The Focusing in Crossed Fields of Charged Particles at Relativistic Energies

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The motion of charged particles traveling with relativistic energies in a uniform magnetic field superimposed at right angles to the electric field of a cylindrical condenser has been investigated. Expressions for the angle at which first-order focusing of a divergent beam of particles occurs and first approximations to the line breadth as well as dispersion in velocity, momentum, and energy are given.

INTRODUCTION

THE motion of charged particles traveling with non-relativistic energies in crossed electric and magnetic fields has been investigated by W. Henneberg.[1] His formulas are in a convenient form and are sufficiently accurate to be used in the design of an apparatus employing ions traveling with kinetic energies which are very much less than their proper energy. However, if these formulas are applied to systems employing particles with kinetic energies which are of the order of five percent or more of their proper energy, an error may result.

The purpose of this paper is to extend Henneberg's results to include particles with relativistic energies. The notation used here will be the same as that used by Henneberg.

EQUATIONS OF MOTION

The field configuration considered is a superposition of the electric field E of a cylindrical condenser and a homogeneous magnetic field H directed parallel to the cylinder axis. A schematic cross section of the field is shown in Fig. 1. The equations of motion will be described in terms of the cylindrical coordinates r and φ with origin at the center of curvature of the condenser and z axis along the axis of the condenser.

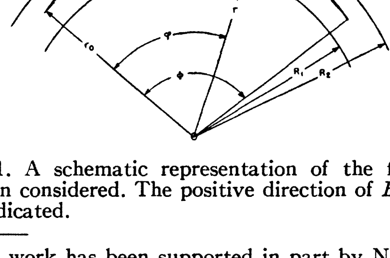

FIG. 1. A schematic representation of the field configuration considered. The positive direction of E, H, and φ are indicated.

In such a field a charged particle of relativistic mass m and charge e at r, moving with angular velocity $\omega = d\varphi/dt$ and radial velocity $\dot{r}$, will satisfy:**

$$d(m\dot{r})/dt = mr\omega^2 - eE - Her\omega/c, \qquad (1)$$

$$d(mr^2\omega - \tfrac{1}{2}Her^2/c)/dt = 0, \qquad (2)$$

$$d(mc^2)/dt = -eE\dot{r}. \qquad (3)$$

The positive direction of E, H, and φ is indicated in Fig. 1.

From Eq. (1) it is seen that the electric and magnetic field strengths may be adjusted so that the electric force and the centrifugal force have a resultant which is just canceled by the radial magnetic force acting on the particle, thereby making the radial momentum constant in time. With the field strengths adjusted so that this condition is realized, a charged particle traveling initially with a velocity $v_0 = r_0\omega_0$ which has no radial component will describe a circular orbit of radius r_0.

In order to satisfy this condition for circular orbits, it is clear that the ratio of the electric force to the centrifugal force may be any positive or negative number so long as the radial magnetic force is adjusted to cancel the combined electric and centrifugal forces. Because of this generality, Henneberg introduced the parameter y which he defined as follows:

$$y = (eE/mr_0\omega_0^2)_0.\text{***} \qquad (4)$$

* This work has been supported in part by Navy Contract N5 ori-76, Task Order IV, Part D.

[1] W. Henneberg, Ann. d. Physik **19**, 335 (1934).

** This consideration will be confined to particles moving in the plane $z=0$.

*** The zero subscript on an expression in parentheses indicates that the expression is evaluated at $t=0$. The zero subscript on a single symbol not in parentheses denotes the

The notation may be simplified slightly by making the following substitutions:

$$|eH/mc|=h;\quad \pm|eE/m|=\mathcal{E}/r. \tag{5}$$

($\mathcal{E}$ will be positive or negative depending on whether the electric force acting on the particle is directed inward or outward.) It is to be noted that in this form the spatial variation of E is stated explicitly.

By expanding the left member of Eq. (1) and making use of Eqs. (3) and (5), Eq. (1) becomes

$$\ddot{r}-(\mathcal{E}/r)(\dot{r}^2/c^2)=r\omega^2-\mathcal{E}/r-hr\omega. \tag{6}$$

Integration of Eq. (2) gives

$$mr^2\omega-\tfrac{1}{2}mhr^2=C=(mr^2\omega-\tfrac{1}{2}mhr^2)_0. \tag{7}$$

From Eq. (3) it follows that m may be expressed as a function of r alone. Then ω may be expressed as the following function of r,

$$\omega=C/mr^2+\tfrac{1}{2}h, \tag{8}$$

and eliminated from (6) giving

$$\ddot{r}-(\mathcal{E}/r)(\dot{r}^2/c^2)=C^2/m^2r^3-\tfrac{1}{4}h^2r-\mathcal{E}/r. \tag{9}$$

FIRST-ORDER SOLUTION: FOCUSING

Consider a source of particles located at $r=r_0$, $\varphi=0$. The field strengths are adjusted so that a particle with speed $r_0\omega_0$ will describe a circular orbit if it leaves the source with zero radial velocity. Particles which satisfy these initial conditions must by (6) satisfy

$$r_0\omega_0^2=\mathcal{E}_0/r_0+h_0r_0\omega_0 \tag{10}$$

as well as

$$C_0=(m)_0r_0^2\omega_0-\tfrac{1}{2}(mh)r_0^2. \tag{11}$$

The motion of particles which diverge initially from the circular orbit by an angle α (see Fig. 1) but have the same speed $r_0\omega_0$ may be described in terms of α and the parameters characterizing the circular orbit. To do this, the variables r and ω and the parameter C are expressed in terms of power series in α, namely:

$$\begin{aligned} r&=r_0+\alpha r_1+\alpha^2 r_2+\cdots,\\ \omega&=\omega_0+\alpha\omega_1+\alpha^2\omega_2+\cdots,\\ C&=C_0+\alpha C_1+\alpha^2 C_2+\cdots. \end{aligned} \tag{12}$$

If α is small, the terms involving α to a power

value of the quantity associated with the stable or circular orbit. The only exception is m_0 which is taken to be the symbol for the rest mass of the particle.

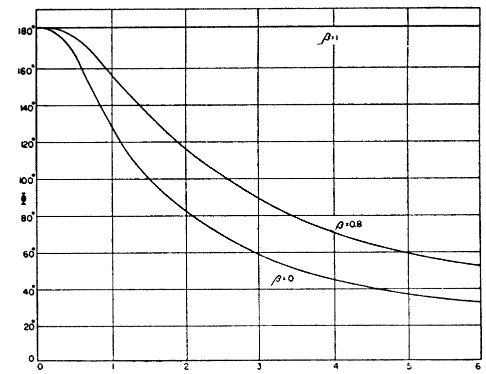

FIG. 2. A plot of the angle of focus Φ *versus* the absolute value of the field parameter y for three values of the relative speed of the particle, $\beta=r_0\omega_0/c$.

higher than the first may be neglected in Eqs. (12).

In order to find the differential equation r_1 must satisfy, terms containing powers of α higher than the first are neglected and Eq. (9) is expanded by Taylor's series. That is,

$$f(r_0+\alpha r_1)\doteq f(r_0)+(df/dr)_0\alpha r_1.$$

Then upon equating coefficients of α,

$$\begin{aligned} \ddot{r}_1=\;&2C_0C_1/(m^2)_0r_0^3-(3C^2/m^2r_0^4)_0r_1\\ &-(2C^2/m^3r_0^3)_0(dm/dr)_0r_1-\tfrac{1}{4}(h^2)_0r_1\\ &-\tfrac{1}{2}(hr_0)_0(dh/dr)_0r_1-[d(\mathcal{E}/r)/dr]_0r_1. \end{aligned} \tag{13}$$

This equation may be simplified by rewriting the expression on the right in terms of ω_0, y, and β, where $\beta=r_0\omega_0/c$. In regard to C^2 it should be noted that from Eq. (7) it follows that

$$(\omega_1)_0=(C_1/mr^2)_0.$$

However, since only particles which have the speed $r_0\omega_0$ are being considered, it is seen that

$$\begin{aligned} (r\omega)_0&\doteq r_0(\omega_0+\alpha\omega_1+\alpha^2\omega_2)_0\\ &=r_0\omega_0\cos\alpha\doteq r_0(\omega_0-\tfrac{1}{2}\alpha^2\omega_0). \end{aligned} \tag{14}$$

Then $(\omega_1)_0=0$, hence $C_1=0$ and $C^2=C_0^2$ to terms of first order in α.

The simplified form of Eq. (13) is

$$\ddot{r}_1=-[1+y^2(1-\beta^2)]\omega_0^2r_1. \tag{15}$$

The solution of this well-known differential equation subject to the initial conditions $r_1=0$, $\dot{r}_1=r_0\omega_0$ at $t=0$ may be written at once:

$$r_1=\frac{r_0\sin\omega_0[1+y^2(1-\beta^2)]^{\frac{1}{2}}t}{[1+y^2(1-\beta^2)]^{\frac{1}{2}}}. \tag{16}$$

So, to terms of first order in α, r becomes equal to r_0 when the particle has swept through the angle

$$\varphi=\Phi=\pi[1+y^2(1-\beta^2)]^{-\frac{1}{2}}. \quad (17)$$

A plot of Φ *versus* the absolute value of y is given for three values of β in Fig. 2.

SECOND-ORDER SOLUTION: LINE BREADTH

When second-order terms are taken into account it will be found that the image of the source formed at $\varphi=\Phi$ is "smeared out" on one side or the other of r_0 by an amount which depends on α^2. The extent of the smearing out of the image is referred to as the line breadth and is denoted by B. As a first approximation of B, r_2 will be found and evaluated at $\varphi=\Phi$.

To find the differential equation r_2 must satisfy, Taylor's expansion is applied again to (9). Here

$$f(r_0+\alpha r_1+\alpha^2 r_2)\doteq f(r_0)+(df/dr)_0(\alpha r_1+\alpha^2 r_2)+\tfrac{1}{2}(d^2f/dr^2)_0(\alpha^2 r_1^2+\cdots),$$

or upon equating the coefficients of α^2,

$$\ddot{r}_2=2C_0C_2/(m^2)_0 r_0^3+(\mathcal{E}/r_0)_0(\dot{r}_1^2/c^2)_0+(df/dr)_0 r_2+\tfrac{1}{2}(d^2f/dr^2)_0 r_1^2, \quad (18)$$

where from Eq. (15) it is seen that

$$(df/dr)_0=-[1+y^2(1-\beta^2)]\omega_0^2.$$

By differentiation and simplification,

$$(d^2f/dr^2)_0=(2\omega_0^2/r_0)[3/2+2y(1-\beta^2)+3y^2(1-\beta^2)/2-2y^3\beta^2(1-\beta^2)].$$

Then

$$\ddot{r}_2=A_0-\gamma^2 r_2+A_1\sin^2\gamma t, \quad (19)$$

where

$$A_0=-\tfrac{1}{2}[1+y(1-2\beta^2)]r_0\omega_0^2,$$
$$\gamma^2=[1+y^2(1-\beta^2)]\omega_0^2,$$

and

$$A_1=\frac{3+4y+3y^2(1-\beta^2)-6y\beta^2[1+y^2(1-\beta^2)]}{2[1+y^2(1-\beta^2)]}r_0\omega_0^2.$$

Subject to the initial conditions $r_2=\dot{r}_2=0$ at $t=0$, the solution is

$$r_2=a_1\sin^2\gamma t+a_2(1-\cos\gamma t), \quad (20)$$

where

$$a_1=-r_0\frac{3+4y+3y^2(1-\beta^2)-6y\beta^2[1+y^2(1-\beta^2)]}{6[1+y^2(1-\beta^2)]^2} \quad (21)$$

and

$$a_2=r_0\frac{3+5y+3y^2(1-\beta^2)-3y^3(1-\beta^2)-6y\beta^2[1+y^2(1-\beta^2)]}{6[1+y^2(1-\beta^2)]^2}. \quad (22)$$

But now to get the line breadth it is necessary to evaluate r_2 at the time when $\varphi=\Phi$. To do this, an expression for ω_1 must be obtained. A simple way to obtain this expression is to carry out the differentiation indicated in Eq. (2). The result of this operation may be written in the form of

$$\dot{\omega}_1=-[1+y(1-\beta^2)]\omega_0\cos\gamma t.$$

Subject to the initial conditions $\omega_1=0$ for $t=0$ it is seen that

$$\omega_1=-\gamma^{-1}\omega_0^2[1+y(1-\beta^2)]\sin\gamma t.$$

Then, by integration,

$$\varphi=\omega_0 t+\alpha\frac{1+y(1-\beta^2)}{1+y^2(1-\beta^2)}(\cos\gamma t-1). \quad (23)$$

For $\varphi=\Phi$,

$$t=T=\frac{\pi}{\gamma}+\frac{2\alpha}{\omega_0}\cdot\frac{1+y(1-\beta^2)}{1+y^2(1-\beta^2)}. \quad (24)$$

Then set

$$r=r_0+B \text{ for } \varphi=\Phi, \quad t=T,$$

where B is the line breadth.

So from (20) and (24)

$$B\doteq 2\alpha^2 r_0\left[\frac{a_2}{r_0}-\frac{1+y(1-\beta^2)}{1+y^2(1-\beta^2)}\right].$$

By (22) this becomes

$$B\doteq -\alpha^2 r_0\frac{1+\frac{1}{3}y+y^2(1-\beta^2)+3y^3(1-\beta^2)}{[1+y^2(1-\beta^2)]^2}.\dagger$$

DISPERSION AND REDUCED DISPERSION

The line breadth is a quantity which is useful in the description of the performance of a system. Still other useful quantities are the dispersion

† The expression given in the minutes of the 1947 annual meeting (Phys. Rev. **73**, 1259 (1948)) is incorrect.

TABLE I. Expressions for the angle of focus Φ, line breadth B, velocity dispersion D_v, potential difference between the plates of the cylindrical condenser V, and the magnetic field strength H for the four field parameters most frequently used.

	Magnetic $y=0$	Electric $y=1$	Achromatic $y=-1/(1-\beta^2)$	Wein velocity filter $r_0=\pm\infty$; $r_0/y=(mc^2\beta^2/eE)_0$
Φ	π	$\frac{\pi}{\sqrt{2}}\cdot\frac{\sqrt{2}}{(2-\beta^2)^{\frac{1}{2}}}$	$\frac{\pi}{\sqrt{2}}\cdot\left(\frac{2(1-\beta^2)}{2-\beta^2}\right)^{\frac{1}{2}}$	$L=\frac{m_0c^2\beta^2\pi}{eE}\cdot\frac{1}{1-\beta^2}$ $=\pi R_H\cdot\frac{1}{(1-\beta^2)^{\frac{1}{2}}}$
B	$-\alpha^2r_0$	$\frac{-4\alpha^2r_0}{3}\cdot\frac{4-3\beta^2}{(2-\beta^2)^2}$	$\frac{\alpha^2r_0}{3}\cdot\frac{4+8\beta^2-3\beta^4}{(2-\beta^2)^2}$	$-\frac{3\alpha^2m_0c^2\beta^2}{eE}\cdot\frac{1}{(1-\beta^2)^{\frac{3}{2}}}$ $=-3\alpha^2R_H\cdot\frac{1}{1-\beta^2}$
D_v	$2r_0\cdot\frac{1}{1-\beta^2}$	$2r_0\cdot\frac{1}{1-\beta^2}$	0	$\frac{2m_0c^2\beta^2}{eE}\cdot\frac{1}{(1-\beta^2)^{\frac{3}{2}}}$ $=2R_H\cdot\frac{1}{1-\beta^2}$
V	0	$\frac{m_0c^2\beta^2}{e}\ln\frac{R_2}{R_1}\cdot\frac{1}{(1-\beta^2)^{\frac{1}{2}}}$	$\frac{m_0c^2\beta^2}{e}\ln\frac{R_2}{R_1}\cdot\frac{1}{(1-\beta^2)^{\frac{3}{2}}}$	$\frac{(R_2-R_1)m_0c^2\beta^2\pi}{eL}\cdot\frac{1}{1-\beta^2}$
H	$\frac{m_0c^2\beta}{er_0}\cdot\frac{1}{(1-\beta^2)^{\frac{1}{2}}}$	0	$\frac{m_0c^2\beta}{er_0}\cdot\frac{2-\beta^2}{(1-\beta^2)^{\frac{3}{2}}}$	$H=E/\beta$

and reduced dispersion in velocity, momentum, and energy. The expressions for the dispersion in velocity and in momentum should become equivalent in the limit of small velocities. The velocity dispersion should be equal to β^2 times the energy dispersion in the same limit.

Consider first what is meant by velocity dispersion.

A particle which leaves the source at r_0 along the tangent to the circular orbit not with velocity $v_0=r_0\omega_0$ but v_0+dv will be a distance dx from r_0 after it has swept through the angle $\varphi=\Phi$. This distance will be proportional to the fractional increment in velocity, i.e., $dx=D_vdv/v_0$. The proportionality constant D_v is called the velocity dispersion. A first approximation of D_v is obtained by setting $r=r_0+x$ and determining r to the first order, setting $\varphi=\Phi$ and comparing with $r=r_0+D_vdv/v_0$.

Setting $r=r_0+x$, $v=v_0+dv$, and applying Taylor's expansion for functions of two variables to Eq. (9) results in

$$\ddot{x}=-\gamma^2x+adv, \tag{26}$$

where

$$a=\omega_0[1+y(1-\beta^2)]/(1-\beta^2)$$

and γ^2 is as defined in connection with Eq. (19). The solution for initial condition $x=0$ at $t=0$ is

$$x=a\gamma^{-2}(1-\cos\gamma t)dv. \tag{27}$$

Then for $\varphi=\Phi$, $x=2a\gamma^{-2}dv$. Upon comparison with $r-r_0=x=D_vdv/v_0$ it is seen that a first approximation to the velocity dispersion is

$$D_v\doteq\frac{2r_0}{1-\beta^2}\cdot\frac{1+y(1-\beta^2)}{1+y^2(1-\beta^2)}. \tag{28}$$

Expressions for the momentum dispersion D_p and the energy dispersion D_m may be obtained in a manner similar to that indicated above. The momentum dispersion is defined by the following equation:

$$dx=D_pdp/p_0;$$

the energy dispersion by

$$dx=D_md(mc^2)/mc^2=D_mdm/m.$$

The resulting expressions are

$$D_p\doteq 2r_0\frac{1+y(1-\beta^2)}{1+y^2(1-\beta^2)} \tag{29}$$

and

$$D_m\doteq\frac{2r_0}{\beta^2}\cdot\frac{1+y(1-\beta^2)}{1+y^2(1-\beta^2)}. \tag{30}$$

It is interesting to note that

$$D_p^{-1}=D_m^{-1}+D_v^{-1},$$

an equality which may be obtained directly from the definition of momentum.

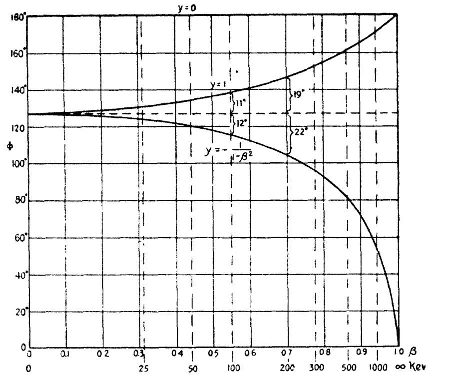

FIG. 3. A plot of the angle of focus Φ *versus* the relative speed $\beta = r_0\omega_0/c$ for three common values of the field parameter y. A kinetic energy scale for electrons is indicated along the β-axis.

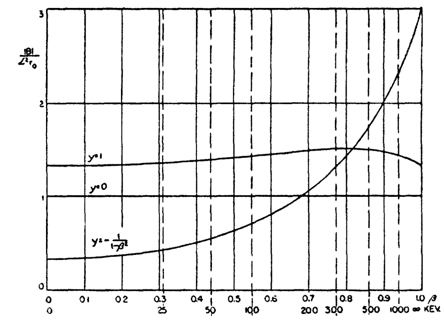

FIG. 4. A plot of the absolute value of the line breadth divided by the square of the angle of divergence and the radius of the circular orbit $|B|/\alpha^2 r_0$ *versus* the relative speed of the particle $\beta = r_0\omega_0/c$ for three common values of the field parameter y. A kinetic energy scale for electrons is indicated along the β-axis.

All three of these expressions vanish when $y = -1/(1-\beta^2)$. It can be shown that $dr_0/dv = dr_0/dp = dr_0/dm = 0$ for $y = -1/(1-\beta^2)$.

The reduced dispersion δ is a quantity which serves as a better figure of merit than D or B alone. It is defined by

$$\delta = |D/B|\alpha^2.$$

This analysis indicates that a first approximation of δ_v is

$$\delta_v \doteq \frac{2}{1-\beta^2} \cdot \frac{1+y(1-\beta^2)+y^2(1-\beta^2)+y^3(1-\beta^2)^2}{1+\frac{1}{3}y+y^2(1-\beta^2)+3y^3(1-\beta^2)}.$$

The expressions for Φ, B, D_v, and δ_v reduce to those obtained by Henneberg if β is set equal to zero.

DISCUSSION OF RESULTS

It is well known that for the pure magnetic case ($y=0$) the focusing angle is equal to π independent of β, as it is here. Referring to Table I it is seen that B is also independent of β for this case. This is as it should be since in this case the mass remains unchanged over the trajectory. On the other hand, D_v should be expected to increase as β increases, since a small fractional change in velocity gives rise to an increasingly larger fractional change in mass as β increases.

The symbol V in Table I refers to the potential which must be applied across the cylindrical condenser. The general expressions for V and H are

$$V = (m_0c^2/e)(\ln R_2/R_1)[\beta^2/(1-\beta^2)^{\frac{1}{2}}]y,$$
$$H = (m_0c^2/er_0)[\beta/(1-\beta^2)^{\frac{1}{2}}](1-y).$$

Some of the expressions in this table contain a dot. That part of the expression to the left of the dot is the non-relativistic term while that to the right is the relativistic correction.

The second column of the table refers to the pure electric case ($y=1$) which was first investigated by Hughes and Rojansky.[2] The expressions for Φ and β are plotted against β in Figs. 3 and 4. Along the β coordinate the kinetic energy of electrons in kilo-electron volts is indicated.

The case for which there is both direction focusing and velocity focusing was first investigated by Bartky and Dempster.[3] This is the case for which $D_v=0$, ($y=-1/(1-\beta^2)$), which Henneberg refers to as the "achromatic" case. The expressions for Φ and B are plotted against β in Figs. 3 and 4 for this case, too.

Prior to the present calculations, Professor J. S. Schwinger has, in an unpublished note to Professor K. T. Bainbridge, calculated the relativistic expression for Φ for the achromatic case. Schwinger imposed the condition that dr_0/dv vanish. This is equivalent to the vanishing of D_v. The results obtained here are in agreement with his results. A study of Professor Schwinger's note was of great aid in the solution of the more general problem.

It should be pointed out that the power series given here converge for a limited range of values

[2] A. L. Hughes and V. Rojansky, Phys. Rev. **34**, 284 (1929).

[3] W. Bartky and A. J. Dempster, Phys. Rev. **33**, 1019 (1929).

of α. This range of α decreases in extent as $y\beta^2$ increases. When $y\beta^2$ goes to infinity the range of α goes to zero. Consequently, the expressions for Φ, B, and $\dot{D}$ for the achromatic case are not valid in the limit as $\beta \rightarrow 1$.

The expressions for the Wien velocity filter are given in the last column of the table. In a Wien velocity filter a parallel plate condenser replaces the cylindrical condenser. The length of the filter, L, i.e., the distance between source and image, may be obtained from the limit of $r_0\Phi$ as $r_0 \rightarrow \infty$, with $r_0/y = (mv^2/eE)_0$. The calculation of the relativistic value of L was also contained in the note of Schwinger which was referred to above.

The expressions for B and D_v for the Wien filter result from the same limiting process. The R_H which appears in the table stands for the radius the particle would describe if the magnetic field alone were present. The minus sign in the expression for B is included to indicate that B and D_v have opposite signs.

To Professor K. T. Bainbridge, who suggested this problem, the author is deeply indebted for valuable suggestions and aid. Henneberg's article as well as problems arising in the relativistic treatment were discussed with Professor Bainbridge and Mr. F. L. Niemann. Professor E. M. Purcell and Professor W. H. Furry have read this article and suggested several changes.

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Contributions to the Nuclear Processes Induced in Magnesium by Polonium Alpha-Particles

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Mg metal was bombarded in hemispherical symmetrical arrangements from the center by a strong Po source. The excitation function of the induced Al^{28} activity and of the induced γ-radiation was determined with an improved resolving power and accuracy. The absolute yields were determined carefully. Possibilities for the origin of the γ-radiation are discussed.

As is known, Mg consists of 3 isotopes ($A = 24$, 25, 26) amounting to 77.4, 11.5 and 11.1 percent. Six possible nuclear transformations may occur in the case of the immigration of an α-particle:

1. $Mg^{24}(\alpha,p)Al^{27}$
2. $Mg^{25}(\alpha,p)Al^{28}$
3. $Mg^{26}(\alpha,p)Al^{29}$
4. $Mg^{24}(\alpha,n)Si^{27}$
5. $Mg^{25}(\alpha,n)Si^{28}$
6. $Mg^{26}(\alpha,n)Si^{29}$

and there is the possibility of the inelastic scattering of the α-particle followed by the γ-radiation of the excited Mg nucleus.

The main subject of this paper is to investigate the exact shape of the excitation function and the absolute yield of the short-living artificial radioactivity (Al^{28}), and the excitation function, quantum energy, absolute yield and origin of the γ-radiation at α-energies below 5.3 Mev.

APPARATUS

Po Preparation

The technics of Po preparations used at this Institute enabled us—by means of a volatilization method[1]—to obtain very pure Po-sources with a strength of about 10 mC or more, on a highly polished Pt-Ir disk of 3 mm diameter.

Activation Apparatus

The Po-source was located in the center of the activation apparatus and brass hemispheres of 5 cm diameter, coated on the inside with a thick pressed Mg metal plate, were placed over it (Fig. 1). Then the air was removed and CO_2 gas of suitable pressure was let in to keep the α-particles down to the energy required. The geometry

[1] A. Szalay, Zeits. f. Physik 112, 29 (1939).