lg\frac{2K}{mc^2} + \frac{1}{2}\right)^{-1}\right]. \quad \text{(A10)}$$

The *absorption coefficient* for photons is obtained from the combination of Compton and pair production cross sections.

Knock-on Probability

$$\sigma_k(E, E') = \frac{2BdE}{E'^2}\left(1 - \frac{2E'}{E}\right), \quad \text{(A11)}$$

where B is the same as for the Compton effect.

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Microwave Spectra of Some Linear *XYZ* Molecules

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Measurements of the pure rotational spectra of a number of isotopes and vibrational states of OCS, ClCN, BrCN, and ICN near one centimeter wave-length have been made. Experimental techniques used and interpretation of the spectra of linear *XYZ* molecules are discussed. Tables include frequencies and intensities of lines and comparison with theoretical values, rotational constants B_0, rotation-vibration constants α, l-type doubling constants, internuclear distances, half-width parameters of lines, quadrupole coupling constants, and nuclear quadrupole moments. Agreement between experimental results and available theory is good in all cases except for the values of l-type doubling constants.

SINCE much of the pure rotational spectra of molecules lies in the microwave region, and since microwave techniques have been developed to give both high resolution and accurate frequency measurements, microwave spectroscopy may be expected to make a considerable contribution to the study of molecular rotational spectra.

Among the simplest of the molecules which can be studied with microwave techniques are those of the linear *XYZ* type, when their end-to-end dissymmetry is sufficient to give them an appreciable dipole moment. Their spectra are relatively simple and intense, so that they provide a good test of theory and techniques, and a good introduction into more complex types of molecules. Only brief statements of some of the more important observations on this type of spectra have so far been published. Even though microwave study of these simple linear molecules may be regarded as just well begun, it seems

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