Neutron-Proton and Proton-Proton Scattering at High Energies*

JULIUS ASHKIN
*University of Rochester, Rochester, New York**

AND

TA-YOU WU**
University of Michigan, Ann Arbor, Michigan

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The tensor force has been taken into account in calculating the neutron-proton and proton-proton scattering at energies in the range from 100 to 200 Mev. Most of the calculations are performed in the Born approximation using a square well potential and the exchange models of Rarita and Schwinger. The rigorous scattering theory has also been applied to the neutron-proton scattering with the symmetrical force model at 100 and 200 Mev. The results show that the Born approximation is inaccurate at 100 Mev but not bad at 200 Mev. Comparison with the experimental neutron-protron scattering at 90 Mev shows that the theoretical cross section for symmetrical forces is too large by a factor of about 1.6.

THE scattering of neutrons by protons at energies up to 80 Mev has recently been calculated by Camac and Bethe.[1] Throughout this work central forces alone were used, the tensor force being consistently neglected. The primary object was to investigate the scattering under different assumptions concerning the exchange character of the forces and their range.

In the present article the scattering has been calculated in the higher energy range from 100 to 200 Mev, taking account of the tensor forces. For this we have used the square well potential adjusted by Rarita and Schwinger[2] to fit the ground state of the deuteron and the low energy neutron-proton scattering. Calculations were made for three types of forces:[2] ordinary forces, Majorana (spatial) exchange forces, and "symmetric" exchange forces. The work may therefore be regarded as an extrapolation of the results of Rarita and Schwinger to the high energy range.

Preliminary measurements at Berkeley of the neutron-proton scattering at 90 Mev indicate important qualitative differences from all three of the above exchange models. The experiment is consistent with a force containing equal amounts of ordinary interaction and spatial exchange.[3] We feel, nevertheless, that the present calculations are worth reporting. The lack of agreement with experiment is in itself interesting. Further, the exact calculations at 100 and 200 Mev, which were made with the symmetrical forces, should serve as a valuable check on any approximate method for obtaining the cross section.

In Section I the neutron-proton and the proton-proton scattering has been calculated in the Born approximation for energies 100, 150, and 200 Mev. In IA we obtain the amplitude scattered by the tensor force alone in the Born approximation, and in IB and C the amplitudes are combined according to the three exchange models to give the differential cross section. Section II contains an *exact* calculation of the neutron-proton scattering for the symmetric force model. Comparison with I shows that the Born approximation is fairly good at 200 Mev but is untrustworthy at 100 Mev.

I. SCATTERING IN THE BORN APPROXIMATION

A. Effect of the Tensor Forces

Since the operator representing the tensor force between neutron and proton gives zero when applied to any singlet spin function, we need consider only the triplet scattering in this section. Separating out the motion of the center of mass, the wave function in the relative coor-

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** At present at Columbia University, New York, New York.

[1] M. Camac and H. A. Bethe, Phys. Rev. **73**, 191 (1948).

[2] W. Rarita and J. Schwinger, Phys. Rev. **59**, 436 (1941); Phys. Rev. **59**, 556 (1941).

[3] R. Serber, private communication.

dinate system representing the initial incident wave will be of the form

$$\psi_{\rm inc}=\exp(ik\mathbf{n}_0\cdot\mathbf{r})\sum_{m_s=-1}^{1}a_{m_s}\chi^{m_s}. \quad (1)$$

Here $\mathbf{r}$ is the vector from proton to neutron, $\mathbf{n}_0$ is a unit vector in the direction of the initial relative motion, and $\hbar k\mathbf{n}_0$ is the momentum of the incident neutron in the center of mass coordinate system. The χ^{m_s} ($m_s=1$, 0, -1) are the three triplet spin functions taken for convenience with respect to $\mathbf{n}_0$ as axis of quantization, and the a_{m_s} are three arbitrary constants.

Under the influence of the tensor force the spin function will, in general, change upon scattering, giving rise to a scattered wave of asymptotic form

$$\psi_{\rm scatt}\sim(1/r)e^{ikr}\sum_{m_s'=-1}^{1}\chi^{m_s'}\times\sum_{m_s=-1}^{1}\mathcal{S}_{m_s'm_s}(\theta,\varphi)a_{m_s}. \quad (2)$$

The amplitude of the scattered wave in the direction θ, φ with respect to $\mathbf{n}_0$ depends on the azimuthal angle φ as well as on the polar angle θ, since the tensor force is non-central. The angular dependence of the matrix of coefficients $\mathcal{S}_{m_s'm_s}(\theta,\varphi)$ gives a complete description of the scattering of any wave with definite initial spin state. In this section we shall calculate this matrix in the Born approximation. In II we shall use the rigorous scattering theory.

In most experiments the neutrons in the incident beam will be unpolarized with respect to spin. To find the triplet scattering cross section per unit solid angle in this case we must, therefore, average the square modulus of the scattered amplitude in (2) over the phases of the three amplitudes a_{m_s} of the triplet spin functions. Since the χ^{m_s} are orthogonal and normalized to unity, and since $a_{m_s}{}^*a_{m_s''}$ averages to zero if $m_s\neq m_s''$ and to $\frac{1}{3}$ if $m_s=m_s''$, we find

$$\sigma_{\rm trip}=\text{Average}\Big(\sum_{m_s}\sum_{m_s'}\sum_{m_s''}\mathcal{S}^*{}_{m_s'm_s}\mathcal{S}_{m_s'm_s''}a^*{}_{m_s}a_{m_s''}\Big)$$
$$=\tfrac{1}{3}\sum_{m_s'}\sum_{m_s''}|\mathcal{S}_{m_s'm_s''}|^2. \quad (3)$$

The Born approximation for scattering problems[4] gives a simple explicit expression for the asymptotic scattered wave in terms of the interaction potential V between the neutron and proton.

$$\psi_{\rm scatt}\sim-(1/r)e^{ikr}(M/4\pi h^2)\times\int\exp(-ik\mathbf{n}\cdot\mathbf{r})V\psi_{\rm inc}d\mathbf{r}, \quad (4)$$

M is the neutron or proton mass and $\mathbf{n}$ is the unit vector in the direction of scattering. If we define a set of matrix elements $V_{m_s'm_s}(r)$ by the equation

$$V\chi^{m_s}=\sum_{m_s'=-1}^{1}V_{m_s'm_s}(r)\chi^{m_s'}, \quad (5)$$

we find, from (1) and (4)

$$\mathcal{S}_{m_s'm_s}(\theta,\varphi)=-(M/4\pi h^2)\int\exp(-ik\mathbf{n}\cdot\mathbf{r})\times V_{m_s'm_s}(r)\exp(ik\mathbf{n}_0\cdot\mathbf{r})d\mathbf{r}. \quad (6)$$

In the case of exchange forces, $V_{m_s'm_s}(r)$ will in general contain an operator which exchanges proton and neutron, i.e., changes $\mathbf{r}$ to $-\mathbf{r}$. It is therefore essential to keep the order of the factors in (6) as it is instead of combining the exponentials as is usually done in atomic scattering problems.

The linearity of the scattered amplitude in the interaction potential V allows us to calculate separately the amplitude scattered by the tensor force and by the central forces, and then to combine the two at the end according to whatever model of the forces one wishes to consider. Apart from a possible Majorana (spatial) exchange operator whose effect will be discussed later, the term in the potential containing the tensor force will be of the form $-S_{12}J(r)$, where $-J(r)$ gives the shape of the potential and

$$S_{12}=3\frac{(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})}{r^2}-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2. \quad (7)$$

$\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are the spin operators of neutron and proton, respectively.

In calculating the amplitude scattered by the tensor force, however, we shall not expand $S_{12}\chi^{m_s}$ as in (5), but rather perform the integral

[4] Mott and Massey, *Theory of Atomic Collisions* (The Clarendon Press, Oxford, 1933), p. 88.

in (4) directly treating the spin operators simply as constant vectors in the spatial integration. The result is:

$$(M/4\pi\hbar^2)\int \exp(ik(\mathbf{n}_0-\mathbf{n})\cdot\mathbf{r})J(r)\times\{(3/r^2)(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\}d\mathbf{r}=C(\theta)\mathcal{T}(\theta,\varphi), \quad (8)$$

where

$$C(\theta)=-(M/\hbar^2)\int_0^\infty r^2J(r)\times\left(\frac{\sin Kr}{Kr}-3\frac{\sin Kr-Kr\cos Kr}{(Kr)^3}\right)dr,$$

$$K=k|\mathbf{n}_0-\mathbf{n}|=2k\sin\tfrac{1}{2}\theta, \quad (8a)$$

and

$$\mathcal{T}(\theta,\varphi)=\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2-3\sigma_{1,\,\mathbf{n}_0-\mathbf{n}}\sigma_{2,\,\mathbf{n}_0-\mathbf{n}}. \quad (8b)$$

$\mathcal{T}$ is an angular average of the tensor operator weighted by $\exp[ik(\mathbf{n}_0-\mathbf{n})\cdot\mathbf{r}]$. $\sigma_{1,\,\mathbf{n}_0-\mathbf{n}}$ and $\sigma_{2,\,\mathbf{n}_0-\mathbf{n}}$ are the components of $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ in the direction of $\mathbf{n}_0-\mathbf{n}$, i.e., in the direction of the momentum transfer.

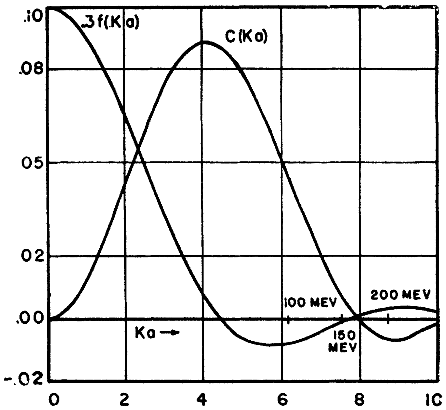

FIG. 1. The amplitude $c(Ka)$ scattered by the tensor force for a square well potential. The central force amplitude $f(Ka)$ (multipled by 0.3) is given for comparison.

To find the contribution to the matrix $\mathcal{S}$ of (6) we need to know the result of applying the operator $\mathcal{T}$ to the triplet spin functions. Defining the matrix elements of $\mathcal{T}$ by the equation

$$\mathcal{T}\chi_{m_s}=\sum_{m_s'=-1}^{1}\mathcal{T}_{m_s'm_s}(\theta,\varphi)\chi_{m_s'},$$

we find, after a simple calculation,

$$\{\mathcal{T}_{m_s'm_s}(\theta,\varphi)\}=\begin{array}{c|ccc} m_s'\backslash m_s & 1 & 0 & -1\\ \hline 1 & -\tfrac{1}{2}(1-3\cos\theta) & (3/\sqrt{2})\sin\theta e^{-i\varphi} & -3\cos^2(\theta/2)e^{-2i\varphi}\\ 0 & (3/\sqrt{2})\sin\theta e^{i\varphi} & 1-3\cos\theta & -(3/\sqrt{2})\sin\theta e^{-i\varphi}.\\ -1 & -3\cos^2(\theta/2)e^{2i\varphi} & -(3/\sqrt{2})\sin\theta e^{i\varphi} & -\tfrac{1}{2}(1-3\cos\theta)\end{array} \quad (9)$$

The azimuthal angle φ of the direction of scattering enters only in the off-diagonal elements where it enters in terms of unit modulus. Since the remaining central interaction in V contributes only additional diagonal terms to the matrix $\mathcal{S}$ of (6), we see that in the averaging process (3) for the cross-section there is no dependence on φ. For an unpolarized incident beam the scattering is axially symmetric.

It is also evident from (9) that the sum of the diagonal elements of $\mathcal{T}$ vanishes. This is important since it leads to the result that in the Born approximation the cross section (3) is composed additively of a contribution from the tensor forces and one from the central forces. The central forces give the same contribution to each diagonal element of $\mathcal{S}$ since they do not mix the triplet spin functions. In the calculation of the triplet cross section (3) the sum of the squares of the diagonal elements of $\mathcal{S}$ will contain cross terms between tensor force and central force amplitudes of the form: central force amplitude multiplied by the sum of the diagonal elements of $\mathcal{T}$. Since this vanishes, the above-mentioned additivity follows.

If the function $J(r)$, giving the shape of the nuclear potential, is taken as a square well of range a and depth V, evaluation of the function $C(\theta)$ of (8a) is simple:

$$C(\theta)=a\frac{MVa^2}{\hbar^2}\frac{1}{(Ka)^3}\left\{3\int_0^{Ka}\frac{\sin t}{t}dt-4\sin Ka+Ka\cos Ka\right\}. \quad (10)$$

Thus, $C(\theta)$ is proportional to a dimensionless function of Ka which may be calculated once for all energies. This function, which we call $c(Ka)$, is plotted in Fig. 1. For any given energy, Ka varies from 0 to $2ka$ as the angle of scattering in the center of mass system varies from 0 to π. On the graph there are indicated the values of $2ka$ for 100, 150, and 200 Mev calculated for the range $a=2.80\times10^{-13}$ cm. In this energy range $C(\theta)$ has its maximum value somewhere between 50° and 80° as we can see readily from the graph. This has the effect of bolstering up the triplet scattering cross section for intermediate angles. The vanishing of $C(\theta)$ for $\theta=0$ is not a special property of the square well, but holds for any potential $J(r)$ which does not have too bad a singularity at $r=0$. In particular it holds for the Yukawa-type potential.

B. Neutron-Proton Cross Sections Corresponding to Various Types of Exchange

In this section we shall calculate the neutron-proton scattering cross section using the nuclear force models of Rarita and Schwinger.[2] For states of even parity all the models must give the same potential since they are adjusted to fit the ground state of the deuteron, and the low energy neutron-proton scattering. The former gives the potential for triplet spin states and the latter for singlet. These have the form

$$\begin{aligned} {}^3V_{\text{even}} &= -(1+\gamma S_{12})J(r), \\ {}^1V_{\text{even}} &= -(1-2g)J(r). \end{aligned} \tag{11}$$

$J(r)$ is taken as a square well of depth 13.89 Mev and range $a=2.80\times10^{-13}$ cm, $\gamma=0.775$ and $1-2g=0.857$. S_{12} is the tensor operator (7).

The differences between the models arise in the states of odd parity which play an increasingly important role as the energy increases. Three cases are chosen. (i) *Ordinary forces* in which the potential is the same for odd as it is for even parity, (ii) *Majorana exchange forces* in which the potential changes sign in going from even to odd parity, and (iii) "*Symmetric*" *exchange forces* described by

$${}^3V_{\text{odd}} = -\tfrac{1}{3}\,{}^3V_{\text{even}}, \quad {}^1V_{\text{odd}} = -3\,{}^1V_{\text{even}}.$$

This dependence of the potential on the parity of the wave function may be described alternatively in terms of the Majorana exchange operator, P_M, which interchanges the spatial coordinates of the two particles.[5] Thus, since P_M does not affect wave functions of even parity but changes the sign of a wave function of odd parity, we have

exchange forces:

$${}^3V_{\text{exch}} = P_M\,{}^3V_{\text{even}}, \quad {}^1V_{\text{exch}} = P_M\,{}^1V_{\text{even}}; \tag{12a}$$

symmetric forces:[6]

$$\begin{aligned} {}^3V_{\text{sym}} &= \tfrac{1}{3}(1+2P_M)\,{}^3V_{\text{even}}, \\ {}^1V_{\text{sym}} &= -(1-2P_M)\,{}^1V_{\text{even}}. \end{aligned} \tag{12b}$$

The effect of the P_M operator on the scattered amplitude is strikingly demonstrated in the exchange force case (12a). Insertion of this potential into (4) produces a change[7] in ψ_{inc} from $\exp[ik\mathbf{n}_0\cdot\mathbf{r}]$ to $\exp[-ik\mathbf{n}_0\cdot\mathbf{r}]$. This has the same effect, however, as changing $\mathbf{n}$ to $-\mathbf{n}$ in $\exp[-ik\mathbf{n}\cdot\mathbf{r}]$ and leaving ψ_{inc} alone. As a result, the amplitude scattered in the direction $-\mathbf{n}$ with exchange forces is identical in the Born approximation with the amplitude in the direction $\mathbf{n}$ with ordinary forces. The exchange forces produce a simple exchange of identity between the interacting particles. Thus, if a beam of high energy neutrons is incident on a material containing protons in the laboratory system, the pattern of scattered particles will contain a large fraction of energetic protons peaked in the forward direction. Recent experiments at Berkeley[3] show such an effect but indicate an exchange interaction more complicated than the simple exchange model (12a). The possibility of such a phenomenon was first pointed out by Wick and Bhabha soon after the exchange forces were introduced into nuclear physics.

With this observation on the effect of the operator P_M we may write down immediately the scattered amplitudes for each model. Denoting by

$$\begin{aligned} F(\theta) &= (M/4\pi\hbar^2)\int \exp[ik(\mathbf{n}_0-\mathbf{n})\cdot\mathbf{r}]J(r)d\mathbf{r} \\ &= (M/\hbar^2)\int_0^\infty (\sin Kr/Kr)J(r)r^2dr, \end{aligned} \tag{13}$$

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[5] H. A. Bethe, *Elementary Nuclear Theory* (John Wiley and Sons, Inc., New York, 1947), pp. 80–84.

[6] In the isotopic spin notation the potential for symmetric forces may be written $-\frac{1}{3}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 V_{\text{even}}$.

[7] ψ_{inc} is now considered to include the singlet as well as the triplet part of the incident wave.

the amplitude scattered in the direction $\mathbf{n}$ by a central force of potential $-J(r)$, we have for the triplet amplitudes (6), according to (11), (12), and (8):

ordinary:

$$\mathcal{S}_{m_s'm_s}(\theta,\varphi)=F(\theta)\delta_{m_s'm_s}+\gamma C(\theta)\mathcal{T}_{m_s'm_s}(\theta,\varphi),$$

exchange:

$$\mathcal{S}_{m_s'm_s}(\theta,\varphi)=F(\pi-\theta)\delta_{m_s'm_s}+\gamma C(\pi-\theta)\mathcal{T}_{m_s'm_s}(\pi-\theta,\pi+\varphi),$$

symmetric:

$$\mathcal{S}_{m_s'm_s}(\theta,\varphi)=\tfrac{1}{3}\{F(\theta)+2F(\pi-\theta)\}\delta_{m_s'm_s}+\tfrac{1}{3}\gamma\{C(\theta)\mathcal{T}_{m_s'm_s}(\theta,\varphi)+2C(\pi-\theta)\mathcal{T}_{m_s'm_s}(\pi-\theta,\pi+\varphi)\}. \quad (14)$$

$\pi-\theta$ and $\pi+\varphi$ are the polar and azimuthal angles of $-\mathbf{n}$ with respect to the incident direction $\mathbf{n}_0$. $\delta_{m_s'm_s}$ is unity or zero according as m_s' and m_s are equal or unequal.

For the singlet amplitudes we have, according to (11), (12), and (13):

$$\begin{aligned}&\text{ordinary:} && (1-2g)F(\theta),\\ &\text{exchange:} && (1-2g)F(\pi-\theta), \qquad (15)\\ &\text{symmetric:} && -(1-2g)\{F(\theta)-2F(\pi-\theta)\}.\end{aligned}$$

The complete cross section for the neutron-proton scattering is obtained from the singlet and triplet cross sections by

$$\sigma(\theta)=\tfrac{1}{4}\sigma_{\text{sing}}(\theta)+\tfrac{3}{4}\sigma_{\text{trip}}(\theta).$$

Squaring the amplitudes (15) for the singlet cross section and performing the sum in (3) we find:

ordinary:

$$\sigma(\theta)=\tfrac{1}{4}(3+(1-2g)^2)F^2(\theta)+6\gamma^2C^2(\theta),$$

exchange:

$$\sigma(\theta)=\tfrac{1}{4}(3+(1-2g)^2)F^2(\pi-\theta)+6\gamma^2C^2(\pi-\theta),$$

symmetric:

$$\begin{aligned}\sigma(\theta)=&(\tfrac{1}{3}+(1-2g)^2)(\tfrac{1}{4}F^2(\theta)+F^2(\pi-\theta))\\&+(\tfrac{1}{3}-(1-2g)^2)F(\theta)F(\pi-\theta)\\&+(8/3)\gamma^2(\tfrac{1}{4}C^2(\theta)+C^2(\pi-\theta))\\&-(4/3)\gamma^2C(\theta)C(\pi-\theta). \qquad (16)\end{aligned}$$

Tensor forces and central forces contribute separately to the cross section, as has been noted earlier.

For a square well potential $F(\theta)$ is proportional to a dimensionless function, $f(Ka)$, of the variable Ka.

$$F(\theta)=a\frac{MVa^2}{\hbar^2}\frac{\sin Ka-Ka\cos Ka}{(Ka)^3}. \qquad (17)$$

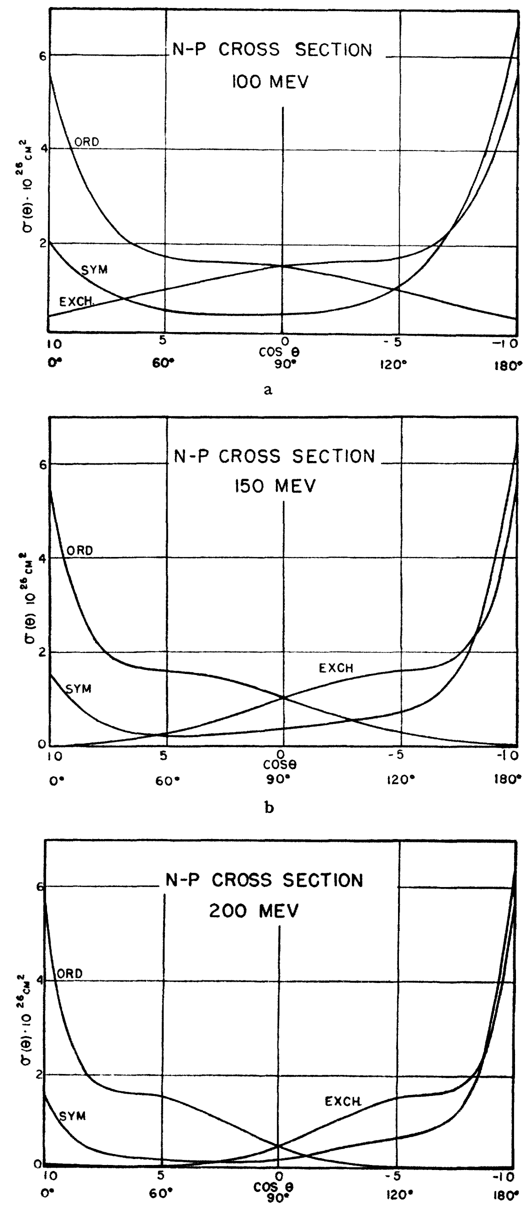

FIG. 2. Neutron-proton scattering cross section (center of mass system) in the Born approximation for a square well potential including the tensor forces. (a) 100 Mev, (b) 150 Mev, (c) 200 Mev.

This is plotted in Fig. 1 (multiplied by 0.3 for convenience) to show the contrast between the central and tensor force amplitudes. The cross sections based on these functions $f(Ka)$ and $c(Ka)$ are shown in Fig. 2a, b, c for the energies 100, 150, and 200 Mev. The general qualitative features are evident from (16) and Fig. 1. Thus the exchange curve is the reflection of the ordinary cross section about the 90° position. It is clear also, that as the energy increases the differences between the different models become sharper. In Table I we have summarized some of

TABLE I. Neutron-proton scattering in the Born approximation for different exchange models.

Neutron energy (lab.) Mev	Total cross section (10^{-24} cm²) Ord.	Exch.	Symm.	$\sigma(180°)/\sigma(90°)$ Ord.	Exch.	Symm.
100	0.205	0.205	0.167	0.31	3.69	13.1
150	0.143	0.143	0.114	0.011	5.43	17.6
200	0.108	0.108	0.090	0.030	12.1	35.5

the information to be obtained from the curves. The comparison with the Berkeley experiments will be made later in Section II where the cross section is computed more accurately than in the Born approximation.

Table I shows that for the exchange and symmetric forces the ratio of the scattered intensities at 180° and 90° is a rapidly increasing function of the energy as expected. For ordinary forces, however, this ratio shows an erratic behavior which is clarified by the graphs in Fig. 1. Thus the low value 0.011 at 150 Mev is due to the accidentally low values of $f(Ka)$ and $c(Ka)$ for 180° relative to 90°. The subsequent *increase* to 0.030 at 200 Mev seems at first sight surprising, but can be readily explained in terms of the oscillatory behavior of $f(Ka)$ and may, be attributed, therefore, to the special nature of the square well potential. This effect was first pointed out by Camac and Bethe[1] who investigated the ratio $\sigma(180°)/\sigma(90°)$ in the absence of tensor forces and found an extraordinary increase from 1.1 to 93 in going from 40 Mev to 80 Mev. From Fig. 1 we see that this may occur if the value of Ka for 90° at some energy happens to be near the node of $f(Ka)$. In the Born approximation the node occurs at the 90° angle for 106 Mev. With tensor forces the effect is not so pronounced since $f(Ka)$ and $c(Ka)$ do not have a node for a common value of Ka, in general.

As we shall see in Section II, the Born approximation is a poor one at 100 Mev. Apart from numerical disagreements, however, it is possible at this point to notice an important ***qualitative*** deficiency in the approximation which may be of importance in making comparisons with experiment. Thus for exchange forces the curves in Fig. 2 are the reflections of the ordinary curves about 90°, the behavior near 0° being identical with the behavior of the ordinary cross section near 180°. In actual fact the exact cross section for exchange forces shows a pronounced maximum at 0° which is entirely lacking in the Born approximation.

The explanation for this is given most directly if we restrict the consideration to ordinary and exchange central forces leaving out the tensor forces and the spin dependence. In the Born approximation the phase shift for the partial wave of angular momentum l is given in terms of the potential V by[8]

$$\delta_l = -(\pi M/2h^2)\int_0^\infty V(r)J_{l+\frac{1}{2}}{}^2(kr)rdr. \quad (18)$$

Due to the change of sign of V for odd l in the case of exchange forces, the phase shifts are alternately the same and opposite to the phase shifts for ordinary forces as l goes through the sequence 0, 1, 2, 3, $\cdots$. If the energy is high enough and the potential V is not too large, the approximate phase shifts (18) will not be very different from the exact values. In terms of the δ_l the scattered amplitude in the direction θ is given rigorously by

$$f(\theta) = (1/2ik)\sum_{l=0}^{\infty}(2l+1)(e^{2i\delta_l}-1)P_l(\cos\theta).$$

If we take an energy like 200 Mev and V approximately 20 Mev, the phase shifts are comparatively small, of the order 0.4 radians, and are appreciable up to $l=5$. It is therefore legitimate to expand $\exp(2i\delta_l)-1$. Taking the exchange force amplitude at $\theta=0$,

$$f(0) = (1/2ik)\sum_{l=0}^{\infty}(2l+1)(2i\delta_l - 2\delta_l^2+\cdots).$$

[8] Reference 4, p. 28.

As is well known,[9] the Born approximation results if only the terms $2i\delta_l$ are kept in the expansion, with δ_l given by (18). Since these terms alternate in sign like $(-1)^l$ they contribute very little to the sum. They give the same contribution at $\theta=0$ as the similar terms at $\theta=180°$ for ordinary forces where the Legendre polynomials $P_l(-1)$ contribute the alternation of sign instead of the δ_l. However, the δ_l^2 terms all contribute with the same sign to $f(\theta)$ and give maximum effect at $\theta=0$ where the Legendre polynomials give the strongest constructive interference. Some idea of the size of the effect can be seen from Fig. 3 where the energy 200 Mev and a square well of depth 20 Mev and range $a=e^2/mc^2$ were chosen. With tensor forces the same phenomenon should manifest itself. As a result the angular dependence of the neutron-proton cross section for exchange forces and for symmetric forces should be qualitatively the same, contrary to the Born approximation.

C. Neutron-Neutron or Proton-Proton Cross Section

If it is assumed that the forces between neutron and neutron and between neutron and proton are the same if the pairs of particles are in the same state, we may use the same set of scattered amplitudes (14) and (15), appropriately modified by the Pauli exclusion principle, to calculate the $n-n$ or the $p-p$ scattering. For singlet scattering the spatial part of the wave function must be symmetric and all three of the models give the amplitude:

singlet: $(1-2g)(F(\theta)+F(\pi-\theta))$.

For the triplet case the spatial wave function must be antisymmetric giving:

triplet: $\mathcal{S}_{m_s'm_s}(\theta,\varphi)-\mathcal{S}_{m_s'm_s}(\pi-\theta,\pi+\varphi)$.

The cross sections resulting from these amplitudes are:

ordinary:

$$\begin{aligned}\sigma(\theta)=\tfrac{1}{4}(3+(1-2g)^2)(F^2(\theta)+F^2(\pi-\theta))\\+6\gamma^2(C^2(\theta)+C^2(\pi-\theta))\\-\tfrac{1}{2}(3-(1-2g)^2)F(\theta)F(\pi-\theta)\\+6\gamma^2C(\theta)C(\pi-\theta);\end{aligned}$$

[9] Reference 4, p. 90.

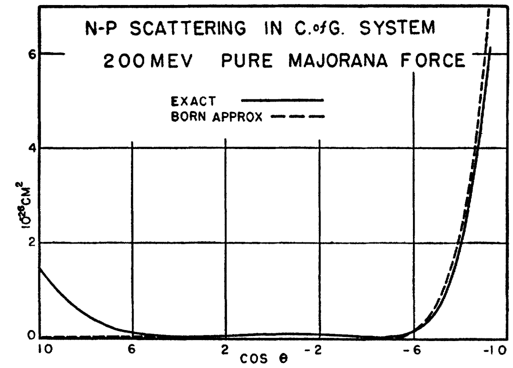

FIG. 3. Existence of a forward maximum in the scattered intensity for exchange forces. Spin dependence and tensor forces are neglected in this illustration.

exchange: same as ordinary;

symmetric: (19)

$$\begin{aligned}\sigma(\theta)=(\tfrac{1}{12}+\tfrac{1}{4}(1-2g)^2)(F^2(\theta)+F^2(\pi-\theta))\\+(\tfrac{1}{2}(1-2g)^2-\tfrac{1}{6})F(\theta)F(\pi-\theta)\\+\tfrac{2}{3}\gamma^2(C^2(\theta)+C^2(\pi-\theta)+C(\theta)C(\pi-\theta)).\end{aligned}$$

From the curves in Fig. 4 we see that the cross section in the symmetrical force theory is much smaller than the one corresponding to ordinary forces or exchange forces. This is due to the fact that for symmetric forces in the odd triplet states the interaction is one third as large as for ordinary or exchange forces. Since the even triplet states are excluded for identical particles, the triplet cross section with symmetrical forces is reduced by a factor 9. The admixture of the singlet cross section, which is the same in all three models, makes the ratio somewhat less than 9 but still large. An absolute measurement of the

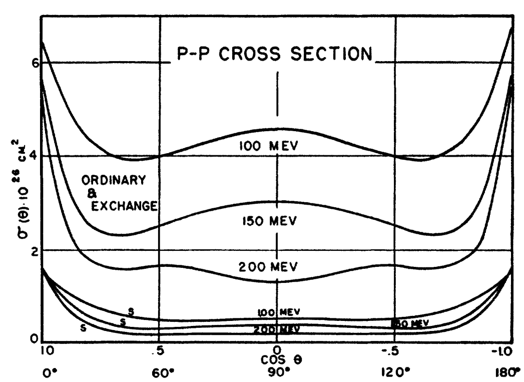

FIG. 4. Proton-proton cross section in the Born approximation for the same potentials as Fig. 2. Coulomb scattering is neglected.

proton-proton scattering cross section at some intermediate angle should, therefore, give an indication of the character of the exchange forces involved. Tensor forces have the effect of bringing up the cross section in the intermediate angular range. The change from maximum to minimum at 90° as the energy increases can be explained in terms of the slope and curvature of the curve for $c(Ka)$ in Fig. 1.

Table II gives the total $p-p$ (or $n-n$) cross section and its comparison with the $n-p$ cross section. A simple estimate of the ratio of cross sections $\sigma_{n-p}/\sigma_{p-p}$ can be made if the energy is large enough so that angular integrals of the type $\int F(\theta)F(\pi-\theta)d\omega$ or $\int C(\theta)C(\pi-\theta)d\omega$ may be neglected. Comparison of (16) and (19) shows that the ratio should be approximately 1.0 for ordinary or exchange forces and 5.0 for symmetric forces. It should be noted that the total proton-proton cross section is obtained from (19) by integrating over all solid angles and then dividing by two, since in each scattering event two protons are ejected with no means for distinguishing the initial particle from the target particle.

TABLE II. Proton-proton and neutron-proton scattering compared in the Born approximation.

Energy (lab.) Mev	σ_{p-p} (10^{-24} cm²) Ord.	Exch.	Symm.	$\sigma_{n-p}/\sigma_{p-p}$ Ord.	Exch.	Symm.
100	0.278	0.278	0.0414	0.738	0.738	4.04
150	0.182	0.182	0.0289	0.788	0.788	3.96
200	0.115	0.115	0.0208	0.944	0.944	4.34

II. NEUTRON-PROTON CROSS SECTION IN THE RIGOROUS SCATTERING THEORY

The extension of the method of partial waves to scattering problems with non-central forces, like the tensor force, has been made by Rarita and Schwinger.[2] Since their final results are somewhat condensed, however, we shall first put them in a form more suitable for computation and then apply them to the neutron-proton scattering with the symmetric force model.

We consider the triplet $n-p$ scattering under a non-central force of potential $V(\mathbf{r}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$ which commutes with the total angular momentum $\mathbf{J}=\mathbf{L}+\mathbf{S}$, with the magnitude of the spin S^2, and with the space exchange operator P_M. V will, in general, not commute with any of the components of the orbital angular momentum $\mathbf{L}$, or of the spin angular momentum $\mathbf{S}$. These are the commutation properties of the tensor force. To facilitate the phase shift analysis, which is the basis of the method of partial waves, it is convenient to transform to the $SLJm$ representation in which states are labeled according to the value of total spin S, total orbital angular momentum L, total angular momentum J, and z-component of J equal to m. In this representation we define a set of functions $F_L{}^{Jm}$, the "normalized spherical harmonics with spin," which are eigenfunctions of the operators S^2, L^2, J^2, J_z, and P_M. The total spin $S=1$ and the eigenvalue of the exchange operator, P_M, is the parity $(-1)^L$.

These functions, $F_L{}^{Jm}$, may be obtained by a unitary transformation applied to the orthogonal functions $Y_{Lm_L}(\theta,\varphi)\chi_{m_s}$, which are appropriate for the SLm_Lm_s representation, where the Y_{Lm_L} are the ordinary spherical harmonics and the χ_{m_s} are the triplet spin functions ($m_s=1, 0, -1$).

$$F_L{}^{Jm}=\sum_{m_s=-1}^{1} Y_{L,m-m_s}\chi_{m_s} \times(SLm-m_sm_s|SLJm), \tag{20}$$
$$Y_{Lm_L}\chi_{m_s}=\sum_{J=L-1}^{L+1} F_L{}^{Jm_L+m_s} \times(SLJm_L+m_s|SLm_Lm_s).$$

For a given value of J we must have $|m|\leqq J$ and L one of the values $J-1$, J, $J+1$ as is shown by the vector model. Condon and Shortley[10] give a table for the transformation coefficients which is reproduced in (21) for the special case of $m_L=0$.

$(SL0m_s|SLJm_s)$ (21)

$J \backslash m_s$	± 1	0
$J=L+1$	$\left(\frac{L+2}{2(2L+1)}\right)^{\frac{1}{2}}$	$\left(\frac{L+1}{2L+1}\right)^{\frac{1}{2}}$
$J=L$	$-1/\sqrt{2}$	0
$J=L-1$	$\left(\frac{L-1}{2(2L+1)}\right)^{\frac{1}{2}}$	$-\left(\frac{L}{2L+1}\right)^{\frac{1}{2}}$

[10] Condon and Shortley, *Theory of Atomic Spectra* (The Macmillan Company, New York, 1935), p. 76, Table 2³.

Taking the z axis as the direction of propagation of the incident triplet wave and using this same direction as axis of quantization for the angular momentum, we make the usual expansion:

$$e^{ikz}\sum_{m_s=-1}^{1} a_{m_s}\chi_{m_s}=\sum_{L=0}^{\infty}\sum_{m_s=-1}^{1}(2L+1)i^L \times[g_L(kr)/kr][4\pi/(2L+1)]^{\frac{1}{2}} \times Y_{L0}(\theta)\chi_{m_s}a_{m_s}, \tag{22}$$

where

$$g_L(kr)=(\pi kr/2)^{\frac{1}{2}}J_{L+\frac{1}{2}}(kr)\sim\sin(kr-\tfrac{1}{2}L\pi)$$

is the Lth radial wave function in the absence of interaction. The complete wave function for the scattering problem is obtained by first writing the incident plane wave as a sum of partial waves in the $SLJm$ representation and then replacing the radial function $g_L(kr)$ by

$$\exp(i\delta_L{}^{Jm})u_L{}^{Jm}(r),$$

where $u_L{}^{Jm}(r)$ is the radial function with the non-central interaction and $\delta_L{}^{Jm}$ is the phase shift for this partial wave:

$$u_L{}^{Jm}(r)\sim\sin(kr-\tfrac{1}{2}L\pi+\delta_L{}^{Jm}).$$

Subtracting the incident wave gives for the asymptotic form of the scattered wave, in the customary way,

$$\psi_{\text{scatt}}\sim\frac{e^{ikr}}{2ikr}\sum_{m_s=-1}^{1}\sum_{L=0}^{\infty}\sum_{J=L-1}^{L+1}[4\pi(2L+1)]^{\frac{1}{2}}\{\exp(2i\delta_L{}^{Jm_s})-1\}(SLJm_s|SL0m_s)F_L{}^{Jm_s}a_{m_s}$$

$$=\frac{e^{ikr}}{2ikr}\sum_{m_s,m_s'}\sum_{L,J}[4\pi(2L+1)]^{\frac{1}{2}}\{\exp(2i\delta_L{}^{Jm_s})-1\}(SLm_s-m_s'm_s'|SLJm_s) \times(SLJm_s|SL0m_s)Y_{L,\,m_s-m_s'}(\theta,\varphi)\chi_{m_s'}a_{m_s}.$$

The coefficient of a_{m_s} in this formula is the matrix element $\mathcal{S}_{m_s'm_s}(\theta,\varphi)$ of (2) given in the Born approximation by (6).

Application of the table of Condon and Shortley gives:

$$m_s=1\begin{cases}\mathcal{S}_{11}=\dfrac{1}{2ik}\displaystyle\sum_{L=0}^{\infty}[4\pi(2L+1)]^{\frac{1}{2}}Y_{L0}(\theta)\left\{\dfrac{L-1}{2(2L+1)}[\exp(2i\delta_L{}^{L-1,1})-1]+\tfrac{1}{2}[\exp(2i\delta_L{}^{L,1})-1]+\dfrac{L+2}{2(2L+1)}[\exp(2i\delta_L{}^{L+1,1})-1]\right\},\\[2ex] \mathcal{S}_{01}=\dfrac{1}{2ik}\displaystyle\sum_{L=1}^{\infty}[4\pi(2L+1)]^{\frac{1}{2}}\dfrac{Y_{L1}(\theta,\varphi)}{[2L(L+1)]^{\frac{1}{2}}}\left\{-\dfrac{L^2-1}{2L+1}[\exp(2i\delta_L{}^{L-1,1})-1]-[\exp(2i\delta_L{}^{L,1})-1]+\dfrac{L(L+2)}{2L+1}[\exp(2i\delta_L{}^{L+1,1})-1]\right\},\\[2ex] \mathcal{S}_{-11}=\dfrac{1}{2ik}\displaystyle\sum_{L=2}^{\infty}[4\pi(2L+1)]^{\frac{1}{2}}Y_{L2}(\theta,\varphi)[(L-1)L(L+1)(L+2)]^{\frac{1}{2}}\times\left\{\dfrac{\exp(2i\delta_L{}^{L-1,1})-1}{2L(2L+1)}-\dfrac{\exp(2i\delta_L{}^{L,1})-1}{2L(L+1)}+\dfrac{\exp(2i\delta_L{}^{L+1,1})-1}{(2L+1)(2L+2)}\right\};\end{cases} \tag{23}$$

$$m_s=0\begin{cases}\mathcal{S}_{10}=\dfrac{1}{2ik}\displaystyle\sum_{L=1}^{\infty}[4\pi(2L+1)]^{\frac{1}{2}}Y_{L-1}(\theta,\varphi)\left[\dfrac{L(L+1)}{2}\right]^{\frac{1}{2}}\left\{-\dfrac{\exp(2i\delta_L{}^{L-1,0})-1}{2L+1}+\dfrac{\exp(2i\delta_L{}^{L+1,0})-1}{2L+1}\right\},\\[2ex] \mathcal{S}_{00}=\dfrac{1}{2ik}\displaystyle\sum_{L=0}^{\infty}[4\pi(2L+1)]^{\frac{1}{2}}Y_{L0}(\theta)\left\{\dfrac{L}{2L+1}[\exp(2i\delta_L{}^{L-1,0})-1]+\dfrac{L+1}{2L+1}[\exp(2i\delta_L{}^{L+1,0})-1]\right\},\\[2ex] \mathcal{S}_{-10}=\dfrac{1}{2ik}\displaystyle\sum_{L=1}^{\infty}[4\pi(2L+1)]^{\frac{1}{2}}Y_{L1}(\theta,\varphi)\left[\dfrac{L(L+1)}{2}\right]^{\frac{1}{2}}\left\{-\dfrac{\exp(2i\delta_L{}^{L-1,0})-1}{2L+1}+\dfrac{\exp(2i\delta_L{}^{L+1,0})-1}{2L+1}\right\};\end{cases}$$

Equation (23) continued on next page)

$$m_s=-1\left\{\begin{array}{l}\mathcal{S}_{1-1}\text{: change } Y_{L2} \text{ in } \mathcal{S}_{-11} \text{ to } Y_{L-2} \text{ and replace the phase shifts by those for } m_s=-1;\\ \mathcal{S}_{0-1}\text{: change } Y_{L1} \text{ in } \mathcal{S}_{01} \text{ to } Y_{L-1} \text{ and replace the phase shifts;}\\ \mathcal{S}_{-1-1}\text{: replace the phase shifts in } \mathcal{S}_{11} \text{ by those for } m_s=-1.\end{array}\right.$$

Actually, for interactions of the symmetry of the tensor force the phase shifts for $m_s=1$ and

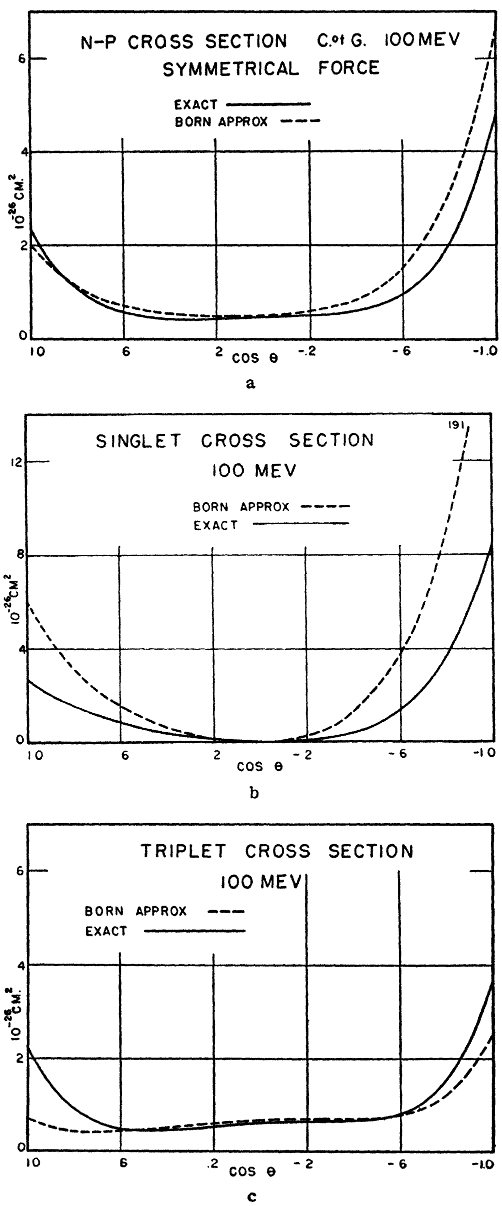

FIG. 5. The rigorous scattering theory and the Born approximation compared at 100 Mev for the neutron-proton scattering with symmetrical forces. (a) the complete differential cross section; (b) the singlet cross section; (c) the triplet cross section.

for $m_s=-1$ are the same. In the averaging process (3) the amplitudes with $m_s=1$ and -1 give the same contribution to the sum. It is also understood in the formulas that the phase shifts $\delta_L^{Jm_s}$ which appear formally must obey the conditions $|m_s|\leqq J$ and $0\leqq J$; otherwise they are to be replaced by zero.

To obtain numerical values for the phase shifts it is necessary to integrate the radial wave equations for the functions $u_L^{Jm}(r)$. These equations are obtained from the Schrödinger equation

$$(\nabla^2+k^2)\psi-(M/\hbar^2)V(\mathbf{r},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2)\psi=0$$

by expanding ψ in the form (24)

$$\psi=\sum_{J=0}^{\infty}\sum_{m=-J}^{J}\sum_{L=J-1}^{J+1}(1/r)u_L^{Jm}(r)F_L^{Jm}.$$

Using the orthogonality of the spin angular functions F_L^{Jm} and the fact that V commutes with $\mathbf{J}$, we find for each Jm three coupled equations for u_L^{Jm} with $L=J-1,\ J,\ J+1$:

$$\frac{d^2u_L^{Jm}}{dr^2}+\left(k^2-\frac{L(L+1)}{r^2}\right)u_L^{Jm}-\frac{M}{\hbar^2}\sum_{L'=J-1}^{J+1}V_{LL'}{}^J(r)u_{L'}{}^{Jm}(r)=0. \quad (25)$$

$V_{LL'}{}^J(r)$ is the matrix element $(F_L^{Jm},\ VF_{L'}{}^{Jm})$. These matrix elements are easily shown to be independent of m and are given for the tensor force S_{12} in the appendix of Rarita's and Schwinger's paper:

$$S_{LL'}{}^J=\begin{array}{c|ccc} L \backslash L' & J-1 & J & J+1\\ \hline J-1 & -2\dfrac{(J-1)}{2J+1} & 0 & \dfrac{6[J(J+1)]^{\frac{1}{2}}}{2J+1}\\ J & 0 & 2 & 0\\ J+1 & \dfrac{6[J(J+1)]^{\frac{1}{2}}}{2J+1} & 0 & -\dfrac{2(J+2)}{2J+1}\end{array}. \quad (26)$$

Actually the three Eqs. (25) split into two coupled equations for u_{J-1}^{Jm} and u_{J+1}^{Jm} and one equation for u_J^{Jm} alone. This is an immediate consequence of the fact that V commutes with the spatial exchange operator and hence cannot couple states of different parity.

Integration of the equations is greatly simplified for the case of a square well potential. Following the procedure of Rarita and Schwinger, one can express the solutions inside the well for u_{J-1}^{Jm} and u_{J+1}^{Jm} by power series[11] involving two arbitrary constants which must be adjusted to give continuity of the functions and their derivatives at the boundary. The solution outside will be a linear combination of the regular solution $g_L(kr)$ and the irregular solution $g_{-L-1}(kr)$ which has asymptotic form

$$\sin(kr-\tfrac{1}{2}L\pi+\delta_L^{Jm}).$$

Since

$$g_{-L-1}(kr)=[\pi kr/2]^{\frac{1}{2}}J_{-L-\frac{1}{2}}(kr)\sim(-1)^L\cos(kr-\tfrac{1}{2}L\pi),$$

the outside solutions are:

$$u_L^{Jm}=A_L^{Jm}\{\cos\delta_L^{Jm}g_L(kr)+(-1)^L\sin\delta_L^{Jm}g_{-L-1}(kr)\}. \quad (27)$$

This solution must represent the incident wave plus an outgoing spherical wave. From (22), (20), (21), and (24) the incident wave is represented by

$$(u_L^{Jm})_{\text{inc}}=C_L^{Jm}g_L(kr), \quad (28)$$

where

$$C_L^{Jm}=\frac{[4\pi(2L+1)]^{\frac{1}{2}}}{k}i^L(SLJm|SL0m). \quad (28a)$$

If the difference between (27) and (28) is to be an outgoing wave proportional to $\exp(ikr)$, the constant A_L^{Jm} must be chosen so that

$$u_L^{Jm}=C_L^{Jm}\exp(i\delta_L^{Jm})\{\cos\delta_L^{Jm}g_L(kr)+(-1)^L\sin\delta_L^{Jm}g_{-L-1}(kr)\}. \quad (29)$$

In this formula the index m is identical with m_s, the projection of the spin in the incident direction, since the orbital angular momentum of the incident wave with respect to this direction is zero. Equating the values and derivatives on both sides of the boundary gives the pair of phase shifts $\delta_{J-1}^{Jm_s}$ and $\delta_{J+1}^{Jm_s}$ for each J and $m_s=\pm1$, 0. The phase shifts[12] $\delta_J^{Jm_s}$ and the singlet phase shifts are obtained in the usual way.

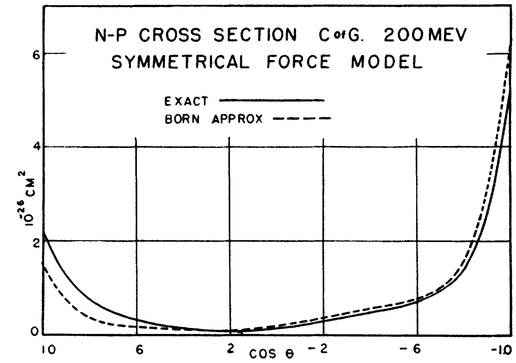

FIG. 6. Neutron-proton cross section at 200 Mev for symmetrical forces. Comparison of the rigorous theory with the Born approximation.

The method given above has been applied to the neutron-proton scattering with the symmetrical force model and calculations performed for 100 and 200 Mev. Figure 5a, b, c and Fig. 6 give the results in graphical form. For the total scattering cross section one finds:

Symm. forces:

$$\begin{aligned}\sigma_{n-p}&=0.129\times10^{-24}\text{ cm}^2, \quad 100\text{ Mev},\\ \sigma_{n-p}&=0.090\times10^{-24}\text{ cm}^2, \quad 200\text{ Mev}.\end{aligned} \quad (30)$$

Experiments by E. M. McMillan and coworkers[13] with 90-Mev neutrons give

$$(0.083\pm0.004)\times10^{-24}\text{ cm}^2$$

for the total cross section, in serious disagreement with the theory. Judging from the Born approximation, the discrepancy should be even larger for the pure exchange forces. Furthermore, the measured angular distribution is qualitatively very much in disagreement with the symmetrical theory. Experimentally the ratio

$$\sigma(180°)/\sigma(90°)$$

is about 3, whereas the curve of Fig. 5a shows about 10. Although the angular distribution has not been measured to very small angles in the center of mass system for the neutrons (angles near 90° in the laboratory for the recoil protons)

[11] W. Rarita and J. Schwinger, Phys. Rev. 59, 436 (1941), Eqs. (10) and (11).

[12] Only the phases $\delta_J^{J\pm1}$ enter. This is because $(SL=JJm|SJ0m)$ vanishes for $m=0$ as can be seen from (21).

[13] L. J. Cook, E. M. McMillan, J. M. Peterson, and D. C. Sewell, Phys. Rev. 72, 1264 (1947).

TABLE III. Phase shifts (in radians) for the neutron-proton scattering at 100 Mev with the symmetrical force model.

	Coupled triplet phases	Exact	Born		Exact	Born
$J=0$	K_1^{00}	0.3022	0.2920			
	ζ_1^{00}	0	0			
$J=1$	K_0^{10}	0.5743	0.9137	K_0^{11}	0.4191	0.1976
	ζ_0^{10}	0.0771	0	ζ_0^{11}	0.0498	0
	K_2^{10}	0.1047	0.1538	K_2^{11}	−0.2602	−0.5624
	ζ_2^{10}	0.0500	0	ζ_2^{11}	−0.0773	0
$J=2$	K_1^{20}	−0.1553	−0.1860	K_1^{21}	−0.0064	−0.0358
	ζ_1^{20}	−0.0087	0	ζ_1^{21}	0.0059	0
	K_3^{20}	−0.0578	−0.0578	K_3^{21}	0.0911	0.0923
	ζ_3^{20}	0.0059	0	ζ_3^{21}	−0.0087	0
$J=3$	K_2^{30}	0.2413	0.1416	K_2^{31}	0.1292	0.0443
	ζ_2^{30}	−0.0122	0	ζ_2^{31}	0.0095	0
	K_4^{30}	0.0515	0.0413	K_4^{31}	−0.0610	−0.0559
	ζ_4^{30}	0.0095	0	ζ_4^{31}	−0.0122	0
$J=4$	K_3^{40}	−0.0055	−0.0068	K_3^{41}	−0.0016	−0.0029
	ζ_3^{40}	0	0	ζ_3^{41}	0	0
	K_5^{40}	−0.0017	−0.0018	K_5^{41}	0.0022	0.0022
	ζ_5^{40}	0	0	ζ_5^{41}	0	0
$J=5$	K_4^{50}	0.0026	0.0018	K_4^{51}	0.0016	0.0010
	ζ_4^{50}	0	0	ζ_4^{51}	0	0
	K_6^{50}	0.0004	0.0003	K_6^{51}	−0.0006	−0.0004
	ζ_6^{50}	0	0	ζ_6^{51}	0	0

Uncoupled triplet phases				Singlet phases		
J	$\delta_J{}^{J1}$	Exact	Born	L	Exact	Born
1	δ_1^{11}	−0.3214	−0.3546	0	0.3193	0.3740
2	δ_2^{21}	0.6777	0.3937	1	−0.7438	−1.0724
3	δ_3^{31}	−0.0219	−0.0241	2	0.1577	0.1323
4	δ_4^{41}	0.0096	0.0079	3	−0.0562	−0.0730
5	δ_5^{51}	−0.0002	−0.0002	4	0.0028	0.0027
				5	−0.0007	−0.0006

the indication is that the cross section is symmetrical about 90°. This would imply a potential of the type

$$V=\tfrac{1}{2}(1+P_M)J(\mathbf{r},\,\boldsymbol{\sigma}_1,\,\boldsymbol{\sigma}_2),$$

in which ordinary and exchange forces enter with equal weight. Investigation of the cross section with such a potential, taking $(1/r)\exp(-r/a)$ for the spatial dependence, is being carried on by Serber.

Relativistic corrections to the cross section would be expected to play a role at energies comparable with the rest energy of the meson. Since it is the energy in the center of mass system which is important for this consideration it is unlikely that at 100 Mev in the laboratory any relativistic effects could account for the discrepancy between the theory and the experiment. An estimate of the corrections to the total cross section can be obtained from the formulas of Snyder and Marshak[14] which are valid in the Born approximation. At 100 Mev the differences are only 5–10 percent but they increase rapidly with increasing energy

[14] H. Snyder and R. E. Marshak, Phys. Rev. **72**, 1253 (1947).

From the curves of Fig. 5 and Fig. 6 we can estimate the reliability of the Born approximation used in Section I. The weakness of the approximation is especially evident from Fig. 5b, showing the singlet scattering cross section at 100 Mev. In the Born approximation, the total singlet cross section is 0.39×10^{-24} cm², while the exact calculation gives 0.17×10^{-24} cm². Such large differences have already been noticed by Camac and Bethe.[1] The primary reason for the discrepancy is the failure of the approximation for the $L=1$ phase shift. Comparison of the exact phase shifts and the Born phase shifts as given in Table III shows that this is the only serious difference for the singlet scattering. This may be attributed to the fact that for odd singlet states with the symmetrical model the potential is especially large, $-3\,{}^1V_{\text{even}}$. For the triplet phases the differences between the exact calculation and the Born approximation are much more erratic but the resulting cross sections, as shown in Fig. 5c, agree somewhat better than for the singlet case. Presumably, this is due to accidental cancellation of errors. We must conclude that the Born approximation is untrustworthy for energies as low as 100 Mev. At 200 Mev, however, the situation is much improved as is shown by the curves in Fig. 6. Both the singlet and triplet parts of the cross section are also in much better agreement with the approximation.

The triplet phases in the Born approximation are obtained from a generalization of the well-known formula (18) valid for central forces. Since this formula may be of some value for estimating phases at energies high enough where the Born approximation is valid it is included in the Appendix. Table III gives the real and imaginary parts of the phases using the notation

$$\delta_L{}^{Jm_s}=K_L{}^{Jm_s}+i\zeta_L{}^{Jm_s}.$$

A check on the calculations can be made by seeing if the real and imaginary parts of the phases obey the relations (42), (43), and (44) of Rarita and Schwinger:[15]

$$K_{J-1}{}^{J0}-K_{J+1}{}^{J0}=K_{J-1}{}^{J1}-K_{J+1}{}^{J1},$$

$$\zeta_{J-1}{}^{J0}=\zeta_{J+1}{}^{J1},\quad \zeta_{J-1}{}^{J1}=\zeta_{J+1}{}^{J0},$$

$$\exp(2i\delta_{J-1}{}^{J0})-\exp(2i\delta_{J+1}{}^{J0})=\exp(2i\delta_{J-1}{}^{J1})-\exp(2i\delta_{J+1}{}^{J1}).$$

[15] W. Rarita and J. Schwinger, Phys. Rev. **59**, 436 (1941), especially page 445.

In conclusion we should like to thank Professor R. E. Marshak and Professor G. E. Uhlenbeck for interesting discussions.

APPENDIX

For central forces the phase shifts for different angular momenta L are given in the Born approximation by the formula (18) as obtained in Mott and Massey, page 28. Applying the same method to the Eqs. (25), starting from the unperturbed solution

$$u_L^{Jm_s}=C_L^{Jm_s}g_L(kr),$$

given by (28), we find an expression for the phase shift, correct to the first order in the potential:

$$\delta_L^{Jm_s}=-\frac{M}{k\hbar^2}\frac{\sum_{L'=J-1}^{J+1}[2(2L'+1)]^{\frac{1}{2}}i^{L'-L}(SL'Jm_s|SL'0m_s)\int_0^\infty g_L(kr)g_{L'}(kr)V_{LL'}{}^J(r)dr}{[2(2L+1)]^{\frac{1}{2}}(SLJm_s|SL0m_s)}$$

In spite of the factor $i^{L'-L}$ in the summation, these phase shifts are real since this factor is real for $L'-L$ equal to 0 or 2 and $V_{LL'}{}^J$ is equal to zero if L and L' differ by unity. In the higher approximation the phase shifts are in general complex. It is not very difficult to show that if in the rigorous expressions (23) for the $s_{m_s'm_s}(\theta,\varphi)$ one replaces $\exp(2i\delta_L^{Jm_s})-1$ by $2i\delta_L^{Jm_s}$ with $\delta_L^{Jm_s}$ given by the approximate formula above, the Born approximation of Section I results. This is completely analogous to the results for central forces.

As an example, for the symmetrical force model we find, using (26), (11), (12b), and (21):

Triplet phases

L even:

$$\delta_L^{L-1,1}=\frac{M}{k\hbar^2}\left[\left(1-\gamma\frac{2L+2}{2L-1}\right)\int_0^\infty g_L^2J(r)dr-\gamma\frac{6L}{2L-1}\int_0^\infty g_Lg_{L-2}J(r)dr\right],$$

$$\delta_L^{L,1}=\frac{M}{k\hbar^2}\left[(1+2\gamma)\int_0^\infty g_L^2J(r)dr\right],$$

$$\delta_L^{L+1,1}=\frac{M}{k\hbar^2}\left[\left(1-\gamma\frac{2L}{2L+3}\right)\int_0^\infty g_L^2J(r)dr-\gamma\frac{6(L+1)}{2L+3}\int_0^\infty g_Lg_{L+2}J(r)dr\right],$$

$$\delta_L^{L-1,0}=\frac{M}{k\hbar^2}\left[\left(1-\gamma\frac{2L+2}{2L-1}\right)\int_0^\infty g_L^2J(r)dr+\gamma\frac{6(L-1)}{2L-1}\int_0^\infty g_Lg_{L-2}J(r)dr\right],$$

$$\delta_L^{L+1,0}=\frac{M}{k\hbar^2}\left[\left(1-\gamma\frac{2L}{2L+3}\right)\int_0^\infty g_L^2J(r)dr+\gamma\frac{6(L+2)}{2L+3}\int_0^\infty g_Lg_{L+2}J(r)dr\right].$$

L odd: Phases are $-\frac{1}{3}$ times those for even L.

Singlet phases

L even:

$$\delta_L=\frac{M}{k\hbar^2}(1-2g)\int_0^\infty g_L^2J(r)dr,$$

L odd: Phases are -3 times those for even L.