at 1.82 Mev (~67 percent) and 0.56 Mev (~33 percent). The maximum gamma-ray energy was found to be 1.12 Mev by coincidence absorption of Compton recoil electrons produced in an aluminum target.

Figure 4 shows the result of beta-gamma-coincidence measurements. It is evident that no coincidences are present for beta-ray absorption thicknesses greater than 0.23 g/cm^2 of aluminum, which corresponds to 0.62 Mev. Hence the transitions which involve the 1.76 Mev beta-ray group lead directly to the ground state of Sr^{86}. Those coincidences which appear between this absorber thickness and zero absorber thickness are to be ascribed to disintegrations which involve the low-energy beta-ray group. No evidence of gamma-gamma-coincidence could be found; it is to be concluded that the low energy group of beta-rays leads to a 1.12 Mev level of Sr^{86}, which in turn goes to the ground state with the emission of a single gamma-ray.

The spectrum of Rb^{86} has recently been measured in a magnetic lens by Zaffarano, Kern, and Mitchell,[15] who found one gamma-ray of energy 1.081 Mev and two beta-ray groups with end points at 1.822 and 0.716 Mev. The results described in the present paper confirm their results.

VI. ACKNOWLEDGMENTS

The author expresses his thanks to Dr. R. G. Wilkinson who generously provided the use of Sc^{46} and Au^{198} sources which he had prepared for spectrographic measurements, and to Professor A. C. G. Mitchell for the use of his purified Rb^{86} source. He also wishes to thank Professor Mitchell for interest and suggestions throughout the course of the experiments. This work was supported by the Office of Naval Research under Contract N6ori-48, T. O. 1.

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[15] D. J. Zaffarano, B. D. Kern, and A. C. G. Mitchell, Phys. Rev. **74**, 682 (1948).

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Additional Rotational Energy Levels of H_2O and D_2O Molecules*

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An extension of the pure rotational energy levels for H_2O and D_2O is made in order to obtain new possible microwave transitions. For H_2O the levels are given through $J=14$ and for D_2O through $J=15$. Additional assignments are made in the previously observed absorption spectra. The Wang secular determinant for the asymmetric rotator is expanded, and equations for the the energy levels of $J=12$, 13, 14, and 15 are given in terms of the asymmetry parameter.

INTRODUCTION

THE calculation by Van Vleck[1] on the absorption of microwaves by water vapor showed an absorption greater than the amount predicted by the theory as developed. An effort to account for the residual absorption on the basis of line widths was unsuccessful. It was thought that the additional intensity might be accounted for by an accidental coincidence with a line corresponding to a transition value between levels of J higher than 11. The absorption spectrum of water vapor had previously been observed and analyzed by several investigators, the most satisfactory and complete analysis being that of Randall, Dennison, Ginsburg, and Weber[2] (hereafter referred to as RDGW). They obtained the pure rotational spectrum in the region from 140 to 555 cm^{-1} and combined their results with those of Wright and Randall[3] in the longer wave-

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* The calculations and energy levels for H_2O were presented at the Symposium on Molecular Structure and Spectroscopy, Ohio State University, Columbus, Ohio (June, 1947).

[1] J. H. Van Vleck, Phys. Rev. **71**, 425 (1947).

[2] Randall, Dennison, Ginsburg, and Weber, Phys. Rev. **52**, 160 (1937).

[3] N. Wright and H. M. Randall, Phys. Rev. **44**, 39 (1933).

TABLE I. The rotational energy levels of the H_2O molecule. The levels for the $J=11$ and the lower levels for the higher J values are taken from RDGW. W is the difference between levels obtained from the spectrum and those calculated from solution of the rigid asymmetric top equations.

J / τ	11		12		13		14	
	W	δW	W	δW	W	δW	W	δW
14							4971.3	648.2
13					4353.6	503.0	4971.3	648.2
12			3768.88	380.49	4353.6	503.0	4701.5	494.6
11	3218.55	279.19	3768.88	380.49	4090.9	373.6	4701.5	494.6
10	3218.55	279.19	3514.88	273.90	4090.9	373.6	4432.0	372.7
9	2974.88	193.68	3514.88	273.90	3832.0	272.7	4432.0	372.7
8	2974.88	193.68	3267.69	192.57	3832.0	272.7	4172.5	273.0
7	2741.42	130.06	3267.69	192.57	3584.1	192.9	4172.5	273.0
6	2741.42	130.06	3033.78	130.31	3584.1	192.9	3943.5	197.9
5	2523.08	83.77	3033.78	130.31	3351.5	130.5	3943.5	197.9
4	2523.08	83.77	2815.03	85.73	3351.5	130.5	3689.0	136.4
3	2322.82	52,51	2815.03	85.71	3128.5	91.9	3689.0	136.4
2	2322.77	52.47	2612.94	58.39	3128.5	91.8	3466.9	99.1
1	2145.17	33.80	2613.21	57.79	2933.2	62.3	3466.9	99.2
0	2144.06	33.26	2438.16	41.51	2930.4	62.4	3273.8	69.8
− 1	2000.71	25.74	2435.69	39.46	2765.6	44.0	3273.9	66.0
− 2	1984.20	26.65	2297.61	41.02	2756.7	42.1	3118.7	55.6
− 3	1898.56	27.39	2273.36	34.52	2636.1	45.3	3094.1	53.7
− 4	1840.14	22.39	2209.98	32.98	2594.2	34.3	3003.2	49.7
− 5	1811.21	25.36	2116.21	34.64	2543.1	39.7	2928.7	42.9
− 6	1694.71	14.26	2107.08	28.68	2431.0	28.6	2896.2	46.2
− 7	1691.13	14.64	1962.20	18.55	2419.2	32.4	2754.6	33.5
− 8	1525.65	7.68	1961.00	18.30	2243.5	27.9	2746.3	37.8
− 9	1525.36	7.80	1775.26	10.36	2242.7	28.1	2545.7	35.0
−10	1327.58	4.33	1775.26	10.30	2043.0	13.5	2545.4	34.9
−11	1327.58	4.33	1558.42	6.03	2043.0	13.5	2332.6	18.8
−12			1558.42	6.03	1807.4	8.2	2332.6	18.8
−13					1807.4	8.2	2074.3	10.9
−14							2074.3	10.9

length region to obtain all energy levels through $J=11$, together with some for higher J values. Some of the lines they observed were not assigned. In addition there was available the prism measurements made by Weber and Randall[4] which extended the absorption spectrum of water vapor to shorter wave-lengths. It was thought possible, by using these measurements, to extend the analysis.

A number of absorption lines in the spectrum of D_2O as observed by Fuson, Randall, and Dennison[5] (hereafter referred to as FRD) had not been assigned. It was thought worth while to extend the calculations made for H_2O to include the D_2O molecule as well and obtain assignments for as many of these lines as possible.

CALCULATION

It was necessary to extend the equations obtained by Nielsen[6] from the Wang secular determinant for the asymmetrical rotator. He had obtained the equations through $J=10$, and in the paper of RDGW that for $J=11$ was added. The equations for the higher J values, through $J=15$, are given in the Appendix.

Making use of the values previously assigned by RDGW for the moments of inertia of the water molecule, namely

$$h/8\pi^2cA=27.8055,\quad h/8\pi^2cB=14.499724,$$
$$h/8\pi^2cC=9.279276$$

giving the parameter of b the value -0.164 exactly, these equations were solved for the energy levels. The values through $J=14$ were obtained for H_2O and through $J=15$ for D_2O. Since the mass of the molecule is less for the former than for the latter, the spectrum extends to higher frequencies. In this region the wave-lengths, and consequently the frequencies, are known with much less accuracy since they were obtained by prism spectrometer measurements.

TABLE II. New assignments in the spectrum of H_2O. The last fourteen lines are from Weber and Randall's prism measurements.

cm^{-1}	Assignment $J_{\tau'}'\leftarrow J_\tau$	cm^{-1}	Assignment $J_{\tau'}'\leftarrow J_\tau$
550.33	$12_{12}-11_{10}$	280.46	$13_{-9}-12_{-7}$
	$12_{11}-11_{11}$	267.73	$13_{-10}-12_{-10}$
	$13_{7}-12_{5}$	266.94	$12_{-6}-11_{-4}$
	$13_{6}-12_{6}$	266.29	$12_{-8}-11_{-6}$
536.48	$13_{5}-12_{3}$	200.44	$13_{-3}-12_{-1}$
	$13_{4}-12_{4}$	188.28	$14_{-2}-13_{0}$
	$14_{2}-13_{0}$	152.55	$13_{-1}-12_{1}$
526.27	$12_{8}-11_{6}$	139.84	$12_{-2}-11_{0}$
	$12_{7}-11_{7}$		
519.81	$13_{-3}-12_{-5}$	617.7	$14_{14}-13_{12}$
518.09	$14_{0}-13_{-2}$		$14_{13}-13_{13}$
515.27	$12_{-4}-11_{-6}$	610.6	$14_{12}-13_{10}$
	$13_{3}-12_{1}$		$14_{11}-13_{11}$
	$13_{2}-12_{2}$	600.0	$14_{10}-13_{8}$
502.45	$14_{-10}-13_{-8}$		$14_{9}-13_{9}$
492.22	$12_{4}-11_{2}$	592.0	$14_{8}-13_{6}$
	$12_{3}-11_{3}$		$14_{7}-13_{7}$
	$13_{0}-12_{0}$	584.7	$13_{13}-12_{11}$
	$13_{-1}-12_{-3}$		$13_{12}-12_{12}$
472.68	$13_{-2}-12_{-2}$		$14_{6}-13_{4}$
468.88	$12_{2}-11_{0}$		$14_{5}-13_{5}$
468.04	$12_{1}-11_{1}$	580.7	$13_{-5}-12_{-7}$
458.05	$14_{-3}-13_{-3}$	576.0	$13_{11}-12_{9}$
453.96	$12_{0}-11_{-2}$		$13_{10}-12_{10}$
443.90	$12_{-2}-11_{-4}$	564.3	$13_{9}-12_{7}$
434.98	$12_{-1}-11_{-1}$		$13_{8}-12_{8}$
385.58	$14_{-5}-13_{-5}$	560.5	$14_{4}-13_{2}$
384.21	$13_{-4}-12_{-4}$		$14_{3}-13_{3}$
374.80	$12_{-3}-11_{-3}$	540.0	$12_{10}-11_{8}$
335.34	$14_{-7}-13_{-7}$		$12_{9}-11_{9}$
323.90	$13_{-6}-12_{-6}$	533.7	$14_{1}-13_{1}$
303.00	$13_{-7}-12_{-5}$	497.5	$13_{1}-12_{-1}$
	$14_{-9}-13_{-9}$	315.32	$14_{-8}-13_{-6}$
302.04	$14_{-6}-13_{-4}$	305.00	$12_{-5}-11_{-5}$
289.65	$14_{-11}-13_{-11}$		
282.45	$13_{-8}-12_{-8}$		

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[4] L. R. Weber and H. M. Randall, Phys. Rev. **40**, 835 (1932).

[5] N. Fuson, H. M. Randall, and D. M. Dennison, Phys. Rev. **56**, 983 (1939).

[6] H. H. Nielsen, Phys. Rev. **38**, 1432 (1931).

TABLE III. The rotational levels of the D_2O molecule. The $J=11$ levels are taken from FRD as are some of the levels for the higher J values. The levels with subscript c are calculated from series. $W-W_k$ is the difference from the "key" energy levels and W the difference from the rigid asymmetric rotator calculations.

J \ τ	11		12		13		14		15	
	W	$W-W_k$	W	$W-W_k$	W	$W-W_k$	W	$W-W_k$	W	δW
15									3280.37_c	271.95
14							2885.21	−12.56	3280.37_c	271.95
13					2512.54	−8.60	2885.21	−12.56	3069.20	213.73
12			2164.14	−6.86	2512.54	−8.60	2685.50	−12.47	3069.20	213.73
11	1837.89	−4.45	2164.14	−6.86	2324.30	−7.13	2685.50	−12.47	2871.84	162.44
10	1837.89	−4.45	1986.60	−4.66	2324.30	−7.13	2499.17	−5.23	2871.84	162.44
9	1672.71	−1.68	1986.60	−4.66	2147.87	−3.01	2499.17	−5.23	2680.94	122.50
8	1672.71	−1.68	1821.62	−0.38	2147.87	−3.01	2319.74	−3.08	2680.94	122.50
7	1519.60	1.14	1821.62	−0.38	1982.04	0.31	2319.74	−3.08	2502.97	88.54
6	1519.60	1.14	1668.15	2.16	1982.04	0.31	2154.61	2.03	2502.97	88.54
5	1378.56	2.45	1668.15	2.16	1828.36	2.18	2154.61	2.03	2340.65	58.96
4	1378.56	2.45	1526.34	2.77	1828.36	2.18	2001.15	3.86	2340.65	58.96
3	1250.97	2.72	1526.34	2.77	1687.26	4.62	2001.15	3.86	2189.98	37.29
2	1250.95	2.70	1400.00	4.70	1687.26	4.62	1861.08	7.09	2189.98	37.29
1	1139.78	3.56	1400.00	4.70	1561.23	5.43	1861.08	7.09	2045.71	25.53
0	1139.63	3.49	1287.55	3.54	1561.23	5.55	1737.00	7.87	2045.71	25.53
−1	1046.21	2.96	1287.50	3.57	1449.93	2.73	1737.19	7.36	1922.11	16.44
−2	1042.03	2.39	1196.28	3.61	1448.37	2.16	1625.05	2.46	1921.60	15.95
−3	979.36	3.15	1190.75	3.32	1367.36	3.10	1624.73	5.68	1819.58	12.14
−4	960.32	2.84	1136.25	2.32	1352.40	1.91	1549.64	5.25	1815.36	7.95
−5	931.60	1.21	1105.18	3.60	1307.28	1.90	1526.80	3.55	1750.13	8.15
−6	882.26	2.43	1087.58	3.64	1265.40	3.01	1490.80	1.72	1712.21	9.60
−7	876.94	1.89	1022.15	2.82	1253.78	4.06	1431.50	1.95	1688.50	9.76
−8	794.33	1.48	1019.15	2.25	1172.80	5.39	1427.90	5.46	1616.18	3.60
−9	793.96	1.46	925.15	1.91	1170.75	4.68	1331.60	5.23	1613.05	2.90
−10	692.57	1.39	924.95	1.86	1065.25	2.10	1331.35	5.61	1508.28	2.20
−11	692.57	1.39	813.35	1.65	1065.25	2.10	1215.2	3.01	1507.79	2.00
−12			813.35	1.65	943.75	1.99	1215.0	3.01	1374.7	2.5
−13					943.75	1.99	1083.8	2.47	1374.8	2.5
−14							1083.8	2.47	1233.4	0.82
−15									1233.4	0.82

Thus it was not thought worth while to go beyond $J=14$ values for H_2O.

The calculated energy levels were then corrected for centrifugal stretching by the method of RDGW and compared with the observed spectrum. It was possible to extend the energy level scheme through $J=14$. The results are given in Table I. The table includes the results for $J=11$ as given by RDGW. δW is the difference between the experimentally determined energy, W, and the calculated value as obtained from the solution of the equations. The progressions of the δW continue in much the same fashion as for smaller J values. Similarly, various series approximating the symmetric rotator[7] with rotation about either the axis of greatest or least moment of inertia can be extended. Both methods apply to these additional energy levels. Table II gives the observed rotational lines used in this extension with the new assignments.

For the case of the rotational levels of D_2O, a similar method was applied. As in the paper of FRD, it was possible to apply the proper mass correction to the δW's as obtained from H_2O. This gave a set of "key" levels which were then used to obtain the proper transition frequencies. It was possible to do this for all levels through $J=14$. For the 31 levels of $J=15$, only the extrapolated progression of the δW's was possible since no terms for H_2O were available. Table III gives the additional energy levels thus obtained for D_2O and the difference between the actual and "key" levels. For $J=15$, δW is given. Table IV gives the observed lines used and their new assignments.

DISCUSSION AND SUMMARY

The results for H_2O are a straightforward extension of previous work. There are one or two

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[7] The two levels of greatest τ value degenerate into a single level with $k=J$, the next two have a $k=J-1$, and so on. For the rotation about the axis of greatest moment the two levels with the largest negative value of τ degenerate into a single level with $k=J$, the next two higher into a level with $k=J-1$, and so on.

TABLE IV. New assignments in the spectrum of D_2O.

cm⁻¹	Assignment $J_{\tau'}'\leftarrow J_\tau$	cm⁻¹	Assignment $J_{\tau'}'\leftarrow J_\tau$	cm⁻¹	Assignment $J_{\tau'}'\leftarrow J_\tau$	cm⁻¹	Assignment $J_{\tau'}'\leftarrow J_\tau$
383.70	$15_{13}-14_{11}$	297.38	$15_{-1}-14_{-3}$	328.90	$13_{-7}-12_{-9}$	216.05	$13_{-4}-12_{-4}$
	$15_{12}-14_{12}$	296.52	$15_{-2}-14_{-2}$		$15_{3}-14_{1}$	211.35	$12_{-3}-11_{-3}$
372.67	$14_{14}-13_{12}$	293.25	$12_{-6}-11_{-8}$		$15_{2}-14_{2}$	188.28	$15_{-8}-14_{-8}$
	$14_{13}-13_{13}$	289.59	$12_{6}-11_{4}$	326.25	$12_{12}-11_{10}$	181.55	$15_{-9}-14_{-7}$
	$15_{11}-14_{9}$		$12_{5}-11_{5}$		$12_{11}-11_{11}$	177.72	$13_{-6}-12_{-6}$
	$15_{10}-14_{10}$	288.56	$14_{0}-13_{-2}$		$13_{9}-12_{7}$		$14_{-7}-13_{-7}$
361.20	$14_{12}-13_{10}$	287.26	$13_{3}-12_{1}$		$13_{8}-12_{8}$	176.93	$15_{-10}-14_{-10}$
	$14_{11}-13_{11}$		$13_{2}-12_{2}$		$14_{6}-13_{4}$	176.19	$15_{-11}-14_{-9}$
	$15_{9}-14_{7}$		$14_{-1}-13_{-1}$		$14_{5}-13_{5}$	173.58	$12_{-5}-11_{-5}$
	$15_{8}-14_{8}$	285.13	$13_{-5}-12_{-7}$	313.89	$12_{10}-11_{8}$	162.50	$14_{-8}-13_{-6}$
351.30	$14_{10}-13_{8}$	284.22	$14_{-4}-13_{-6}$		$12_{9}-11_{9}$	160.95	$14_{-9}-13_{-9}$
	$14_{9}-13_{9}$	273.48	$13_{1}-12_{-1}$		$13_{7}-12_{5}$	158.55	$14_{-10}-13_{-8}$
348.36	$13_{13}-12_{11}$		$13_{0}-12_{0}$		$13_{6}-12_{6}$	153.65	$13_{-8}-12_{-8}$
	$13_{12}-12_{12}$	272.65	$14_{-2}-13_{-4}$		$14_{4}-13_{2}$	148.60	$13_{-9}-12_{-7}$
	$15_{7}-14_{5}$	265.54	$15_{-4}-14_{-4}$		$14_{3}-13_{3}$		$13_{-7}-12_{-5}$
	$15_{6}-14_{6}$	262.18	$13_{-3}-12_{-5}$	308.65	$15_{1}-14_{-1}$	138.40	$14_{-6}-13_{-4}$
339.50	$15_{5}-14_{3}$	259.18	$13_{-1}-12_{-3}$		$15_{0}-14_{0}$	127.36	$12_{-6}-11_{-4}$
	$15_{4}-14_{4}$	257.37	$14_{-3}-13_{-3}$	302.02	$12_{8}-11_{6}$	125.40	$15_{-5}-14_{-3}$
337.70	$13_{11}-12_{9}$	252.09	$13_{-2}-12_{-2}$		$12_{7}-11_{7}$	121.95	$15_{-7}-14_{-5}$
	$13_{10}-12_{10}$	235.96	$12_{-2}-11_{-4}$		$13_{5}-12_{3}$	101.28	$14_{-5}-13_{-2}$
	$14_{8}-13_{6}$	221.41	$15_{-6}-14_{-6}$		$13_{4}-12_{4}$		
	$14_{7}-13_{7}$	219.52	$14_{-5}-13_{-5}$				

slight irregularities in the symmetric top approximation series necessitated by the probable inaccuracies of the prism data. It will be observed that the δW values increase until for the highest τ value for $J=14$ group of levels, it reaches 648.2 cm⁻¹. Most of this is a centrifugal stretching correction and its magnitude indicates that higher order terms than the first must be used in a perturbation calculation.

In the case of the D_2O rotation terms, the calculations revealed that some changes should be made in the FRD assignments for the levels beyond $J=11$. Those for which new values have been obtained are 12_{12}, 12_{11}, 13_3, 13_2, 13_1, 13_0, 13_{-8}, and 14_2. The changes in these levels were made in order to satisfy observed δW progressions and at the same time keep the series regular. An additional 42 lines observed by FRD were given assignments. The top levels for $J=15$ were calculated since the transitions from the comparable levels for $J=14$ were beyond the range of the observations of FRD. This is indicated in Table III.

It will be noticed that most of the levels are based on the assignment of one transition only. Calculations on intensities have shown that lines other than those given are weaker by a factor of five at best. Since the intensities of the lines used are nearly at the weakest observable strength, it is probable that the others would not have been observed.

On the basis of these extended assignments for the energy levels of H_2O, it will be observed that only one other transition could be expected to contribute absorption in the spectral region now available to microwave technique. This is the transition $13_{-5}\rightarrow 14_{-9}$. It would give rise to an absorption at 2.5 cm⁻¹. By virtue of Boltzmann factors only, the lower state for this transition, which is at 2543.1 cm⁻¹, has a population 0.33 percent of the population of the lower state for the well-known 1.35 cm⁻¹ ($5_{-1}\rightarrow 6_{-5}$) absorption. It is difficult to see how this 2.5 cm⁻¹ transition could influence the Van Vleck calculation sufficiently to account for the observed difference.

In the D_2O energy level extension, it can be seen that several more transitions would be allowed on the basis of selection rules. A group of transitions should appear at 4.56 cm⁻¹. These are $12_5\rightarrow 11_9$, $12_6\rightarrow 11_8$,[8] $13_7\rightarrow 12_9$, $13_6\rightarrow 12_{10}$, $14_8\rightarrow 13_{10}$, 14_7-13_{11}, 15_9-14_{11}, 15_8-14_{12}. Another pair of allowed transitions, $14_{10}-15_6$ and 14_9-15_7, should give rise to an absorption at 3.80 cm⁻¹.

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[8] Previously calculated by G. W. King, R. M. Hainer, and P. C. Cross, Phys. Rev. **71**, 433 (1947).

APPENDIX

The extension of the asymmetric rotator equations, which were solved for the H_2O and D_2O molecules, are here given.

Equations for $J=12$

$$\begin{aligned}&W^6-W^5(286\pm 78b)+W^4(28,743\pm 22,230b-17,121b^2)-W^3(1,234,948\pm 2,219,724b-3,980,340b^2\\&\mp 884,988b^3)+W^2(21,967,231\pm 94,106,220b-301,092,246b^2\mp 185,257,332b^3+53,971,983b^4)\\&-W(128,816,766\pm 1,619,337,798b-8,499,796,500b^2\mp 11,599,949,556b^3+7,751,468,790b^4\\&\pm 1,304,412,174b^5)+108,056,025\pm 8,428,369,950b-72,069,547,473b^2\mp 213,220,651,644b^3\\&+246,759,212,007b^4\pm 113,153,886,846b^5-15,740,160,975b^6=0.\end{aligned}$$

$$\begin{aligned}&W^6-364W^5+(48,048-14,196b^2)W^4-(2,846,272-4,256,928b^2)W^3+(75,851,776-424,427,328b^2\\&+32,271,408b^4)W^2-(791,691,264-16,313,696,256b^2+6,226,294,464b^4)W+2,123,366,400\\&+2,123,366,400-198,078,676,992b^2+267,622,115,328b^4-4,247,640,000b^6=0.\end{aligned}$$

$$\begin{aligned}&W^7-364W^6+(48,048-26,208b^2)W^5-(2,846,272-8,581,248b^2)W^4+(75,851,776-984,282,624b^2\\&+137,928,960b^4)W^3-(791,691,264-48,263,694,336b^2+137,928,960b^4)W^3-(791,691,264\\&-48,263,694,336b^2+34,624,392,192b^4)W^2+(2,123,366,400-981,410,217,984b^2\\&+2,586,048,049,152b^4-99,221,815,296b^6)W+6,376,469,299,200b^2-56,815,669,886,976b^4\\&+11,330,631,696,384b^6=0.\end{aligned}$$

Equations for $J=13$

$$\begin{aligned}&W^6-364W^5+(48,048-21,476b^2)W^4-(2,846,272-6,272,032b^2)W^3+(75,851,776-611,444,288b^2\\&+83,727,280b^4)W^2-(791,691,264-23,066,394,624b^2+14,889,551,040b^4)W+2,123,366,400\\&-275,898,286,080b^2+599,267,911,680b^4-29,688,120,000b^6=0.\end{aligned}$$

$$\begin{aligned}&W^7-364W^6+(48,048-37,856b^2)W^5-(2,846,272-12,168,832b^2)W^4+(75,851,776-1,374,883,328b^2\\&+312,981,760b^4)W^3-(791,691,264-66,634,573,824b^2+74,560,711,680b^4)W^2+(2,123,366,400\\&-1,344,077,660,160b^2+5,339,879,976,960b^4-440,565,350,400b^6)W+8,695,185,408,000b^2\\&-113,553,686,937,600b^4+43,584,616,857,600b^6=0.\end{aligned}$$

$$\begin{aligned}&W^7-(455\pm 91b)W^6+(77,077\pm 41,314b-25,571b^2)W^5-(6,092,515\pm 6,972,693b-10,048,623b^2\\&\mp 1,606,241b^3)W^4+(230,673,443\pm 547,446,172b-1,395,802,590b^2\mp 592,098,052b^3\\&+134,427,475b^4)W^3-(3,841,278,805\pm 20,443,837,141b-83,663,110,622b^2\mp 74,088,692,678b^3\\&+39,368,434,105b^4\pm 4,473,628,705b^5)W^2+(21,878,089,479\pm 329,112,534,114b-2,094,866,377,839b^2\\&\mp 3,674,515,896,324b^3+3,427,662,556,665b^4\pm 1,042,576,770,690b^5-100,140,527,025b^6)W\\&-(18,261,468,225\pm 1,661,793,608,475b-16,727,518,198,755b^2\mp 59,825,780,628,705b^3\\&+87,396,258,340,755b^4\pm 55,150,844,763,105b^5-13,131,815,922,225b^6\mp 705,556,726,875b^7)=0.\end{aligned}$$

Equations for $J=14$

$$\begin{aligned}&W^7-(455\pm 105b)W^6+(77,077\pm 47,670b-36,939b^2)W^5-(6,092,515\pm 8,045,415b-14,419,895b^2\\&\mp 2,764,755b^3)W^4+(230,673,443\pm 631,668,660b-1,940,497,902b^2\mp 994,665,420b^3\\&+303,873,507b^4)W^3-(3,841,278,805\pm 23,589,042,855b-114,588,599,790b^2\mp 121,872,781,170b^3\\&+83,968,308,585b^4\pm 13,053,862,395b^5)W^2+(21,878,089,479\pm 379,745,231,670b-2,834,763,235,959b^2\\&\mp 5,937,152,549,580b^3+6,965,688,307,113b^4\pm 2,821,662,850,230b^5-437,000,843,097b^6)W\\&-(18,261,468,225\pm 1,917,454,163,625b-22,431,187,875,915b^2\mp 95,235,419,445,075b^3\\&+170,680,813,811,235b^4\pm 140,314,118,838,075b^5-49,036,115,840,985b^6\mp 5,524,796,502,225b^7)=0\end{aligned}$$

$$\begin{aligned}&W^7-560W^6+(119,392-31,584b^2)W^5-(12,263,680-15,110,400b^2)W^4+(633,721,088\\&-2,605,220,352b^2+202,551,552b^4)W^3-(15,658,639,360-198,942,965,760b^2+71,275,438,080b^4)W^2\\&+(157,294,854,144-6,582,989,144,064b^2+7,586,985,443,328b^4-181,233,303,552b^6)W\\&-(416,179,814,400-73,342,526,423,040b^2+241,683,710,607,360b^4-28,054,235,381,760b^6)=0.\end{aligned}$$

$$\begin{aligned}
&W^8-560W^7+(119,392-53,424b^2)W^6-(12,263,680-27,253,440b^2)W^5+(633,721,088\\
&\quad-5,164,169,472b^2+671,805,792b^4)W^4-(15,658,639,360-456,545,940,480b^2\\
&\quad+280,317,784,320b^4)W^3+(157,294,854,144-19,393,045,807,104b^2+39,714,028,697,088b^4\\
&\quad-1,752,049,619,712b^6)W^2-(416,179,814,400-364,086,983,393,280b^2+2,239,771,868,651,520b^4\\
&\quad-463,515,825,669,120b^6)W-2,272,341,786,624,000b^2+41,917,266,749,030,400b^4\\
&\qquad-27,807,458,557,132,800b^6+292,220,641,440,000b^8=0.
\end{aligned}$$

Equations for $J=15$

$$\begin{aligned}
&W^7-560W^6+(119,392-45,024b^2)W^5-(12,263,680-21,131,520b^2)W^4+(633,721,088\\
&\quad-3,582,761,472b^2+446,683,392b^4)W^3-(15,658,639,360-269,665,566,720b^2+148,917,557,600b^4)W^2\\
&\quad+(157,294,854,144-8,815,336,341,504b^2+15,143,568,703,488b^4-765,344,858,112b^6)W\\
&\quad-(416,179,814,400-97,254,038,568,960b^2+464,234,875,453,440b^4-102,410,704,650,240b^6)=0.
\end{aligned}$$

$$\begin{aligned}
&W^8-560W^7+(119,392-73,584b^2)W^6-(12,263,680-37,010,880b^2)W^5+(633,721,088-6,929,079,552b^2\\
&\quad+1,351,635,552b^4)W^4-(15,658,639,360-606,530,995,200b^2+543,110,088,960b^4)W^3\\
&\quad+(157,294,854,144-25,566,948,900,864b^2+74,548,539,606,528b^4-5,852,150,007,552b^6)W^2\\
&\quad-(416,179,814,400-477,458,328,453,120b^2+4,096,623,324,487,680b^4-1,417,629,505,582,080b^6)W\\
&\quad-2,971,523,874,816,000b^2+75,109,687,846,502,400b^4-79,245,961,379,020,800b^6\\
&\qquad+2,531,209,111,200,000b^8=0.
\end{aligned}$$

$$\begin{aligned}
&W^8-(680\pm120b)W^7+(179,452\pm81,480b-52,164b^2)W^6-(23,434,840\pm21,452,760b-31,318,200b^2\\
&\quad\mp4,588,920b^3)W^5+(1,601,489,318\pm2,790,728,040b-7,040,225,052b^2\mp2,634,185,880b^3\\
&\quad+651,554,982b^4)W^4-(55,742,803,480\pm189,387,990,120b-745,037,531,280b^2\mp553,580,725,680b^3\\
&\quad+312,586,120,440b^4\pm35,246,395,080b^5)W^3+(886,165,820,604\pm6,499,748,427,480b\\
&\quad-38,237,034,993,084b^2\mp52,593,125,379,120b^3+50,971,318,438,788b^4\pm14,699,387,983,320b^5\\
&\quad-1,727,702,682,372b^6)W^2-(4,940,831,601,000\pm99,840,150,045,000b-873,799,877,112,120b^2\\
&\quad\mp2,225,322,266,189,400b^3+3,306,291,533,482,680b^4\pm1,880,691,146,285,400b^5\\
&\quad-517,541,613,851,880b^6\mp37,269,826,288,200b^7)W+4,108,830,350,625\pm493,059,642,075,000b\\
&\quad-6,630,772,426,582,500b^2\mp32,929,424,899,875,000b^3+70,864,421,323,062,150b^4\\
&\quad\pm73,249,613,224,893,000b^5-35,034,611,605,251,300b^6\mp6,726,596,373,189,000b^7\\
&\qquad+306,917,176,190,625b^8=0.
\end{aligned}$$