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Elastic and Inelastic Scattering of 100- to 200-Mev Protons or Neutrons by Deuterons

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Expressions have been obtained, on the Born approximation, for the elastic and the total scattering cross section of protons or neutrons by deuterons, on the basis of the three forms of nucleon potentials suggested by Rarita and Schwinger, including tensor forces but with a Gaussian function for the radial part $V(r)$. Numerical calculations have, however, been carried through without tensor forces for (i) the angular distribution and total cross section of elastic scattering at 100, 150, and 200 Mev, (ii) the energy spectrum of protons scattered (elastically plus inelastically) in different directions from that of the incident protons and total cross section, and (iii) the energy spectra of neutrons ejected in directions making angles 0° and 30° with the direction of the incident protons, for 200 Mev. The three potentials predict considerable differences in both the differential and the total cross sections for both elastic and total scattering. The proton-deuteron total cross sections for 100 Mev is 0.424, 0.212, 0.089×10^{-24} cm^2 according to the ordinary, exchange, and symmetrical force, respectively, as compared with the observed value 0.117×10^{-24} recently reported from Berkeley for 90 Mev. The calculations show that the interference of the amplitudes scattered by the two particles of the deuteron plays an important role in the determination of the total cross section for proton-deuteron scattering. It is therefore not a good approximation to set the total *p-d* cross section equal to the sum of the *p-n* and *p-p* cross sections for free particle collisions.

I. INTRODUCTION

THERE has been a number of calculations of the cross section of the elastic scattering of a proton or neutron by a deuteron, with the view of testing the various forms of nucleon interactions by comparison with experimental data. Motz and Schwinger and Buckingham and Massey have calculated the cross sections on the basis of nucleon interaction potentials which are linear combinations of the Heisenberg, Majorana, Bartlett, and Wigner types.[1] All these calculations are made for incident protons or neutrons of energy of the order 15 Mev. For such energies, the calculation is usually very lengthy and the difference between the various forms of potentials is rather small so that a definite choice of the interaction potential by comparison with the experimental data on the angular distribution of the scattered intensities is difficult.

For higher energies of 100 to 200 Mev the calculation can be considerably simplified by the use of Born's approximation. Recent calculations[2] of the cross sections of proton-neutron collisions at these energies show that the angular distribution of the scattered intensities is quite different for the three forms of the interaction potential suggested by Rarita and Schwinger.[3] As high energy protons and neutrons (100 Mev) are now available, it seems desirable to carry out similar calculations, i.e., to Born's approximation, for the elastic and inelastic collisions between a deuteron and a proton or neutron. In the following, with some simplifying assumptions about the radial dependence of the potentials and the wave function of the ground state of the deuteron, expressions have been obtained for the cross sections of both elastic and total (elastic plus inelastic) scattering of a proton or neutron by a deuteron on the basis of the

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[2] J. Ashkin and T. Y. Wu, Phys. Rev. **73**, 973 (1948).

[3] William Rarita and Julian Schwinger, Phys. Rev. **59**, 557 (1941).

three potentials of Rarita and Schwinger. As these expressions entail considerable amounts of computational labor, the actual numerical calculations have been carried out by omitting the tensor forces from the general formulae obtained. The result of the present calculation and that of the proton-neutron scattering, which includes the tensor forces, show that the difference among the three potentials is quite pronounced at these high energies.

II. FORMULATION OF THE PROBLEM

A. Wave Function for the Scattering Problem

Let 1 denote the incoming proton, 2 and 3 the proton and the neutron originally forming the deuteron.[4] Let us introduce the relative coordinates

$$\mathbf{r}=\mathbf{r}_1-\tfrac{1}{2}(\mathbf{r}_2+\mathbf{r}_3),\quad \boldsymbol{\rho}=\mathbf{r}_3-\mathbf{r}_2. \qquad (1)$$

The wave equation, after separating off the motion of the system as a whole, is

$$\left[\frac{3\hbar^2}{4M}\Delta_r+\frac{\hbar^2}{M}\Delta_\rho+E-V_{12}(|\mathbf{r}+(\boldsymbol{\rho}/2)|)-V_{13}(|\mathbf{r}-(\boldsymbol{\rho}/2)|)-V_{23}(|\boldsymbol{\rho}|)\right]\Psi(1,2,3)=0, \qquad (2)$$

where M is the proton or neutron mass. The interaction potential V depends on the symmetry property of the state of the pair of nucleons with respect to spin and space exchange. For convenience, we tabulate below the potentials for a pair of unequal nucleons in various states according to the three models of Rarita and Schwinger:

State	Ordinary	Exchange	Symmetrical	
${}^3V^{\text{even}}$	$-(1+\gamma S)U(r)$	$-(1+\gamma S)U(r)$	$-(1+\gamma S)U(r)$	(3)
${}^3V^{\text{odd}}$	$-(1+\gamma S)U(r)$	$+(1+\gamma S)U(r)$	$\frac{1}{3}(1+\gamma S)U(r)$	
${}^1V^{\text{even}}$	$-(1-2g)U(r)$	$-(1-2g)U(r)$	$-(1-2g)U(r)$	
${}^1V^{\text{odd}}$	$-(1-2g)U(r)$	$+(1-2g)U(r)$	$3(1-2g)U(r)$	

For two equal nucleons, only the ${}^3V^{\text{odd}}$ and ${}^1V^{\text{even}}$ states are allowed by the Pauli principle. Here ${}^3V^{\text{even}}$ refers to a state which is symmetrical in the spins and even in the space coordinates of the two particles, etc. S is the tensor interaction,

$$S_{ij}=\frac{3(\boldsymbol{\sigma}_i\cdot\mathbf{r})(\boldsymbol{\sigma}_j\cdot\mathbf{r})}{r^2}-(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j), \qquad (4)$$

$U(r)$ is a central force potential. g and γ are constants. The potential for ${}^3V^{\text{even}}$ has been chosen to be the same in the three theories to fit the data on the ground state of the deuteron. The potential for ${}^1V^{\text{even}}$ has also been given the same value in the three theories to fit the data on proton-proton scattering at low energies.

The wave function $\Psi(1, 2, 3)$ of the system may be developed in a series of products of a spin wave function and a space wave function. There are eight linearly independent and orthogonal spin wave functions. Of these there are four which are totally symmetrical in the three particles. These may be put in the form

$$\begin{aligned}\chi_1&=\alpha_1\alpha_2\alpha_3,\\ \chi_2&=\beta_1\beta_2\beta_3,\\ \chi_3&=(1/\sqrt{3})(\alpha_1\beta_2\alpha_3+\beta_1\alpha_2\alpha_3+\alpha_1\alpha_2\beta_3),\\ \chi_4&=(1/\sqrt{3})(\beta_1\alpha_2\beta_3+\alpha_1\beta_2\beta_3+\beta_1\beta_2\alpha_3).\end{aligned} \qquad (5)$$

For the other four, we shall construct

$$\begin{aligned}\chi_5&=(1/\sqrt{6})(\alpha_1\alpha_2\beta_3+\alpha_1\beta_2\alpha_3-2\beta_1\alpha_2\alpha_3),\\ \chi_6&=(1/\sqrt{6})(\beta_1\beta_2\alpha_3+\beta_1\alpha_2\beta_3-2\alpha_1\beta_2\beta_3),\end{aligned} \qquad (6)$$

which are symmetric in 2, 3, and

$$\begin{aligned}\chi_7&=(1/\sqrt{2})\alpha_1(\alpha_2\beta_3-\beta_2\alpha_3),\\ \chi_8&=(1/\sqrt{2})\beta_1(\beta_2\alpha_3-\alpha_2\beta_3),\end{aligned} \qquad (7)$$

which are antisymmetric in 2, 3. These eight spin wave functions are appropriate for the description of the scattering processes treated below.

The wave function $\Psi(1, 2, 3)$ can be expanded

[4] To avoid the awkwardness of expression, we consider the scattering of protons by deuterons in the following. As the Coulomb interaction is neglected throughout, the whole calculation holds also for the case of the scattering of neutrons by deuterons. We then only have to interchange the names of proton and neutron throughout the present paper.

in the form

$$\Psi(1,2,3)=\sum_{i=1}^{6}\chi_i f^{(i)}(\mathbf{r})\psi_0(\boldsymbol{\rho})+\sum_{i=1}^{6}\chi_i\int F_{k''}^{(i)}(\mathbf{r})\psi_{k''}{}^{t}(\boldsymbol{\rho})d\mathbf{k}''+\sum_{j=7}^{8}\chi_j\int F_{k''}^{(j)}(\mathbf{r})\psi_{k''}{}^{s}(\boldsymbol{\rho})d\mathbf{k}'', \quad (8)$$

where $\psi_0(\rho)$ is the wave function of the ground state of the deuteron, $\psi_{k''}{}^{t}(\boldsymbol{\rho})$, $\psi_{k''}{}^{s}(\boldsymbol{\rho})$ are the continuum triplet and singlet state wave function of the deuteron, respectively. These satisfy the equations

$$\left[-\frac{\hbar^2}{M}\Delta_\rho+{}^3V_{23}+\epsilon\right]\chi_i\psi_0(\boldsymbol{\rho})=0,\quad i=1,\cdots,6,$$

$$\left[-\frac{\hbar^2}{M}\Delta_\rho+{}^3V_{23}-\frac{\hbar^2}{M}k''^2\right]\chi_i\psi_{k''}{}^{t}(\boldsymbol{\rho})=0,\quad i=1,\cdots,6, \quad (9)$$

$$\left[-\frac{\hbar^2}{M}\Delta_\rho+{}^1V_{23}-\frac{\hbar^2}{M}k''^2\right]\chi_j\psi_{k''}{}^{s}(\boldsymbol{\rho})=0,\quad j=7,8,$$

where ϵ is the binding energy of the deuteron. The functions $f^{(i)}(\mathbf{r})$, $F_{k''}^{(i)}(\mathbf{r})$, $F_{k''}^{(j)}(\mathbf{r})$ satisfy the following equations obtained by substituting (8) into (2) and making use of the above equations,

$$\left[\frac{3\hbar^2}{4M}\Delta_r+E+\epsilon\right]f^{(i)}(\mathbf{r})=\sum_{\text{spin}}\int\chi_i^*\psi_0^*(\boldsymbol{\rho})(V_{12}+V_{13})\Psi(1,2,3)d\boldsymbol{\rho},\quad i=1,\cdots,6,$$

$$\left[\frac{3\hbar^2}{4M}\Delta_r+E-\frac{\hbar^2}{M}k''^2\right]F^{(i)}(\mathbf{r})=\sum\int\chi_i^*\psi_{k''}^{*t}(\boldsymbol{\rho})(V_{12}+V_{13})\Psi(1,2,3)d\boldsymbol{\rho},\quad i=1,\cdots,6, \quad (10)$$

$$\left[\frac{3\hbar^2}{4M}\Delta_r+E-\frac{\hbar^2}{M}k''^2\right]F^{(j)}(\mathbf{r})=\sum\int\chi_j^*\psi_{k''}^{*s}(\boldsymbol{\rho})(V_{12}+V_{13})\Psi(1,2,3)d\boldsymbol{\rho},\quad j=7,8.$$

To satisfy Pauli's principle, we must employ a wave function which is antisymmetric in the two identical particles 1 and 2. To this end, we construct

$$\Psi_a(1,2,3)=\Psi(1,2,3)-P_{12}\Psi(1,2,3), \quad (11)$$

where P_{12} is the operator permuting both the spin and the space coordinates of 1 and 2. Now $P_{12}\Psi(1,2,3)$ can also be expanded in a series

$$P_{12}\Psi(1,2,3)=\sum_{i=1}^{6}\chi_i g^{(i)}(\mathbf{r})\psi_0(\boldsymbol{\rho})+\sum_{i=1}^{6}\chi_i\int G_{k''}^{(i)}(\mathbf{r})\psi_{k''}{}^{t}(\boldsymbol{\rho})d\mathbf{k}''+\sum_{j=7}^{8}\chi_j\int G_{k''}^{(j)}(\mathbf{r})\psi_{k''}{}^{s}(\boldsymbol{\rho})d\mathbf{k}'', \quad (12)$$

where the functions $g^{(i)}(\mathbf{r})$, $G_{k''}^{(i)}(\mathbf{r})$, $G_{k''}^{(j)}(\mathbf{r})$ satisfy equations similar to (10), with $\Psi(1,2,3)$ in the integrand replaced by $P_{12}\Psi(1,2,3)$. For the wave function $\Psi_a(1,2,3)$ we have, on subtracting the corresponding equations for the $f^{(i)}(\mathbf{r})$, $g^{(i)}(\mathbf{r})$, $F_{k''}^{(i)}(\mathbf{r})$, $G_{k''}^{(i)}(\mathbf{r})$, etc., and denoting by

$$\phi^{(i)}\equiv f^{(i)}(\mathbf{r})-g^{(i)}(\mathbf{r}),\quad \Phi^{(i)}\equiv F^{(i)}(\mathbf{r})-G^{(i)}(\mathbf{r}),\quad \Phi^{(j)}\equiv F^{(j)}(\mathbf{r})-G^{(j)}(\mathbf{r}),\quad i=1,\cdots,6,\quad j=7,8, \quad (13)$$

$$\left[\frac{3\hbar^2}{4M}\Delta_r+E+\epsilon\right]\phi^{(i)}=\sum_{\text{spin}}\int\chi_i^*\psi_0^*(\boldsymbol{\rho})(V_{12}+V_{13})\times(1-P_{12})\Psi(1,2,3)d\boldsymbol{\rho},$$

$$\left[\frac{3\hbar^2}{4M}\Delta_r+E-\frac{\hbar^2}{M}k''^2\right]\Phi^{(i)}=\sum\int\chi_i^*\psi_{k''}^{*t}(\boldsymbol{\rho})\times(1-P_{12})\Psi(1,2,3)d\rho, \quad (14)$$

$$\left[\frac{3\hbar^2}{4M}\Delta_r+E-\frac{\hbar^2}{M}k''^2\right]\Phi^{(j)}=\sum\int\chi_j^*\psi_{k''}^{*s}(\boldsymbol{\rho})(V_{12}+V_{13})\times(1-P_{12})\Psi(1,2,3)d\boldsymbol{\rho}.$$

The functions $\phi^{(i)}$, $\Phi^{(i)}$, $\Phi^{(j)}$ in (13) are the correctly symmetrized wave functions. $\phi^{(i)}$ describes the elastically scattered wave with spin wave function χ_i, together with the incoming wave. The $\Phi^{(i)}(\mathbf{r})$ describe those inelastically scattered waves with spin wave functions which are symmetric in 2 and 3, while $\Phi^{(i)}(\mathbf{r})$ describe those inelastically scattered waves which are antisymmetric in the spins of 2 and 3.

B. Elastic Scattering Cross Section

To obtain the amplitude of the elastically scattered wave of spin wave function χ_i, one calculates as usual the amplitude of the asymptotic solution of $\phi^{(i)}(r)$ for large r and obtains

$$f(\vartheta, \varphi) = -\frac{1}{4\pi}\left(\frac{4M}{3\hbar^2}\right)\Sigma\int\int \exp[-i\mathbf{k}'\cdot\mathbf{r}] \times\chi_i^*\psi_0^*(\boldsymbol{\rho})(V_{12}+V_{13}) \times(1-P_{12})\Psi(1, 2, 3)d\boldsymbol{\rho}d\mathbf{r}, \quad (15)$$

where ϑ, φ are the polar angles of the direction $\mathbf{k}'$ of the scattered wave. To calculate $f(\vartheta, \varphi)$ one replaces, in an approximation, the function $\Psi(1, 2, 3)$ in the integrand by the initial wave function

$$\chi_l f^{(l)}(\mathbf{r})\psi_0(\boldsymbol{\rho}), \quad (16)$$

where χ_l may be one of the six $\chi_1, \cdots\chi_6$. For a given initial spin state χ_l, the amplitude of the scattered wave with spin function χ_i is then given by the matrix element

$$(i|f(\vartheta, \varphi)|l) = -\frac{M}{3\pi\hbar^2}\Sigma\int\int \exp[-i\mathbf{k}'\cdot\mathbf{r}] \times\chi_i^*\psi_0^*(\rho)(V_{12}+V_{13})(1-P_{12}) \times\chi_l f^{(l)}(\mathbf{r})\psi_0(\boldsymbol{\rho})d\boldsymbol{\rho}d\mathbf{r}. \quad (17)$$

The intensity of the scattered waves of all spin states is the sum

$$\sum_{i=1}^{6}|(i|f(\vartheta, \varphi)|l)|^2.$$

For $l=1, 2, 3, 4$, one has the "quartet" state scattering, while for $l=5, 6$, one has the "doublet" state scattering. The intensity to be compared with the observed scattered waves of an unpolarized incident beam is the weighted average

$$I(\vartheta, \varphi) = \tfrac{2}{3}\cdot\tfrac{1}{4}\sum_{l=1}^{4}\sum_{i=1}^{6}|(i|f(\vartheta, \varphi)|l)|^2 + \tfrac{1}{3}\cdot\tfrac{1}{2}\sum_{l=5}^{6}\sum_{i=1}^{6}|(i|f(\vartheta, \varphi)|l)|^2. \quad (18)$$

For central force, all non-diagonal matrix elements $i\neq l$ in (17) vanish.

The differential cross section of a proton being scattered into the solid angle $d\omega = 2\pi\sin\vartheta d\vartheta$ (the polar axis being the direction of the incoming beam) is, since $I(\vartheta, \varphi)$ actually does not depend on φ,

$$\sigma_{el}(E_0, \vartheta)d\omega = 2\pi I(\vartheta, \varphi)\sin\vartheta d\vartheta = 2\pi I(\vartheta)\sin\vartheta d\vartheta, \quad (19)$$

where E_0 denotes the energy of the incident proton.

It is desirable to transform the above expression to the laboratory coordinate system. Let Θ be the angle between the direction of the scattered proton (as seen in the laboratory system) and the direction of the incident beam. It can be shown that

$$\tan\Theta = 2\sin\vartheta/(1+2\cos\vartheta). \quad (20)$$

The differential cross section of a proton being scattered into directions between Θ and $\Theta+d\Theta$ is then given by

$$2\pi\sigma_{el}(E_0, \Theta)\sin\Theta d\Theta = 2\pi g I(\Theta)\sin\Theta d\Theta, \quad (19a)$$

where $I(\Theta)$ is obtained from $I(\vartheta)$ in (19) by means of (20), and where

$$g = (1+2\cos\vartheta)^3/4(2+\cos\vartheta)\cos^3\Theta.$$

C. Inelastic Scattering Cross Section

For inelastic scattering, let us introduce the wave vectors $\mathbf{k}'$ and $\mathbf{k}''$ corresponding to $\mathbf{r}$ and $\boldsymbol{\rho}$, respectively, in (1), namely,

$$E' = \frac{3\hbar^2k'^2}{4M}, \quad E'' = \frac{\hbar^2k''^2}{M}, \quad (21)$$

where E' is the energy of the scattered proton relative to the center of gravity of the other two particles, and E'' is the energy of relative motion of the other two particles. If E_0 denotes the energy of the incident proton in the laboratory system and ϵ the binding energy of the deuteron, the energy relation is

$$\tfrac{2}{3}E_0=E'+E''+\epsilon. \quad (22)$$

Simple energy and momentum considerations show that while the magnitudes of $\mathbf{k}'$ and $\mathbf{k}''$ are restricted by (22), their directions are entirely arbitrary so that the differential cross section must be specified by five variables, such as the directions of both $\mathbf{k}'$ and $\mathbf{k}''$ and the magnitude of either $\mathbf{k}'$ or $\mathbf{k}''$. The differential cross section of a proton being scattered into the solid angle $d\Omega'$ (in the center of gravity system) and $\mathbf{k}''$ lying in the volume element $d\mathbf{k}''=k''^2dk''d\Omega''$ is

$$\sigma(E_0, k'', \Omega')d\Omega'k''^2dk''d\Omega''=(k'/k)I(\Omega', \Omega'')d\Omega'k''^2dk''d\Omega'', \quad (23)$$

where $I(\Omega', \Omega'')$ is given by the weighted average of the "quartet" and the "doublet" scattered intensity

$$I(\Omega', \Omega'')=\tfrac{2}{3}\cdot\tfrac{1}{4}\sum_{l=1}^{4}\sum_{i=1}^{8}|(i|f(\Omega', \Omega'')|l)|^2+\tfrac{1}{3}\cdot\tfrac{1}{2}\sum_{l=5}^{6}\sum_{i=1}^{8}|(i|f(\Omega', \Omega'')|l)|^2, \quad (24)$$

where $(i|f(\Omega', \Omega'')|l)$ is given by

$$(i|f(\Omega', \Omega'')|l)=-\frac{1}{4\pi}\left(\frac{4M}{3\hbar^2}\right)\sum\int\int \exp[-i\mathbf{k}'\cdot\mathbf{r}]\times\chi_i^*\psi_{k''}^*(\boldsymbol{\rho})(V_{12}+V_{13})(1-P_{12})\times\chi_l f^{(l)}(\mathbf{r})\psi_0(\boldsymbol{\rho})d\boldsymbol{\rho}d\mathbf{r}. \quad (25)$$

Here $\psi_{k''}(\boldsymbol{\rho})$ is $\psi_{k''}{}^t(\boldsymbol{\rho})$ for $i=1, \cdots 6$, and is $\psi_{k''}{}^s(\boldsymbol{\rho})$ for $i=7, 8$, as shown in (8) and (12).

The calculation of $I(\Omega', \Omega'')$ and the transformation to the laboratory system will be given in Section IVB.

III. ELASTIC SCATTERING

For the energy range $E_0=100-200$ Mev, we shall employ Born's approximation by replacing the $\Psi(1, 2, 3)$ in the integrand in (15) by the initial wave function (16) in which $f^{(l)}(r)$ is represented by a plane wave, namely,

$$\Psi(1, 2, 3)\equiv\chi_l\psi(1, 2, 3)=\chi_l \exp[i\mathbf{k}\cdot\mathbf{r}]\psi_0(\rho). \quad (26)$$

In integrating over $\mathbf{r}$ and $\boldsymbol{\rho}$ in (17), since the operators V_{12}, V_{13} depend on the symmetry of the wave function of the state with respect to the interchange of the two particles, it is expedient, in dealing with the term V_{13}, to express the wave function $(1-P_{12})\psi(1, 2, 3)$ as a combination of terms which are either symmetric or antisymmetric with respect to 1 and 3 so that for each term the appropriate potential can be immediately picked out from Eqs. (3). Let X_1, X_2, $\cdots X_6$ be the following linearly independent combinations of the basic functions ψ, $P_{12}\psi$, $P_{13}\psi$, $P_{23}\psi$, $P_{12}P_{23}\psi$, $P_{12}P_{13}\psi$:

	ψ	$P_{12}\psi$	$P_{13}\psi$	$P_{23}\psi$	$P_{12}P_{23}\psi$	$P_{12}P_{13}\psi$
X_1	1	1	1	1	1	1
X_2	1	−1	1	1	−1	1
X_3	1	1	1	−1	1	−1
X_4	1	1	−1	1	−1	−1
X_5	1	−1	−1	1	1	−1
X_6	1	−1	−1	−1	1	1

Thus X_1 is totally symmetric, X_6 totally antisymmetric, in 1, 2, 3, X_2, X_3 are symmetric and X_4, X_5 are antisymmetric in 1, 3. Then one finds

$$\psi=\tfrac{1}{4}(X_2+X_3+X_4+X_6),$$
$$P_{12}\psi=\tfrac{1}{4}(X_1-X_2+X_4-X_5).$$

With (26), it can be shown that

$$\psi=P_{23}\psi=\exp[i\mathbf{k}\cdot\mathbf{r}]\psi_0(\boldsymbol{\rho}),$$

$$P_{12}\psi=P_{12}P_{23}\psi=\exp[-i\mathbf{k}\cdot(\tfrac{1}{2}\mathbf{r}+\tfrac{3}{4}\boldsymbol{\rho})]\psi_0(|\mathbf{r}-\tfrac{1}{2}\boldsymbol{\rho}|),$$

$$P_{13}\psi=P_{12}P_{13}\psi=\exp[-i\mathbf{k}\cdot(\tfrac{1}{2}\mathbf{r}-\tfrac{3}{4}\boldsymbol{\rho})]\psi_0(|\mathbf{r}+\tfrac{1}{2}\boldsymbol{\rho}|).$$

In the present paper, the scattered intensity $I(\vartheta, \varphi)$ in (18) has been calculated with the potentials (3) which include the tensor force. But since the actual numerical calculation has been performed after omitting the tensor force on account of the excessive amount of computational work that would be involved, we shall only give the formulas without tensor force in the following and give the complete formulas including tensor force in Appendix I.

Without tensor force, there are only diagonal matrix elements in (17) and the summation over the spin coordinates can be immediately carried out. All integrals in (17) can be shown to reduce to the following three:

$$L_1=\int\int \exp[-i\mathbf{k}'\cdot\mathbf{r}]\psi_0^*(\rho)U(|\mathbf{r}+\tfrac{1}{2}\boldsymbol{\rho}|)\times\exp[i\mathbf{k}\cdot\mathbf{r}]\psi_0(\rho)d\boldsymbol{\rho}d\mathbf{r}, \quad (27a)$$

$$L_2=\int\int \exp[-i\mathbf{k}'\cdot\mathbf{r}]\psi_0^*(\rho)U(|\mathbf{r}+\tfrac{1}{2}\boldsymbol{\rho}|)\times\exp[-i\mathbf{k}\cdot(\tfrac{1}{2}\mathbf{r}+\tfrac{3}{4}\boldsymbol{\rho})]\times\psi_0(|\mathbf{r}-\tfrac{1}{2}\boldsymbol{\rho}|)d\boldsymbol{\rho}d\mathbf{r}, \quad (27b)$$

$$L_3=\int\int \exp[-i\mathbf{k}',\mathbf{r}]\psi_0^*(\rho)U(|\mathbf{r}-\tfrac{1}{2}\boldsymbol{\rho}|)\times\exp[-i\mathbf{k}\cdot(\tfrac{1}{2}\mathbf{r}+\tfrac{3}{4}\rho)]\times\psi_0(|\mathbf{r}-\tfrac{1}{2}\boldsymbol{\rho}|)d\boldsymbol{\rho}d\mathbf{r}. \quad (27c)$$

In order to be able to carry out the integrations in (27) analytically, we shall represent the radial wave function of the ground state of the deuteron by a Gaussian function

$$\psi_0(\rho)=A\exp[-\lambda^2\rho^2]. \quad (28a)$$

For the radial part of the potential $U(|\xi|)$, we shall assume a Gaussian function

$$U(|\xi|)=V_0\exp[-\alpha^2\xi^2]. \quad (28b)$$

Carrying out the integration, we find

$$\begin{aligned}
L_1&=B_1\exp\left[-\frac{\alpha^2+8\lambda^2}{8\lambda^2}x^2\right], \\
x^2&=\left(\frac{k}{\alpha}\right)^2\sin^2\frac{\vartheta}{2}, \\
L_2&=B_2\exp\left[-\frac{\alpha^2}{8\lambda^2}x^2-y^2\right], \\
y^2&=\frac{9k^2}{32(\alpha^2+\frac{1}{2}\lambda^2)}(1+\cos\vartheta), \\
L_3&=B_3\exp\left[-\frac{k^2}{16\lambda^2}(5+4\cos\vartheta)-z^2\right], \\
z^2&=\frac{k^2}{16(\alpha^2+\lambda^2)}(5+4\cos\vartheta),
\end{aligned} \quad (29)$$

where

$$B_1=\frac{A^2V_0\pi^3}{2^{\frac{3}{2}}\alpha^3\lambda^3},\quad B_2=\frac{A^2V_0\pi^3}{2^{\frac{3}{2}}\lambda^3(\alpha^2+\frac{1}{2}\lambda^2)^{\frac{3}{2}}},$$

$$B_3=\frac{A^2V_0\pi^3}{\lambda^3(\alpha^2+\frac{1}{2}\lambda^2)^{\frac{3}{2}}}.$$

After some calculation whose details we shall omit here, it is found that $I(\vartheta,\varphi)$ in (18), which is independent of φ and is hereafter denoted by $I(\vartheta)$, is given, on the three theories of nucleon interaction, by the following expressions.

Ordinary force:

$$\begin{aligned}
(3\pi\hbar^2/M)^2I(\vartheta)=&4(1-g+\tfrac{3}{4}g^2)L_1^2 \\
&+\tfrac{3}{4}(1-\tfrac{2}{3}g+g^2)L_2^2+\tfrac{3}{4}L_3^2 \\
&-2(1+\tfrac{1}{2}g-\tfrac{3}{2}g^2)L_1L_2 \\
&-2(1+\tfrac{1}{2}g)L_1L_3+\tfrac{3}{2}(1-\tfrac{1}{3}g)L_2L_3.
\end{aligned} \quad (30a)$$

Exchange force:

$(3\pi\hbar^2/M)^2I(\vartheta)=$ the expression above, with 1, 2 interchanged. (30b)

Symmetrical force:

$$(3\pi\hbar^2/M)^2I(\vartheta)=\tfrac{1}{4}(11-30g+27g^2)L_2^2+\tfrac{3}{4}L_3^2-\tfrac{1}{2}(1+3g)L_2L_3. \quad (30c)$$

It is to be noticed that of the three integrals L_1, L_2, L_3, L_1 is by far the most important one. It has a strong maximum in the forward direction $\vartheta=0$. The integrals L_2, L_3 contain the exchange operators P_{12} or P_{13} and give but small contributions, L_3 only in the backward direction $\vartheta\simeq\pi$, as is seen from (29). That the scattered intensity $I(\vartheta)$ on the symmetrical theory does not depend on L_1 is a consequence of the potential (3). This can perhaps be brought out more clearly as follows. We may write (3) in the form

$$V_{12}=\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2(1+2P_{12})V_{12}^{\text{even}}.$$

According to (17), the scattered amplitudes are proportional to the matrix elements of the operator:

$$\begin{aligned}
(V_{12}+V_{13})(1-P_{12})
&=\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2V_{12}^{\text{even}}(1+2P_{12})(1-P_{12}) \\
&\quad+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_3V_{13}^{\text{even}}(1+2P_{13})(1-P_{12}) \\
&=\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2V_{12}^{\text{even}}(-1+P_{12}) \\
&\quad+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_3V_{13}^{\text{even}}(1+2P_{13}-P_{12}-2P_{13}P_{12}).
\end{aligned}$$

The terms containing the exchange operators give the L_2 and L_3 of (30c). The remaining terms $-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2V_{12}$ and $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_3V_{13}$ give exactly equal and opposite matrix elements since the ground state of the deuteron is symmetrical in the particles 2 and 3. This cancellation for the symmetrical theory accounts for the very small cross sections shown in Tables I and II.

TABLE I. Differential cross section $\sigma_{el}(E_0,\Theta)$ in 10^{-25} cm², defined in (19a), for elastic proton-deuteron scattering.

Θ	Ordinary	Exchange	Symmetrical
100 Mev			
0	100.5	20.2	0.010,2
10	67.7	13.6	0.010,0
20	19.6	3.82	0.009,9
30	2.79	.506	0.009,9
41	.208	.037,2	0.009,5
63	.002,9	.017,7	0.008,3
90	.003,1	.012,0	0.005,5
106	.006,6	.006,2	0.003,8
126	.025,8	.008,6	0.014,0
150	.086,9	.047,9	0.061,7
180	.129	.082,4	0.099,5
150 Mev			
0	101.2	20.8	0.000,184
10	56.3	11.55	0.000,184
20	9.09	1.86	0.000,184
30	.498	.098	0.000,190
41	.011,4	.001,93	0.000,183
63	.000,07	.000,38	0.000,172
90	.000,06	.000,29	0.000,127
106	.000,02	.000,17	0.000,108
126	.000,80	.000,28	0.000,48
150	.006,23	.004,30	0.004,53
180	.012,0	.009,67	0.010,6
200 Mev			
0	101.0	20.9	0.000,004
10	46.3	9.56	0.000,004
20	4.12	.85	0.000,004
30	.093	.019	0.000,004
41	.000,633	.000,112	0.000,004
63	.000,002	.000,009	0.000,004
90	.000,001	.000,006	0.000,003
106	.000,002	.000,005	0.000,003
126	.000,025	.000,007	0.000,015
150	.000,45	.000,43	0.000,40
180	.001,17	.001,08	0.001,11

TABLE II. Total cross section σ_{el} in 10^{-24} cm² for elastic proton-deuteron scattering.

	Ordinary	Exchange	Symmetrical
100 Mev	0.384	0.077	0.003,8
150 Mev	0.255	0.051	0.000,2
200 Mev	0.185	0.037	0.000,01

IV. TOTAL (ELASTIC PLUS INELASTIC) SCATTERING

A. Total Scattering

We shall again relegate the complete formula including tensor force to Appendix II and confine ourselves to the case of central force in the following.

It is important to notice that since two protons of indeterminate origin are emitted in each inelastic scattering event, we must divide the integral of expression (23) by two to obtain the total inelastic cross section. Representing the incident wave by the plane wave (26), one can express all the matrix elements $(i|f(\Omega',\Omega'')|l)$ in terms of integrals $\mathfrak{L}_1$, $\mathfrak{L}_2$, $\mathfrak{L}_3$ which are obtained from those for L_1, L_2, L_3 in (27) by replacing $\psi_0{}^*(\rho)$ by $\psi_{k''}{}^{*t}(\boldsymbol{\rho})$ or $\psi_{k''}{}^{*s}(\boldsymbol{\rho})$ according as $i=1, 2, \cdots 6$ or $i=7, 8$, and $\mathfrak{L}_1{}^*$, $\mathfrak{L}_2{}^*$, $\mathfrak{L}_3{}^*$ such that $\mathfrak{L}_1{}^*=\mathfrak{L}_1$ with $\mathbf{k}''$ replaced by $-\mathbf{k}''$. Thus for $i=1,\cdots 6$, we have

$$\mathfrak{L}_1{}^t=\int\int \exp[-i\mathbf{k}'\cdot\mathbf{r}]\psi_{k''}{}^{*t}(\boldsymbol{\rho}) \times U(|\mathbf{r}+\tfrac{1}{2}\boldsymbol{\rho}|)\exp[i\mathbf{k}\cdot\mathbf{r}]\psi_0(\rho)d\boldsymbol{\rho}d\mathbf{r}, \quad (31a)$$

and for $i=7, 8$,

$$\mathfrak{L}_1{}^s=\int\int \exp[-i\mathbf{k}'\cdot\mathbf{r}]\psi_{k''}{}^{*s}(\boldsymbol{\rho}) U(|\mathbf{r}+\tfrac{1}{2}\boldsymbol{\rho}|)\exp[i\mathbf{k}\cdot\mathbf{r}]\psi_0(\rho)d\boldsymbol{\rho}d\mathbf{r}. \quad (31b)$$

The actual evaluation of these integrals requires a knowledge of the continuum wave functions $\psi_{k''}{}^t(\boldsymbol{\rho})$ and $\psi_{k''}{}^s(\boldsymbol{\rho})$ of the deuteron. This is essential for any detailed investigation of the energy spectrum or angular distribution of the particles resulting from collisions in which the deuteron disintegrates. For the gross information provided by the total cross section, however, only the following general properties of the continuum wave functions are important, namely, the functions $\{\psi_{k''}{}^t(\boldsymbol{\rho}),\ \psi_0(\boldsymbol{\rho})\}$ and $\psi_{k''}(\boldsymbol{\rho})$ form two complete orthonormal sets, and for large k'', $\psi_{k''}{}^t(\boldsymbol{\rho})$ and $\psi_{k''}{}^s(\boldsymbol{\rho})$ approach the plane wave $1/(2\pi)^{\frac{3}{2}}\exp[i\mathbf{k}''\cdot\boldsymbol{\rho}]$.

If one makes the plane wave approximation

$$\psi_{k''}{}^t(\boldsymbol{\rho})=\psi_{k''}{}^s(\boldsymbol{\rho})=\frac{1}{(2\pi)^{\frac{3}{2}}}\exp[i\mathbf{k}''\cdot\boldsymbol{\rho}], \quad (32)$$

one finds readily

$$\begin{aligned}
&\mathfrak{L}_1{}^t=\mathfrak{L}_1{}^s=\mathfrak{L}_1=D\exp[-q^2-x^2],\\
&\qquad 16\lambda^2q^2=|\mathbf{k}-\mathbf{k}'+2\mathbf{k}''|^2,\\
&\qquad 4\alpha^2x^2=|\mathbf{k}-\mathbf{k}'|^2,\\
&\mathfrak{L}_2{}^t=\mathfrak{L}_2{}^s=\mathfrak{L}_2=D\exp[-q^2-y^2],\\
&\qquad 4\alpha^2y^2=|\mathbf{k}+\tfrac{1}{2}\mathbf{k}'+\mathbf{k}''|^2,\\
&\mathfrak{L}_1{}^*=\mathfrak{L}_1 \text{ with } \mathbf{k}'' \text{ replaced by } -\mathbf{k}'', \text{ etc.},\\
&\qquad D=\frac{AV_0\pi^3}{\alpha^3\lambda^3}.
\end{aligned} \quad (33)$$

The integrals $\mathfrak{L}_3$, $\mathfrak{L}_3^*$ corresponding to L_3 in (27) can be shown to contain a δ-function and contribute nothing to the scattered intensity. They have therefore been left out in the following.

To the approximation (32), the intensity $I(\Omega', \Omega'')$ in (24) is given by the following expressions.

Ordinary force:

$$(2\pi)^3(3\pi\hbar^2/M)^2 I(\Omega', \Omega'') = (1-g+g^2)(\mathfrak{L}_1^2+\mathfrak{L}_2^2+\mathfrak{L}_1^{*2}) - (1+2g-2g^2)\mathfrak{L}_1\mathfrak{L}_2 + (2-2g+g^2)\mathfrak{L}_1\mathfrak{L}_1^* - (1+g-g^2)\mathfrak{L}_2\mathfrak{L}_1^*. \quad (34a)$$

Exchange force:

$$(2\pi)^3(3\pi\hbar^2/M)^2 I(\Omega', \Omega'') = \text{the expression above, with 1, 2 interchanged.} \quad (34b)$$

Symmetrical force:

$$(2\pi)^3(3\pi\hbar^2/M)^2 I(\Omega', \Omega'') = (\tfrac{1}{3}-g+g^2)(\mathfrak{L}_1^2+\mathfrak{L}_2^2+\mathfrak{L}_1^{*2}+4\mathfrak{L}_2^{*2}) + (\tfrac{1}{3}-2g+2g^2)(\mathfrak{L}_1\mathfrak{L}_2-2\mathfrak{L}_1^*\mathfrak{L}_2^*) - (2/9)(1-3g+(9/2)g^2)\mathfrak{L}_1\mathfrak{L}_1^* + (\tfrac{1}{9}-g+g^2)(2\mathfrak{L}_1\mathfrak{L}_2^*-\mathfrak{L}_1^*\mathfrak{L}_2) + (2/9)(5-4g+9g^2)\mathfrak{L}_2\mathfrak{L}_2^*. \quad (34c)$$

It appears that (23), with $I(\Omega', \Omega'')$ given by (34) and (33), gives the cross section for inelastic scattering. The approximation (32), however, has the consequence that the cross section given by (23) and (34) also contains contributions of the elastic scattering, since the plane waves $\psi_{k''}{}^t(\varrho)$ are not orthogonal to the wave function of the ground state of the deuteron and hence contains components of $\psi_0(\rho)$. It will be shown below that (23), (33), and (34) actually form a closer approximation to the total scattering (elastic plus inelastic) than to the inelastic.[5]

To simplify the presentation of the argument, let us first consider that part of $I(\Omega', \Omega'')$ that comes from $\mathfrak{L}_1^2$, in the case of ordinary force (34a). Without making the approximation (32,) one would have obtained instead of $(1-g+g^2)\mathfrak{L}_1^2$ the expression

$$(1-g+\tfrac{3}{4}g^2)(\mathfrak{L}_1{}^t)^2+\tfrac{1}{4}g^2(\mathfrak{L}_1{}^s)^2.$$

[5] Cf. N. F. Mott and H. S. W. Massey, *Theory of Atomic Collisions* (Oxford University Press, New York, 1933), p. 172.

Let us define

$$\mathbf{K}=\mathbf{k}-\mathbf{k}', \quad \boldsymbol{\xi}=\mathbf{r}+\tfrac{1}{2}\boldsymbol{\varrho}. \quad (35)$$

From (31a), one finds

$$\mathfrak{L}_1{}^t = C_{0k''}(\mathbf{K}, \mathbf{k}'')F(K), \quad (36)$$

where

$$C_{0k''}(\mathbf{K}, \mathbf{k}'') = \int \exp[-(i/2)\mathbf{K}\cdot\boldsymbol{\varrho}] \times\psi_{k''}{}^{*t}(\boldsymbol{\varrho})\psi_0(\rho)d\boldsymbol{\varrho}, \quad (37)$$

$$F(K) = \int \exp[i\mathbf{K}\cdot\boldsymbol{\xi}]U(|\boldsymbol{\xi}|)d\boldsymbol{\xi}.$$

$F(K)$ is seen to be a function of the magnitude K of $\mathbf{K}$ alone. Also

$$|C_{0k''}(K)|^2 \equiv \int |C_{0k''}(\mathbf{K}, \mathbf{k}'')|^2 d\Omega'' \quad (37a)$$

can be seen to be a function of the magnitude K of $\mathbf{K}$ alone. On transforming from the variables k', Ω' in (23) to the variable K by means of (35), namely,

$$KdK = kk' \sin\vartheta' d\vartheta' = (1/2\pi)kk'd\Omega',$$

one obtains from (23) that part of the inelastic scattering cross section that comes from $(\mathfrak{L}_1{}^t)^2$

$$2\{\sigma_{\text{ine}}\} = (1-g+\tfrac{3}{4}g^2)(M/3\pi\hbar^2)^2 \times\frac{1}{k}\int\int k'(\mathfrak{L}_1{}^t)^2 d\Omega' d\mathbf{k}''$$
$$= (1-g+\tfrac{3}{4}g^2)(M/3\pi\hbar^2)^2 \times\frac{2\pi}{k^2}\int |F(K)|^2 KdK \times\int |C_{0k''}(K)|^2 k''^2 dk''. \quad (38)$$

Now since the ground state of the deuteron is a triplet state, it is clear from (33), (27), (18), and (24) that every inelastic triplet amplitude $\mathfrak{L}^t$ has a corresponding elastically scattered amplitude L differing from $\mathfrak{L}^t$ in having $\psi_0(\rho)$ in place of $\psi_{k''}{}^t(\rho)$. On adding to (38) the contribution of elastic scattering $(1-g+3/4g^2)L_1$ from (30a), one obtains as a typical contribution to the elastic plus twice the inelastic cross section the expression

$$\sigma_{el}+2\sigma_{ine}=(1-g+\tfrac{3}{4}g^2)(M/3\pi\hbar^2)^2 \times\frac{2\pi}{k^2}\int|F(K)|^2KdK\left\{|C_{00}(K)|^2+\int|C_{0k''}(K)|^2k''^2dk''\right\}, \quad (39)$$

where, analogous to (37),

$$C_{00}(K)=\int\exp\left[-\frac{i}{2}\mathbf{K}\cdot\boldsymbol{\rho}\right]\psi_0^*(\rho)\psi_0(\rho)d\boldsymbol{\rho}. \quad (40)$$

In (39), for any fixed value K of the magnitude of $\mathbf{K}$, the range of integration of k'' is from 0 to the maximum value k_m'' given by the energy equation (22), or, in view of (35),

$$k_m''^2=\tfrac{3}{4}k^2-(M/\hbar^2)\epsilon-\tfrac{3}{4}(k-K)^2. \quad (41)$$

The range of integration of K is from $k-[k^2-(4M/3\hbar^2)\epsilon]^{\frac{1}{2}}$ to $k+[k^2-(4M/3\hbar^2)\epsilon]^{\frac{1}{2}}$.

As $\psi_0(\rho)$ and the $\psi_{k''}{}^t(\boldsymbol{\rho})$ form a complete orthonormal set of functions, one can expand in (37)

$$\exp[-(i/2)\mathbf{K}\cdot\boldsymbol{\rho}]\psi_0(\rho)=C_{00}(K)\psi_0(\rho)+\int C_{0k''}(\mathbf{K},\mathbf{k}'')\psi_{k''}{}^t(\boldsymbol{\rho})d\mathbf{k}''.$$

From this and

$$\int|\exp[-(i/2)\mathbf{K}\cdot\boldsymbol{\rho}]\psi_0(\rho)|^2d\boldsymbol{\rho}=1$$

one obtains the relation

$$|C_{00}(K)|^2+\int_0^\infty|C_{0k''}(K)|^2k''^2dk''=1. \quad (42)$$

In view of (42), we can write in (39)

$$|C_{00}(K)|^2+\int_0^{k_m''}|C_{0k''}(K)|^2k''^2dk''=1-\int_{k_m''}^\infty|C_{0k''}(K)|^2k''^2dk''. \quad (43)$$

Now if one makes the plane wave approximation (32) in (37), then, instead of (37) and (37a), one has

$$D_{0k''}(\mathbf{K},\mathbf{k}'')=\frac{1}{(2\pi)^{\frac{3}{2}}}\int\exp[-(i/2)\mathbf{K}\cdot\boldsymbol{\rho}]\times\exp[-i\mathbf{k}''\cdot\boldsymbol{\rho}]\psi_0(\rho)d\boldsymbol{\rho} \quad (44)$$

and

$$|D_{0k''}(K)|^2=\int|D_{0k''}(\mathbf{K},\mathbf{k}'')|^2d\Omega''. \quad (44a)$$

From (44), it follows that

$$\int_0^\infty|D_{0k''}(K)|^2k''^2dk''=1.$$

The second integral in (38) for the inelastic cross section will be replaced by

$$\int_0^{k_m''}|D_{0k''}(K)|^2k''^2dk''=1-\int_{k_m''}^\infty|D_{0k''}(K)|^2k''^2dk''. \quad (45)$$

But on comparing (45) with (43), it is seen that to the approximation (32), the use of plane waves in (38) leads to (39).

An entirely similar argument holds for the singlet part $\mathfrak{L}_1{}^s$. It cannot be applied to terms containing $\mathfrak{L}_2$ or $\mathfrak{L}_2{}^*$ in (34). Calculations show, however, that $\mathfrak{L}_2$ gives some contribution only for large values of k''. This is expected since it arises from the effect of exchange. For large k'', the plane wave approximation would be valid. Now, for elastic scattering, the contribution from L_2 in (30) is negligibly small. Hence $\mathfrak{L}_2$ in (34), while giving the inelastic scattering, also gives approximately the total scattering. The other terms in (34) are either negligibly small, or give some small contributions only for large values of k''.

Finally let us consider the validity of the approximation (32) which replaces (43) by (45). In (39), the important region in K is when $|F(K)|^2K$ is near its maximum. With the Gaussian potential (28b), this maximum occurs at $K=\alpha$. With the value $\alpha=0.515\times10^{13}$ cm^{-1} in (57), one finds for an incident proton of 100 Mev the corresponding $k_m''(\alpha)$ at 37 Mev. For 200-Mev protons, this $k_m''(\alpha)$ corresponds to 56 Mev. For these k_m'', the approximation of replacing (43) by (45) is justified. For the region of small K in (39), however, the corresponding $k_m''(K)$ is smaller and the plane wave approximation will be bad. Actual calculation shows that the contribution to the total cross section from the region of small k'' is rather small in the case of the symmetrical theory. Hence the error committed by the plane wave approximation in this case is presumably not very great. The situation is less favorable with the ordinary and the exchange force, especially the former.

B. Angular Distribution and Energy Spectrum of Scattered Protons

For comparison with experimental results, it is desirable to transform to the laboratory coordinate system. Let $\mathbf{p}$ be the wave vector corresponding to the momentum of the scattered proton referred to the laboratory system. It can be shown that

$$\mathbf{p}=\tfrac{1}{2}\mathbf{k}+\mathbf{k}'. \tag{46}$$

Let E_p be the energy, in the lab system, of the scattered proton. Then $E_p=\hbar^2p^2/2M$. From (21) and (22), one has $k''dk''=\tfrac{3}{4}k'dk'$. Hence

$$k'k''^2dk''d\Omega'd\Omega''=\tfrac{3}{4}k''k'^2dk'd\Omega'd\Omega''=\tfrac{3}{4}k''d\mathbf{p}d\Omega''$$
$$=(3M/4\hbar^2)pk''dE_pd\Omega d\Omega'',$$

where $d\Omega$ is the solid angle into which $\mathbf{p}$ is directed. Hence the differential cross section of a proton being scattered into the solid angle $d\Omega=2\pi\sin\vartheta d\vartheta$ is given by

$$2\pi\sigma(E_0,E_p,\vartheta)dE_p\sin\vartheta d\vartheta$$
$$=\frac{3M}{4\hbar^2}\frac{k''}{k}pdE_p2\pi\sin\vartheta d\vartheta\int I(\Omega,\Omega'')d\Omega'', \tag{47}$$

where $I(\Omega,\Omega'')$ is obtained from $I(\Omega',\Omega'')$ in (34) by means of the transformation (46).

The evaluation of $\int I(\Omega,\Omega'')d\Omega''$ involves a large amount of work on account of the complicated manner in which the directions of $\mathbf{k}'$ and $\mathbf{k}''$ appear in the expressions (33). In the following, we have carried out a preliminary calculation without the tensor forces. The integral $I(\Omega)\equiv\int I(\Omega,\Omega'')d\Omega''$ is given by the following expressions for the three theories of nucleon interaction.

Ordinary:

$$(2\pi)^3(3\pi\hbar^2/M)^2I(\Omega)$$
$$=(1-g+g^2)(2\Lambda_1^2+\Lambda_2^2)$$
$$-(1+2g-2g^2)\Lambda_1\Lambda_2$$
$$+2(1-g+\tfrac{1}{2}g^2)\Lambda_1\Lambda_1^*$$
$$-(1+g-g^2)\Lambda_1^*\Lambda_2.$$

Exchange:

$$(2\pi)^3(3\pi\hbar^2/M)^2I(\Omega)=\text{expression above with 1, 2 interchanged.} \tag{48}$$

Symmetrical:

$$(2\pi)^3(3\pi\hbar^2/M)^2I(\Omega)$$
$$=(\tfrac{1}{3}-g+g^2)(2\Lambda_1^2+5\Lambda_2^2)$$
$$-(\tfrac{1}{3}-2g+2g^2)\Lambda_1\Lambda_2$$
$$-(2/9)(1-3g+(9/2)g^2)\Lambda_1\Lambda_1^*$$
$$+(\tfrac{1}{9}-g+g^2)\Lambda_1^*\Lambda_2$$
$$+((10/9)-(8/3)g+2g^2)\Lambda_2\Lambda_2^*,$$

where

$$\Lambda_1^2=4\pi D^2\frac{\lambda^2}{xk''}\exp\left[-\frac{x^2}{2\alpha^2}-\frac{(x-2k'')^2}{8\lambda^2}\right]\times\left(1-\exp\left[-\frac{xk''}{\lambda^2}\right]\right),\quad \mathbf{x}=\tfrac{3}{2}\mathbf{k}-\mathbf{p}$$

$$\Lambda_1\Lambda_1^*=4\pi D^2\exp\left[-\frac{k''^2}{2\lambda^2}-\frac{\alpha^2+4\lambda^2}{8\alpha^2\lambda^2}x^2\right],$$

$$\Lambda_2\Lambda_2^*=4\pi D^2\exp\left[-\frac{\alpha^2+\lambda^2}{2\alpha^2\lambda^2}k''^2-\frac{x^2}{8\lambda^2}-\frac{y^2}{2\alpha^2}\right],$$

$$\mathbf{y}=\tfrac{3}{4}\mathbf{k}+\tfrac{1}{2}\mathbf{p}$$

$$\Lambda_1\Lambda_2^*=\Lambda_1^*\Lambda_2=4\pi D^2\frac{\alpha^2}{yk''}\times\exp\left[-\frac{k''^2}{2\lambda^2}-\frac{\alpha^2+2\lambda^2}{8\alpha^2\lambda^2}x^2-\frac{(y-k'')^2}{4\alpha^2}\right](1-\exp[-yk''/\alpha^2]),$$

$$\Lambda_1\Lambda_2=4\pi D^2\frac{\sinh z}{z}\exp\left[-\frac{9}{64}\left(\frac{2\alpha^2+5\lambda^2}{\alpha^2\lambda^2}\right)k^2-\frac{2\alpha^2+5\lambda^2}{16\alpha^2\lambda^2}p^2-\frac{2\alpha^2+\lambda^2}{4\alpha^2\lambda^2}k''^2+\frac{3}{16}\cdot\frac{2\alpha^2+3\lambda^2}{\alpha^2\lambda^2}kp\cos\vartheta\right], \tag{49}$$

$$z^2=\xi^2+\eta^2,\quad \xi=\frac{2\alpha^2-\lambda^2}{4\alpha^2\lambda^2}pk''\sin\vartheta,$$

$$\eta=\frac{3(2\alpha^2+\lambda^2)}{8\alpha^2\lambda^2}kk''-\frac{2\alpha^2-\lambda^2}{4\alpha^2\lambda^2}pk''\cos\vartheta,$$

$$\Lambda_2^2=4\pi D^2\frac{\sinh t}{t}\exp\left[-\frac{9}{32}\frac{\alpha^2+\lambda^2}{\alpha^2\lambda^2}k^2-\frac{\alpha^2+\lambda^2}{8\alpha^2\lambda^2}(p^2+4k''^2)+\frac{3}{8}\frac{\alpha^2-\lambda^2}{\alpha^2\lambda^2}kp\cos\vartheta\right],$$

$$t=\mu^2+\nu^2,\quad \mu=\frac{\alpha^2-\lambda^2}{2\alpha^2\lambda^2}pk''\sin\vartheta,$$

$$\nu=\frac{3(\alpha^2+\lambda^2)}{4\alpha^2\lambda^2}kk''-\frac{\alpha^2-\lambda^2}{2\alpha^2\lambda^2}pk''\cos\vartheta.$$

The numerical values of the total cross section given in Table III were obtained by integrating

TABLE III. Cross sections σ_t for total (elastic plus inelastic) proton-deuteron scattering, in 10^{-24} cm^2.

		Ordinary	Exchange	Symmetrical
100 Mev	σ_{p-n}	0.140	0.140	0.140
	σ_{p-p}	0.130	0.130	0.028
	$\sigma_{p-n}+\sigma_{p-p}$	0.270	0.270	0.168
	$\sigma_{t(p-d)}$	0.424	0.213	0.089
200 Mev	σ_{p-n}	0.070	0.070	0.070
	σ_{p-p}	0.070	0.070	0.014
	$\sigma_{p-n}+\sigma_{p-p}$	0.140	0.140	0.084
	$\sigma_{t(p-d)}$	0.214	0.119	0.043

(47) over the energy and angular distribution of the emerging proton to give $\sigma_{\text{el}}+2\sigma_{\text{ine}}$.

C. Angular Distribution and Energy Spectrum of Neutrons Ejected in Proton-Deuteron Collisions

Experimentally one may have a beam of high energy neutrons impinging on a deuterium target and is interested in the angular distribution of the protons ejected and their spectrum for a given direction. When the Coulomb interaction is neglected, this situation is the same as studying the neutrons ejected in a proton-deuteron collision. To find the differential cross section of a neutron being scattered with energy E_n (in the laboratory system) into the solid angle $d\Omega_n$, one proceeds as follows. It can be shown that the wave vector of the neutron in the laboratory system is given by

$$\mathbf{p}_n=\tfrac{1}{2}\mathbf{k}-\tfrac{1}{2}\mathbf{k}'+\mathbf{k}''. \tag{50}$$

Let us define the vector $\mathbf{s}$,

$$\mathbf{s}=\mathbf{k}''-\tfrac{1}{2}\mathbf{k}' \tag{51}$$

so that $d\mathbf{p}_n=d\mathbf{s}$. Now the volume element in (23) can be written, on introducing a δ-function,

$$d\mathbf{k}''d\Omega'=d\mathbf{k}''d\Omega'dk'\delta(k'-g(k''))$$
$$=d\mathbf{k}''\frac{1}{k'^2}d\mathbf{k}'\delta(k'-g(k'')),$$

where the pole of the δ-function is at the value of k' related to k'' by the energy equation (22), namely,

$$k'=g(k'')\equiv\left(\frac{4M}{3\hbar^2}\right)^{\frac{1}{2}}\left(\frac{2}{3}E_0-\epsilon-\frac{\hbar^2k''^2}{M}\right)^{\frac{1}{2}}. \tag{52}$$

On changing the variables $\mathbf{k}'$, $\mathbf{k}''$ into $\mathbf{k}'$, $\mathbf{s}$, the differential cross section for $\mathbf{k}'$, $\mathbf{s}$ being directed into $d\mathbf{k}'$ and $d\mathbf{s}$, respectively, is

$$\sigma(E_0,\mathbf{s},\mathbf{k}')d\mathbf{s}d\mathbf{k}'=(k'/kk'^2)I(\Omega_s,\Omega')$$
$$\times\delta(k'-g(|\mathbf{s}+\tfrac{1}{2}\mathbf{k}'|))d\mathbf{s}d\mathbf{k}', \tag{53}$$

where $I(\Omega_s,\Omega')$ is obtained from (34) by means of (51). Writing

$$\mu=\cos\mathbf{k}'\mathbf{s},\quad \kappa^2=(M/\hbar^2)(\tfrac{2}{3}E_0-\epsilon)=\tfrac{3}{4}k^2-(M\epsilon/\hbar^2),$$

and inserting (51) into (52), one obtains

$$2k'=-s\mu\pm(s^2\mu^2-4s^2+4\kappa^2)^{\frac{1}{2}}. \tag{54}$$

The condition that at least one of the two roots k_1', k_2' be real and positive imposes a restriction on the values that s and μ may assume. A little consideration shows that for $0\leq s^2\leq\kappa^2$, the root k_1' corresponding to the positive sign in (54) is real and positive for all values of μ between -1 and 1. For $\kappa^2\leq s^2\leq 4/3\kappa^2$, and positive values of μ, both roots are negative and must be excluded. For negative values of μ between -1 and $-2[1-(\kappa^2/s^2)]^{\frac{1}{2}}$, both roots are real and positive. s^2 cannot exceed $(4/3)\kappa^2$ since in that case the roots become complex.

Let us denote by $v(k')$

$$v(k')\equiv k'-g(|\mathbf{s}+\tfrac{1}{2}\mathbf{k}'|).$$

To transform the δ-function $\delta(v(k')-0)$ so that the variable is the same as that of integration k', we have

$$\delta(v-0)=\sum_i(dk'/dv)_{v=0}\delta(k'-k_i')$$

where k_i' is a positive real root of (54), or, $v(k')=0$. For the two ranges of s^2 discussed above, we have

(i) $0\leq s^2\leq\kappa^2,\quad -1\leq\mu\leq 1,$

$$\delta(v-0)=\tfrac{3}{2}k_1'W\delta(k'-k_1'),$$

(ii) $\kappa^2\leq s^2\leq(4/3)\kappa^2,\quad -1\leq\mu\leq-2\left(1-\frac{\kappa^2}{s^2}\right)^{\frac{1}{2}},$

$$\delta(v-0)=\tfrac{3}{2}k_1'W\delta(k'-k_1')+\tfrac{3}{2}k_2'W\delta(k'-k_2'),$$

where

$$W\equiv(4\kappa^2-4s^2+s^2\mu^2)^{-\frac{1}{2}}.$$

Putting these into (53), one finds, for the differential cross section for a neutron being ejected with energy $E_n=\hbar^2p_n^2/2M$ (in the lab system) into the solid angle $d\Omega_n$,

$$\sigma(E_0,\, E_n,\, \Omega_n)dE_nd\Omega_n=\frac{Mp_n}{\hbar^2 k}I(\Omega_n)dE_nd\Omega_n. \quad (55)$$

Here $I(\Omega_n)$ is obtained from $I(\Omega_s)$ by means of (50) and (51), and $I(\Omega_s)$ is given, in the two ranges of values for s^2 above, by

$$\text{(i)}\quad I(\Omega_s)=\frac{3}{2}\int_0^{2\pi}\int_{-1}^{1} k_1'^2 WI(\Omega_s,\, \Omega')d\mu d\varphi',$$

$$\text{(ii)}\quad I(\Omega_s)=\frac{3}{2}\int_0^{2\pi}\int_{-1}^{-\nu} k_1'^2 WI(\Omega_s,\, \Omega')d\mu d\varphi' \quad (56)$$

$$+\frac{3}{2}\int_0^{2\pi}\int_{-1}^{-\nu} k_2'^2 WI(\Omega_s,\, \Omega')d\mu d\varphi',$$

where

$$\nu=2\left(1-\frac{\kappa^2}{s^2}\right)^{\frac{1}{2}}.$$

Calculations show that the important contribution to $I(\Omega)$ comes from those terms in (34) which are significant only for large values of k''. For large k'', the plane wave approximation (32) may not be seriously in error. Hence (55) may be expected to give the cross section for inelastic scattering only.

V. CALCULATION AND RESULT

A. Elastic Scattering

While the cross sections of elastic scattering on the basis of the three theories of nucleon interaction can be calculated from (19), (29), and (30), it is seen that the amount of numerical work is very great. In the present preliminary calculation, we have neglected the tensor force and so chosen the constants g, V_0, α^2 as to fit the data on the proton-proton scattering and the ground state of the deuteron. For the singlet even state, the potential is, according to (3) and (28b),

$$^1V^{\text{even}}=-(1-2g)\,V_0\exp[-\alpha^2r^2].$$

The analysis of Breit *et al.*[6] gives

$$(1-2g)\,V_0=26 \text{ Mev},\quad \alpha^2=0.266\times10^{26}\text{ cm}^{-2}. \quad (57)$$

[6] L. E. Hoisington, S. S. Share, and G. Breit, Phys. Rev. 56, 88 (1939).

For this value of α^2, the binding energy of the deuteron gives the depth V_0:[7]

$$V_0=45 \text{ Mev}. \quad (58)$$

For the constants of the approximate wave function (28b) of the ground state of the deuteron, we take the variational wave function of Gerjuoy and Schwinger,[8]

$$A=0.312\times10^{19},\quad \lambda^2=0.0716\times10^{26}\text{ cm}^{-2}. \quad (59)$$

With these constants, the cross sections for incident protons of 100, 150, and 200 Mev (in the lab system) have been calculated from (19a), (29), and (30). Table I gives the differential cross section $\sigma_{el}(E_0,\,\Theta)$ defined by (19a). Table II gives the total elastic scattering cross section.

$$\sigma_{el}(E_0)=2\pi\int\sigma_{el}(E_0,\,\Theta)\sin\Theta d\Theta.$$

It is seen that the difference predicted by the three theories of nucleon interaction is rather

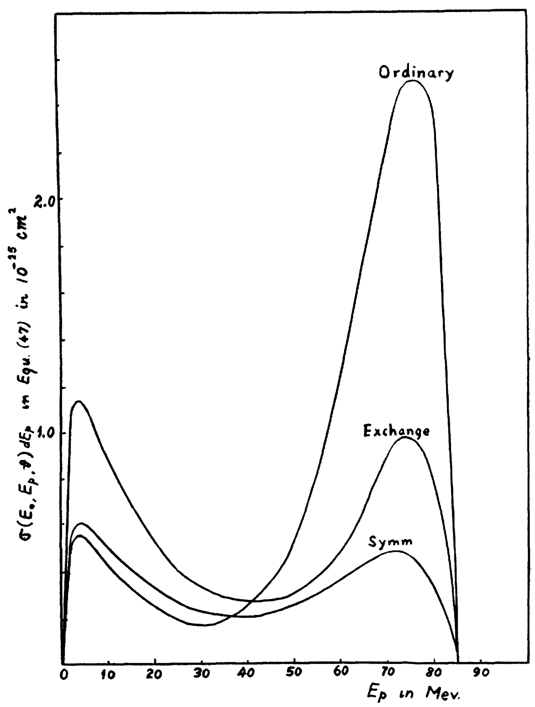

FIG. 1. Differential cross section $\sigma(E_0,\, E_p,\, \vartheta)dE_p$ defined in Eq. (47) of proton-deuteron total scattering. $E_0=100$ Mev, $\vartheta=30°$, and $dE_p=1$ Mev.

[7] H. A. Bethe and R. F. Bacher, Rev. Mod. Phys. 8, 111 (1936).

[8] E. Gerjuoy and J. Schwinger, Phys. Rev. 61, 138 (1942).

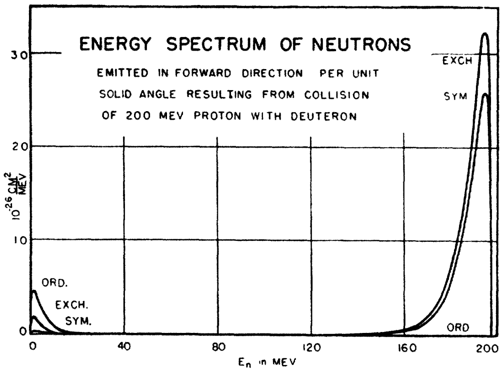

FIG. 2. Energy spectrum of ejected neutrons in the forward direction $\vartheta=0$ in proton-deuteron scattering at 200 Mev.

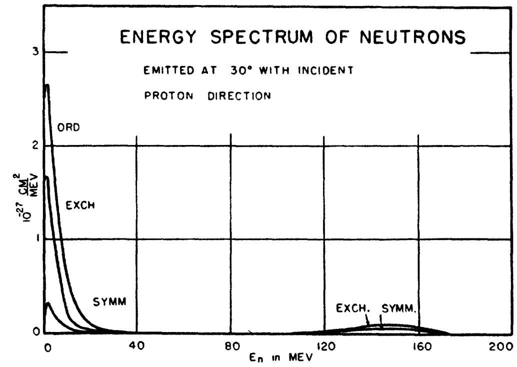

FIG. 3. Energy spectrum of neutrons ejected in directions $\vartheta=30°$ in proton-deuteron scattering at 200 Mev.

large. The very much smaller values of σ_{el} on the symmetrical theory have been discussed at the end of III. The sharp contrast from the two other models lends some interest to an experimental investigation of the elastic scattering at high energies.

B. Total (Elastic Plus Inelastic) Scattering

The differential cross section $\sigma(E_0, E_p, \vartheta)$ defined by (47) has been calculated without tensor forces by using (48), (49) and the constants in (57), (58), and (59). It is found that in (48), the most important terms are Λ_1^2 and $\Lambda_1\Lambda_1^*$ which are large for small ϑ and large values of E_p, i.e., in the neighborhood of $E_0-\epsilon$, and decrease rapidly with decreasing E_p. In Fig. 1 are given as examples the $\sigma(E_0, E_p, \vartheta)dE_p$ for $\vartheta=30°$ and $dE_p=1$ Mev for 100-Mev protons. These curves give the spectral distributions of the scattered protons for this particular direction. For smaller angles ϑ, the maximum at large E_p is further accentuated in all the three theories. For larger angles ϑ, the terms Λ_1^2 and $\Lambda_1\Lambda_1^*$ become negligible and the other terms such as Λ_2^2 give rise to a small maximum at the low energy end of E_p. While no quantitative significance should be attached to these spectral distributions, especially for small ϑ, on account of the failure of the plane wave approximation (32) for large E_p and hence small k'', as discussed in §IVA, the qualitative differences among the three potentials is believed to be of significance for comparison with experimental observations.

In Table III are given the integrated total scattering (elastic plus inelastic) cross sections for 100- and 200-Mev protons on the basis of the three potentials. In the same table are also given for comparison the total cross sections for proton-neutron and proton-proton scatterings calculated with the same potential constants in (57) and (58). The total proton-deuteron cross section σ_t is to be compared with the sum $\sigma_{p-n}+\sigma_{p-p}$ of the proton-neutron and the proton-proton cross sections. It is evident that the effect of interference of the scattered waves in the proton-deuteron scattering plays a large role in determining the final value of the total cross section.

It is of interest to compare the relative importance of elastic and dissociating collisions on the basis of the three theories of nucleon interaction. From σ_{el} in Table II for elastic scattering and σ_t in Table III for total scattering, one finds for the ratio σ_{el}/σ_t the values 0.83, 0.22, and 0.022 for 100 Mev and 0.40, 0.106, and 0.000,06 for 200 Mev, according to the ordinary, the exchange, and the symmetrical potential, respectively.

It is also of interest to make the following indirect comparison with the observed cross sections now available.[9] Recent calculations have shown that in the case of proton-neutron and proton-proton scatterings, the usual Born approximation is quite unsatisfactory for 100 Mev.[1,10] As the above calculation was made to this approximation, we do not expect the numerical values for the σ's to be very accurate at 100

[9] L. J. Cook, E. M. McMillan, J. M. Peterson, and D. C. Sewell, Phys. Rev. **72**, 1264 (1947).
[10] M. Camac and H. A. Bethe, Phys. Rev. **73**, 191 (1948).

Mev. The ratio $\sigma_{p-n}/\sigma_{t(p-d)}$, however, may not be in as serious error as the individual σ's. This ratio for 100 Mev is, for the three potentials,

0.33 (ordinary); 0.66 (exchange);
1.6 (symmetrical).

The observed values are $\sigma_{p-n}=0.083\times10^{-24}$ cm^2 and $\sigma_{t(p-d)}=0.117\times10^{-24}$ cm^2, giving for the ratio $\sigma_{p-n}/\sigma_{t(p-d)}$ the value 0.71. It is seen that the ordinary and symmetrical theories are in definite disagreement with the observed data.

C. Energy Spectrum of Neutrons Ejected in Proton-Deuteron Collisions

By means of (55) and (56) the energy spectra of the neutrons for the forward direction and for directions making an angle of 30° with that of the incident protons have been calculated. The result for 200-Mev incident protons is given in Figs. 2 and 3. As expected, the exchange and the symmetrical potential give neutrons with a well defined narrow energy peak near 200 Mev in the forward direction. These neutrons are the result of a direct exchange between the incident proton and the neutron in the deuteron. That this is essentially a two-body collision is made evident by the fact that the area under the high energy peak for either theory is very nearly equal to the proton-neutron cross section at zero degrees. The low energy peak may be regarded as due to the proton-proton collision in which the outgoing proton pulls the neutron along with it.

At 30°, the high energy peak is very much reduced in intensity and becomes very much wider. The maximum occurs at 150 Mev, corresponding to the conservation of energy and momentum in a two-body collision in which the neutron goes off at 30°, but the spectrum extends all the way to 173 Mev. This upper limit comes from collisions in which both protons come off with the same momentum. All collisions therefore involve the interactions between the three particles in an essential way. The low energy peaks are practically the same as at 0°, as should be expected from their physical interpretation.

The writers wish to thank Professor G. E. Uhlenbeck for suggesting the problem and Professor H. A. Bethe for helpful discussions.

APPENDIX I.†

Elastic Scattering with Tensor Force

When the potentials (3) including tensor force are employed, in summing over the spin coordinates in (17), it is necessary to evaluate the matrix elements $(i|S_{12}|l)$, $(i|S_{13}|l)$ of the operators S_{ij} in (4). These are readily obtained. We shall not give all these except the following few as examples.

$$(1|S_{12}|1)=(1|S_{13}|1)=3\cos^2\mu-1,$$
$$(1|S_{12}|2)=(1|S_{13}|2)=0,$$
$$(3|S_{12}|1)=(3|S_{13}|1)=2\sqrt{3}\sin\mu\cos\mu e^{i\nu},$$

where μ, ν are the polar angles of the line joining the two particles with respect to the direction of quantization of the spin.

The integrals in (17) that arise from the terms S_{12} and S_{13} can be shown to reduce to the following nine:

$$\left.\begin{matrix}I_1\\J_1\\K_1\end{matrix}\right\}=\gamma\int\int\exp[-i\mathbf{k}'\cdot\mathbf{r}]\psi_0^*(\rho)U(|\mathbf{r}+\tfrac{1}{2}\boldsymbol{\rho}|)\begin{Bmatrix}3\cos^2\mu-1\\\sin\mu\cos\mu e^{i\nu}\\\sin^2\mu e^{2i\nu}\end{Bmatrix}\exp[i\mathbf{k}\cdot\mathbf{r}]\psi_0(\rho)d\boldsymbol{\rho}d\mathbf{r},\tag{27A}$$

$$\left.\begin{matrix}I_2\\J_2\\K_2\end{matrix}\right\}=\gamma\int\int\exp[-i\mathbf{k}'\cdot\mathbf{r}]\psi_0^*(\rho)U(|\mathbf{r}+\tfrac{1}{2}\boldsymbol{\rho}|)\begin{Bmatrix}3\cos^2\mu-1\\\sin\mu\cos\mu e^{i\nu}\\\sin^2\mu e^{2i\nu}\end{Bmatrix}\exp[-i\mathbf{k}\cdot(\tfrac{1}{2}\mathbf{r}+\tfrac{3}{4}\boldsymbol{\rho})]\psi_0(|\mathbf{r}-\tfrac{1}{2}\boldsymbol{\rho}|)d\boldsymbol{\rho}d\mathbf{r},\tag{27B}$$

$$\left.\begin{matrix}I_3\\J_3\\K_3\end{matrix}\right\}=\gamma\int\int\exp[-i\mathbf{k}'\cdot\mathbf{r}]\psi_0^*(\rho)U(|\mathbf{r}-\tfrac{1}{2}\boldsymbol{\rho}|)\begin{Bmatrix}3\cos^2\mu-1\\\sin\mu\cos\mu e^{i\nu}\\\sin^2\mu e^{2i\nu}\end{Bmatrix}\exp[-i\mathbf{k}\cdot(\tfrac{1}{2}\mathbf{r}+\tfrac{3}{4}\boldsymbol{\rho})]\psi_0(|\mathbf{r}-\tfrac{1}{2}\boldsymbol{\rho}|)d\boldsymbol{\rho}d\mathbf{r}.\tag{27C}$$

† This appendix refers to Section III.

The angles μ, ν are again the polar angles of the vector $\mathbf{r}+\frac{1}{2}\boldsymbol{\varrho}$ or $\mathbf{r}-\frac{1}{2}\boldsymbol{\varrho}$ in the argument of the potential U with respect to the direction of quantization.

It can be shown that these integrals are given by

$$I_1=\frac{2}{5}\gamma L_1 x^2\left(1-3\sin^2\frac{\vartheta}{2}\right)\,{}_1F_1(1;7/2;x^2),$$

$$I_2=-\frac{1}{5}\gamma L_2 y^2(1+3\cos^2\vartheta)\,{}_1F_1(1;7/2;y^2),$$

$$I_3=\frac{2}{5}\gamma L_3\left[z^2-\frac{3}{16}\frac{k^2}{\alpha^2+\lambda^2}(1+2\cos\vartheta)^2\right]\times{}_1F_1(1;7/2;z^2),$$

$$J_1=\frac{1}{5}\gamma L_1 x^2\sin\vartheta\,{}_1F_1(1;7/2;x^2),$$

$$J_2=\frac{1}{10}\gamma L_2\left(\frac{3k}{4}\right)^2\frac{1}{\alpha^2+\frac{1}{2}\lambda^2}\sin\vartheta(1+\cos\vartheta)\times{}_1F_1(1;7/2;y^2),\quad\text{(29A)}$$

$$J_3=\frac{1}{20}\gamma L_3\frac{k^2}{\alpha^2+\lambda^2}\sin\vartheta(1+2\cos\vartheta)\times{}_1F_1(1;7/2;z^2),$$

$$K_1=\frac{2}{5}\gamma L_1 x^2\cos^2\frac{\vartheta}{2}\,{}_1F_1(1;7/2;x^2),$$

$$K_2=\frac{1}{40}\gamma L_2 x^2\frac{\alpha^2}{\alpha^2+\frac{1}{2}\lambda^2}\cos^2\frac{\vartheta}{2}\,{}_1F_1(1;7/2;y^2),$$

$$K_3=\frac{2}{5}\gamma L_3 x^2\frac{\alpha^2}{\alpha^2+\lambda^2}\cos^2\frac{\vartheta}{2}\,{}_1F_1(1;7/2;z^2),$$

where the x, y, z are given in (29). For the convenience of calculation, the confluent hypergeometric functions can be expressed in terms of the error integral as follows.

$$\frac{4}{15}x^2\,{}_1F_1(1;7/2;x^2)=\frac{1}{x^3}\exp[x^2]\int_0^x\exp[-t^2]dt-\frac{1}{x^2}-\frac{2}{3}.$$

The scattered intensity is given, in the three theories, by the following expressions.

Ordinary force:

$$\begin{aligned}(3\pi\hbar^2/M)^2I(\vartheta)&=\text{expression (30a)}\\&+(2I_1-I_2)^2-(2I_1-I_2)I_3+\tfrac{3}{2}I_3^2\\&+12\{(2J_1-J_2)^2\\&-(2J_1-J_2)J_3+\tfrac{3}{2}J_3^2\}\\&+3\{(2K_1-K_2)^2\\&-(2K_1-K_2)K_3+\tfrac{3}{2}K_3^2\}.\quad\text{(30A)}\end{aligned}$$

Exchange force:

$$(3\pi\hbar^2/M)^2I(\vartheta)=\text{expression above, with 1, 2 interchanged.}\quad\text{(30B)}$$

Symmetrical force:

$$\begin{aligned}(3\pi\hbar^2/M)^2I(\vartheta)&=\text{expression (30c)}\\&+(I_2^2-I_2I_3+I_3^2)\\&+12(J_2^2-J_2J_3+J_3^2)\\&+3(K_2^2-K_2K_3+K_3^2).\quad\text{(30C)}\end{aligned}$$

That the central force and the tensor force in (3) contribute additively to the scattered intensity is due to the fact that the spur of the tensor interaction S_{ij} is zero.

APPENDIX II.††

Total Scattering with Tensor Force

The integrals arising from the terms S_{12} and S_{13} in (25) can be shown to reduce to nine which are obtainable from (27A), (27B), (27C) above by replacing $\psi_0^*(\rho)$ by the continuum functions $\psi_{k''}^{*t}(\boldsymbol{\varrho})$ or $\psi_{k''}^{*s}(\boldsymbol{\varrho})$, and the nine others obtained from these by replacing $\mathbf{k}''$ by $-\mathbf{k}''$. Of these eighteen, the six integrals $\mathcal{I}_3$, $\mathcal{J}_3$, $\mathcal{K}_3$, $\mathcal{I}_3^*$, $\mathcal{J}_3^*$, $\mathcal{K}_3^*$, can be shown to contain a δ-function and contribute nothing to the scattered intensity. They have therefore been omitted in the following. To the approximation (32), these 12 integrals are as follows.

$$\mathcal{I}_1=\frac{2}{5}\gamma\mathcal{L}_1\left[x^2-\frac{3}{4\alpha^2}(k-k'\cos\vartheta')^2\right]\times{}_1F_1(1;7/2;x^2),$$

$$\mathcal{I}_2=\frac{2}{5}\gamma\mathcal{L}_2\left[y^2-\frac{3}{4\alpha^2}(k+K\cos\widehat{\mathbf{kK}})^2\right]\times{}_1F_1(1;7/2;y^2),\quad\mathbf{K}=\tfrac{1}{2}\mathbf{k}'+\mathbf{k}'',$$

†† This appendix refers to Section IV.

$$\mathcal{J}_1=\frac{1}{10}\gamma\mathcal{L}_1\cdot\frac{1}{\alpha^2}k'\sin\vartheta'(k-k'\cos\vartheta')\times{}_1F_1(1;7/2;x^2),$$

$$\mathcal{J}_2=\frac{1}{10}\gamma\mathcal{L}_2\cdot\frac{1}{\alpha^2}K\sin\widehat{\mathbf{kK}}(k+K\cos\widehat{\mathbf{kK}})\times{}_1F_1(1;7/2;y^2), \tag{33A}$$

$$\mathcal{K}_1=-\frac{1}{10}\gamma\mathcal{L}_1\left(\frac{k'}{\alpha}\right)^2\sin^2\vartheta'\ {}_1F_1(1;7/2;x^2),$$

$$\mathcal{K}_2=-\frac{1}{10}\gamma\mathcal{L}_2\left(\frac{K}{\alpha}\right)^2\sin^2\widehat{\mathbf{kK}}\ {}_1F_1(1;7/2;y^2),$$

$\mathcal{J}_1{}^*=\mathcal{J}_1$ with $\mathbf{k}''$ replaced by $-\mathbf{k}''$, etc.,

where x, y, D are those expressions given in (33).

The scattered intensity in the three theories is given below.

Ordinary force:

$$(2\pi)^3(3\pi\hbar^2/M)^2I(\Omega',\Omega'')=\text{expression (34a)}+\tfrac{3}{2}(\mathcal{I}_1-\mathcal{I}_2)^2+(\mathcal{I}_1-\mathcal{I}_2+\tfrac{3}{2}\mathcal{I}_1{}^*)\mathcal{I}_1{}^*+18(\mathcal{J}_1-\mathcal{J}_2)^2+12(\mathcal{J}_1-\mathcal{J}_2+\tfrac{3}{2}\mathcal{J}_1{}^*)\mathcal{J}_1{}^*+(9/2)(\mathcal{K}_1-\mathcal{K}_2)^2+3(\mathcal{K}_1-\mathcal{K}_2+\tfrac{3}{2}\mathcal{K}_1{}^*)\mathcal{K}_1{}^*. \tag{34A}$$

Exchange force:

$(2\pi)^3(3\pi\hbar^2/M)^2I(\Omega',\Omega'')=$ expression above, with 1, 2 interchanged. (34B)

Symmetrical force:

$$(2\pi)^3(3\pi\hbar^2/M)^2I(\Omega',\Omega'')=\text{expression (34c)}+\tfrac{1}{6}(\mathcal{I}_1-\mathcal{I}_2)^2-\tfrac{1}{9}(\mathcal{I}_1-\mathcal{I}_2)(\mathcal{I}_1{}^*+2\mathcal{I}_2{}^*)+\tfrac{1}{6}(\mathcal{I}_1{}^*+2\mathcal{I}_2{}^*)^2+2(\mathcal{J}_1-\mathcal{J}_2)^2-(4/3)(\mathcal{J}_1-\mathcal{J}_2)(\mathcal{J}_1{}^*+2\mathcal{J}_2{}^*)+2(\mathcal{J}_1{}^*+2\mathcal{J}_1{}^*)^2+\tfrac{1}{2}(\mathcal{K}_1-\mathcal{K}_2)^2-\tfrac{1}{3}(\mathcal{K}_1-\mathcal{K}_2)(\mathcal{K}_1{}^*+2\mathcal{K}_2{}^*)+\tfrac{1}{2}(\mathcal{K}_1{}^*+2\mathcal{K}_2{}^*)^2. \tag{34C}$$