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Note on Second Born Approximation and Proton-Neutron and Proton-Proton Scattering

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The scattered amplitude $f(\vartheta)$ is regarded as a series in powers of the interaction potential V, and the second Born approximation consists in calculating the scattered intensity up to the fourth power of V. The calculation can be carried out analytically for a Gaussian potential, leading to a simple way of calculating the second approximation for the phases. The result is applied to the proton-neutron and proton-proton scattering at 100 Mev, on the basis of the three forms of nucleon interaction suggested by Rarita and Schwinger, with a Gaussian $V(r)$ and without tensor forces. The result for the symmetrical theory is: $\sigma_{p-n}=0.96\times 10^{-25}$ cm^2, $\sigma_{p-p}=0.22\times 10^{-25}$ cm^2, as compared with the first approximation values 1.40×10^{-25} and 0.28×10^{-25} cm^2, respectively. These give for $(\sigma_{p-n}+\sigma_{p-p})$ the value 1.18, in much closer agreement with the observed value 1.17 for σ_{p-d} than the other two forms of interaction potential.

I. INTRODUCTION

FOR the scattering by a central field, the scattered amplitude is given by the Faxen-Holtzmark formula, namely,

$$f(\vartheta)=\frac{1}{2ik}\sum(2l+1)(e^{2i\delta_l}-1)P_l(\cos\vartheta)$$
$$=\frac{1}{k}\sum(2l+1)[\delta_l-\tfrac{2}{3}\delta_l{}^3+\cdots+i(\delta_l{}^2-\tfrac{1}{3}\delta_l{}^4\cdots)]P_l(\cos\vartheta). \quad (1)$$

The usual first Born approximation consists in (i) dropping off all terms except δ_l, and (ii) replacing δ_l by its first approximation

$$\delta_l{}^{(1)}=-\frac{\pi M}{2\hbar^2}\int_0^\infty V(r)J_{l+\frac{1}{2}}{}^2(kr)rdr, \quad (2)$$

where M is the proton or neutron mass for proton-neutron collisions.

Recent calculations on the cross section of proton-neutron scattering at high energies show that the first Born approximation is unsatisfactory at 100 Mev.[1] As the exact calculation of the phases δ_l by numerical integration of the wave equation is lengthy for any potential other than the rectangular hole one, it is of some interest to have better approximate methods than the usual Born approximation.

[1] J. Ashkin and T. Y. Wu, Phys. Rev. **73**, 973 (1948); M. Camac and H. A. Bethe, Phys. Rev. **73**, 191 (1948).

One usual method of improving on the Born approximation is to use $\delta_l{}^{(1)}$ in (1), so as to take into account the higher powers of δ_l. This procedure will be justified if the difference $\delta_l{}^{\text{exact}}-\delta_l{}^{(1)}$ is very small compared with $(\delta_l{}^{(1)})^2$. This is, however, in general not the case for low values of l, so that this procedure does not form any consistent approximation in the sense of the perturbation theory.

It has been shown[2] that the phase δ_l can be developed as a series in powers of the interaction potential $V(r)$, namely,

$$\delta_l=\delta_l{}^{(1)}+\delta_l{}^{(2)}+\cdots, \quad (3)$$

where $\delta_l{}^{(1)}$ is given by (2) and

$$\delta_l{}^{(2)}=(-1)^l\left(\frac{M}{2\pi\hbar^2}\right)^2\int_0^\infty V(r)J_{l+\frac{1}{2}}{}^2(kr)rdr\times\int_0^\infty V(r)J_{l+\frac{1}{2}}(kr)J_{-l-\frac{1}{2}}(kr)rdr$$
$$+(-1)^{l+1}\left(\frac{M}{2\pi\hbar^2}\right)^2\int_0^\infty V(r)J_{l+\frac{1}{2}}(kr)rdr\times\int_0^r[J_{l+\frac{1}{2}}(kr)J_{-l-\frac{1}{2}}(k\xi)-J_{-l-\frac{1}{2}}(kr)\times J_{l+\frac{1}{2}}(k\xi)]J_{-l-\frac{1}{2}}(k\xi)V(\xi)\xi d\xi. \quad (4)$$

Calculation of $\delta_l{}^{(2)}$ by means of this expression is very lengthy, if not difficult.

[2] E. J. Hellund, Phys. Rev. **59**, 395 (1941).

Another approximate formula for δ_l has been found by Pais[3] by the variational method. The expression is

$$\frac{2l+1-(2\delta_l/\pi)}{2l+1-(4\delta_l/\pi)}\delta_l = -\frac{\pi M}{2\hbar^2}\int_0^\infty V(r)J^2_{l+\frac{1}{2}-(2\delta_l/\pi)}(kr)rdr. \quad (5)$$

If δ_l is small, this can be put in the form (3), with $\delta_l^{(1)}$ given by (2) and

$$\delta_l^{(2)} = \frac{2}{\pi}\left(a_l - \frac{\delta_l^{(1)}}{2l+1}\right)\delta_l^{(1)}, \quad (6)$$

where

$$a_l = -\left(\frac{\partial \delta_p^{(1)}}{\partial p}\right)_{p=l+\frac{1}{2}} \quad (6a)$$

When δ_l is small, calculation of $\delta_l^{(2)}$ according to (6) is easy. Unfortunately, Pais' method is not valid for low values of l, and $l=0$ in particular, and recourse must be made to numerical solution of the wave equation.

In the present note, we shall obtain the second Born approximation and apply the result to proton-neutron and proton-proton scatterings at 100 Mev.

II. SECOND BORN APPROXIMATION

We shall regard $f(\vartheta)$ as a series in powers of the interaction potential $V(r)$. The scattered intensity $I(\vartheta)$ is, up to the fourth power of $V(r)$,

$$I(\vartheta) = |f^{(1)}(\vartheta)|^2 + |f^{(2)}(\vartheta)|^2 + 2f^{(1)}(\vartheta)f^{(3)}(\vartheta), \quad (7)$$

where

$$f^{(1)}(\vartheta) = \frac{1}{k}\sum(2l+1)\delta_l P_l(\cos\vartheta), \quad (8a)$$

$$f^{(2)}(\vartheta) = \frac{i}{k}\sum(2l+1)\delta_l^2 P_l(\cos\vartheta), \quad (8b)$$

$$f^{(3)}(\vartheta) = -\frac{2}{3k}\sum(2l+1)\delta_l^3 P_l(\cos\vartheta). \quad (8c)$$

In the second and the third term in (7), it is sufficient to employ $\delta_l^{(1)}$ in (8), but in the first term it is necessary to employ[4]

$$\delta_l = \delta_l^{(1)} + \delta_l^{(2)}. \quad (9)$$

Instead of summing (8a) with $\delta_l = \delta_l^{(1)} + \delta_l^{(2)}$ and $\delta_l^{(2)}$ given by (4) or a similar expression obtained by the perturbation method, it is found convenient to calculate the first two terms in (7) together, namely,

$$|f^{(1)}(\vartheta)|^2 + |f^{(2)}(\vartheta)|^2 = |f^{(1)}(\vartheta) + f^{(2)}(\vartheta)|^2.$$

Let ψ be the solution of

$$\Delta\psi + \left[k^2 - \frac{M}{\hbar^2}V(r)\right]\psi = 0,$$

and let

$$\psi = \psi_{(0)} + \psi_{(1)} + \psi_{(2)} + \cdots,$$

where $\psi_{(0)}$ is the solution of the equation

$$\Delta\psi + k^2\psi = 0$$

and represents the incident wave. By successive approximation, one obtains

$$[\Delta + k^2]\psi_{(1)} = \frac{M}{\hbar^2}V(r)\psi_{(0)},$$

$$[\Delta + k^2]\psi_{(2)} = \frac{M}{\hbar^2}V(r)\psi_{(1)},$$

and their solutions

$$\psi_{(1)}(\mathbf{r}'') = -\frac{1}{4\pi}\int e^{ik|\mathbf{r}''-\mathbf{r}'|}\frac{1}{|\mathbf{r}''-\mathbf{r}'|}\cdot\frac{M}{\hbar^2} \times V(r')e^{i\mathbf{k}\cdot\mathbf{r}'}d\mathbf{r}',$$

$$\psi_{(2)}(\mathbf{r}) = -\frac{1}{4\pi}\int e^{ik|\mathbf{r}-\mathbf{r}''|}\frac{1}{|\mathbf{r}-\mathbf{r}''|}\cdot\frac{M}{\hbar^2} \times V(r'')\psi_{(1)}(\mathbf{r}'')d\mathbf{r}''.$$

From the asymptotic solutions $\psi_{(1)}(r)$, $\psi_{(2)}(r)$ for large r, one obtains the scattered amplitude up

[3] A. Pais, Proc. Camb. Phil. Soc. **42**, 45 (1946).

[4] Strictly, one should have included in (9) the next $\delta_l^{(3)}$ which is cubic in V. Including it, however, makes any calculation too difficult. We shall therefore neglect it by assuming the condition (18) below to be valid.

to the second power of V,

$$f(\vartheta)=f_{(1)}(\vartheta)+f_{(2)}(\vartheta)$$
$$=-\frac{M}{4\pi\hbar^2}\int e^{-i\mathbf{k}'\cdot\mathbf{r}'}V(\mathbf{r}')e^{i\mathbf{k}\cdot\mathbf{r}'}d\mathbf{r}'$$
$$+\left(\frac{M}{4\pi\hbar^2}\right)^2\int\int e^{-i\mathbf{k}'\cdot\mathbf{r}''}V(r'')\frac{e^{ik|\mathbf{r}''-\mathbf{r}'|}}{|\mathbf{r}''-\mathbf{r}'|}\times V(r')e^{i\mathbf{k}\cdot\mathbf{r}'}d\mathbf{r}'d\mathbf{r}''. \quad (10)$$

Now it is seen that

$$f_{(1)}(\vartheta)+f_{(2)}(\vartheta)=f^{(1)}(\vartheta)+f^{(2)}(\vartheta). \quad (11)$$

Since $f_{(1)}(\vartheta)$, the first term in (10), is known to be the sum (8a) with δ_l replaced by $\delta_l^{(1)}$, it follows from (11) that the real and the imaginary part of $f_{(2)}(\vartheta)$ in (10) must be, respectively,

real part of
$$f_{(2)}(\vartheta)=\frac{1}{k}\sum(2l+1)\delta_l^{(2)}P_l(\cos\vartheta), \quad (12a)$$
imag. part of
$$f_{(2)}(\vartheta)=\frac{i}{k}\sum(2l+1)(\delta_l^{(1)})^2P_l(\cos\vartheta). \quad (12b)$$

To calculate the last term $f^{(1)}(\vartheta)f^{(3)}(\vartheta)$ in (7), one would have to sum (8c) with δ_l replaced by $\delta_l^{(1)}$. This series cannot be summed in a simple manner as (12b). However, as the δ_0, δ_1, δ_2, $\cdots$ must converge, we may approximate $f^{(3)}(\vartheta)$ by taking the first few terms, say three, in (8c). Up to the approximation desired, the $f^{(1)}(\vartheta)$ in the last term in (7) is given by the usual Born's expression.

To enable $f_{(2)}(\vartheta)$ to be evaluated in analytical form and incidentally to demonstrate the relation (12b), we shall take a Gaussian potential

$$V(r)=V_0\exp(-\alpha^2r^2). \quad (13)$$

The integration in (10) can be effected by a transformation from $\mathbf{r}'$, $\mathbf{r}''$ to $\boldsymbol{\xi}=\mathbf{r}''-\mathbf{r}'$ and $\boldsymbol{\eta}=\mathbf{r}''+\mathbf{r}'$. It can be shown that

$$f_{(2)}(\vartheta)=\frac{(\pi)^{\frac{1}{2}}}{\alpha}\left(\frac{MV_0}{4\alpha^2\hbar^2}\right)^2\left(\frac{k}{\alpha}\cos\frac{\vartheta}{2}\right)^{-1}\times\exp\left(-\frac{k^2}{2\alpha^2}\sin^2\frac{\vartheta}{2}\right)$$
$$\times\Big[x\exp(-x^2)\,{}_1F_1(\tfrac{1}{2};\tfrac{3}{2};x^2)-y\exp(-y^2)\,{}_1F_1(\tfrac{1}{2};\tfrac{3}{2};y^2)+\frac{i(\pi)^{\frac{1}{2}}}{2}(\exp(-y^2)-\exp(-x^2))\Big], \quad (14)$$

where
$$x^2=\frac{2k^2}{\alpha^2}\cos^4\frac{\vartheta}{4},\quad y^2=\frac{2k^2}{\alpha^2}\sin^4\frac{\vartheta}{4},$$

and ${}_1F_1$ is the usual confluent hypergeometric function which for numerical work can be more conveniently expressed in the form

$$x\,{}_1F_1(\tfrac{1}{2};\tfrac{3}{2};x^2)=\int_0^x\exp(t^2)dt.$$

The relation (12b) in this case can then be proved by means of Weber's second exponential integral, together with the addition theorem of Bessel functions.[5]

For Gaussian potential,

$$f_{(1)}(\vartheta)=\frac{(\pi)^{\frac{1}{2}}}{2}\left(\frac{MV_0}{4\alpha^2\hbar^2}\right)\times\exp\begin{Bmatrix}-\dfrac{k^2}{\alpha^2}\sin^2\dfrac{\vartheta}{2}\\[2ex] -\dfrac{k^2}{\alpha^2}\cos^2\dfrac{\vartheta}{2}\end{Bmatrix} \quad (15)$$

for ordinary and Majorana exchange force, respectively. For ordinary force, one readily finds

$$2\pi\int_0^\pi 2f^{(1)}(\vartheta)f^{(3)}(\vartheta)\sin\vartheta d\vartheta$$
$$=-\frac{16\pi^{\frac{3}{2}}}{3\alpha^2}\left(\frac{\alpha}{k}\right)^3\left(\frac{MV_0}{4\alpha^2\hbar^2}\right)\Bigg[\left(\delta_0^3-\frac{5}{2}\delta_2^3\cdots\right)\left(1-\exp\left(-\frac{k^2}{\alpha^2}\right)\right)$$
$$+3(\delta_1^3\cdots)\left\{1-\frac{2\alpha^2}{k^2}+\left(1+\frac{2\alpha^2}{k^2}\right)\exp\left(-\frac{k^2}{\alpha^2}\right)\right\}+\frac{15}{2}(\delta_2^3\cdots)$$
$$\times\left\{1-\frac{4\alpha^2}{k^2}+\frac{8\alpha^4}{k^4}-\left(1+\frac{4\alpha^2}{k^2}+\frac{8\alpha^4}{k^4}\right)\exp\left(-\frac{k^2}{\alpha^2}\right)\right\}+\cdots\Bigg]. \quad (16)$$

[5] G. N. Watson, *Theory of Bessel Functions* (The Macmillan Company, New York, 1944), §§13.31, 11.41.

TABLE I. Phases in proton-neutron and proton-proton scattering. The $\delta_l^{(1)}$ and $\delta_l^{(2)}$ in the table are, for 100 Mev, those given in (22).

	Neutral $\delta_l^{(1)}$	Neutral $\delta_l^{(2)}$	Charged $\delta_l^{(1)}$	Charged $\delta_l^{(2)}$	Symmetrical $\delta_l^{(1)}$	Symmetrical $\delta_l^{(2)}$	
p-n, triplet	$\delta_l^{(1)}$	$\delta_l^{(2)}$	$(-1)^l\delta_l^{(1)}$	$\delta_l^{(2)}$	$\delta_l^{(1)}$ $-\frac{1}{3}\delta_l^{(1)}$	$\delta_l^{(2)}$ $1/9\delta_l^{(2)}$	for l even for l odd
singlet	$(1-2g)\delta_l^{(1)}$	$(1-2g)^2\delta_l^{(2)}$	$(-1)^l(1-2g)\delta_l^{(1)}$	$(1-2g)^2\delta_l^{(2)}$	$(1-2g)\delta_l^{(1)}$ $-3(1-2g)\delta_l^{(1)}$	$(1-2g)^2\delta_l^{(2)}$ $9(1-2g)^2\delta_l^{(2)}$	for l even for l odd
p-p, triplet odd	$2\delta_l^{(1)}$	$4\delta_l^{(2)}$	$-2\delta_l^{(1)}$	$4\delta_l^{(2)}$	$-\frac{2}{3}\delta_l^{(1)}$	$4/9\delta_l^{(2)}$	
singlet even	$2(1-2g)\delta_l^{(1)}$	$4(1-2g)^2\delta_l^{(2)}$	$2(1-2g)\delta_l^{(1)}$	$4(1-2g)^2\delta_l^{(2)}$	$2(1-2g)\delta_l^{(1)}$	$4(1-2g)^2\delta_l^{(2)}$	

For the case of exchange force, one employs the lower expression in (15) and remembers that the $\delta_l^{(1)}$ with odd l change their sign. Hence

$$|f^{(1)}(\vartheta)f^{(3)}(\vartheta)|_{\text{exchange}} = |f^{(1)}(\pi-\vartheta)f^{(3)}(\pi-\vartheta)|_{\text{ordinary}}, \quad (17)$$

and the contribution to the total cross section from the last term in (7) is again given by (16).

For interaction potentials which are mixtures of the ordinary and exchange force, the $f^{(1)}(\vartheta)$, $f^{(2)}(\vartheta)$, $f^{(3)}(\vartheta)$ can be found in a similar manner, as will be illustrated below in III.

A better approximation than the second can be obtained by substituting (9), namely, $\delta_l=\delta_l^{(1)}+\delta_l^{(2)}$, into (1) so as to include the contribution of all powers of δ_l higher than the fourth. While this procedure again does not form any definite approximation in the sense of the perturbation theory, it may be justified and expected to be very good if

$$\delta_l^{\text{exact}}-(\delta_l^{(1)}+\delta_l^{(2)})\ll(\delta_l^{(1)})^3. \quad (18)$$

To obtain the $\delta_l^{(2)}$, one may either calculate them by the perturbation theory, by (4), by (6), or, in the case of the Gaussian potential, make use of the following procedure which is very much shorter. From (12a) and (14), one has

$$\sum(2l+1)\delta_l^{(2)}P_l(\cos\vartheta) = (\pi)^{\frac{1}{2}}\left(\frac{MV_0}{4\alpha^2\hbar^2}\right)^2\frac{1}{\cos(\vartheta/2)}\exp\left(-\frac{k^2}{2\alpha^2}\sin^2\frac{\vartheta}{2}\right) \times[x\exp(-x^2)\,{}_1F_1(\tfrac{1}{2};\tfrac{3}{2};x^2) - y\exp(-y^2)\,{}_1F_1(\tfrac{1}{2};\tfrac{3}{2};y^2)].$$

By taking n different values of ϑ, one can calculate the first n $\delta_l^{(2)}$'s which are not negligible.

III. PROTON-NEUTRON AND PROTON-PROTON SCATTERING

We shall apply the above result to the problem of proton-neutron and proton-proton scattering at 100 Mev on the basis of the three forms of nucleon interaction suggested by Rarita and Schwinger[6] with Gaussian dependence on r and without tensor force, namely,

"Neutral"

$$V(r)=-[1-g+\tfrac{1}{2}g(1+\sigma_1\cdot\sigma_2)]V_0\exp(-\alpha^2r^2),$$

"Charged" (19)

$$V(r)=\tfrac{1}{2}(1+\tau_1\cdot\tau_2)[1-g+\tfrac{1}{2}g(1+\sigma_1\cdot\sigma_2)]V_0\exp(-\alpha^2r^2),$$

"Symmetrical"

$$V(r)=\tfrac{1}{3}\tau_1\cdot\tau_2[1-g+\tfrac{1}{2}g(1+\sigma_1\cdot\sigma_2)]V_0\exp(-\alpha^2r^2).$$

We have chosen the following constants

$$V_0=45 \text{ Mev},\quad (1-2g)V_0=26 \text{ Mev},\quad \alpha^2=0.266\times10^{26}\text{ cm}^{-2} \quad (20)$$

to fit the data on the ground state of the deuteron and the proton-proton scattering at low energies.

The scattered amplitudes $f^{(1)}(\vartheta)$, $f^{(2)}(\vartheta)$, $f^{(3)}(\vartheta)$ in the case of the "N" and the "C" theory can be readily obtained as explained in II. For the symmetrical theory, it can be shown that for the triplet state scattering,

$$\begin{aligned}&{}^3V(r)=-\tfrac{1}{3}(1+2P_{12})V_0\exp(-\alpha^2r^2),\\ &f_{(1)}{}^t(\vartheta)=\tfrac{1}{3}[f_{(1)}(\vartheta)+2f_{(1)}(\pi-\vartheta)],\\ &f_{(2)}{}^t(\vartheta)=\tfrac{1}{9}[5f_{(2)}(\vartheta)+4f_{(2)}(\pi-\vartheta)],\end{aligned} \quad (21a)$$

[6] W. Rarita and J. Schwinger, Phys. Rev. 59, 557 (1941).

and for the singlet state scattering,

$$
\begin{aligned}
{}^1V(r) &= -(1-2g)(-1+2P_{12})V_0\exp(-\alpha^2r^2),\\
f_{(1)}{}^s(\vartheta) &= (1-2g)[-f_{(1)}(\vartheta)+2f_{(1)}(\pi-\vartheta)], \quad (21b)\\
f_{(2)}{}^s(\vartheta) &= (1-2g)^2[5f_{(2)}(\vartheta)-4f_{(2)}(\pi-\vartheta)],
\end{aligned}
$$

where $f_{(2)}(\vartheta)$ is given by (14), and $f_{(1)}(\vartheta)$, $f_{(1)}(\pi-\vartheta)$ are given by the upper and lower expression in (15), respectively. For $f^{(3)}(\vartheta)$, the appropriate phases δ_l are given in Table I.

For the potential (13) with constants as given in (20), we obtain[7]

l	0	1	2	3	4
$\delta_l^{(1)}$	0.833	0.487	0.197	0.0605	0.0127
$\delta_l^{(2)}$	−0.0793	0.068	0.0325	0.00477	0.00211.

(22)

Remembering that to the approximation (3), the $\delta_l^{(1)}$ are linear and the $\delta_l^{(2)}$ quadratic in $V(r)$, one readily obtains the phases for the triplet and singlet state scatterings for the three potentials in (19), as shown in Table I.

The result of the calculation is given in Tables II, III, and IV. Table II gives the differential cross section $2\pi I(\vartheta)$ in the first and the second Born approximation. An important feature of the second approximation is the appearance of a maximum in the forward direction in the case of the exchange force, recently noted by Ashkin and Wu.[1]

Table III gives the total proton-neutron cross section obtained in the first Born approximation, by using $\delta_l^{(1)}$ in (7), and in (1), in the Second Born approximation, and by using $\delta_l^{(1)}+\delta_l^{(2)}$ in (1). Table IV gives the corresponding values for proton-proton scattering.

TABLE II. Differential cross section $2\pi I(\vartheta)$ in 10^{-25} cm² of proton-neutron scattering at 100 Mev.

	Neutral		Charged		Symmetrical	
ϑ	1st Born	2nd Born	1st Born	2nd Born	1st Born	2nd Born
0	6.41	6.45	0.00064	1.066	1.280	1.234
15	5.46	5.51	0.00071	1.035	1.085	1.072
30	3.46	3.41	0.0012	0.849	0.692	0.759
45	1.61	1.58	0.0024	0.614	0.303	0.438
60	0.641	0.530	0.0061	0.388	0.135	0.263
90	0.064	0.052	0.064	0.054	0.062	0.057
120	0.0061	0.018	0.641	0.020	0.51	0.117
135	0.0024	0.0140	1.61	0.51	1.28	0.515
150	0.0012	0.0137	3.46	1.66	2.78	1.40
165	0.00077	0.0094	5.46	3.28	4.38	2.53
180	0.00064	0.0044	6.41	4.12	5.12	3.27

TABLE III. Total cross section in 10^{-25} cm² of proton-neutron scattering at 100 Mev.

	Neutral	Charged	Symmetrical		
	Gaussian	Gaussian	Gauss.	Yukawa	Rect. hole
1st Born	1.40	1.40	1.40	1.39	1.40
Using $\delta_l^{(1)}$ in (7)	1.33	1.25	1.15		
Using $\delta_l^{(1)}$ in (1)	1.22	1.22	1.14		1.20
2nd Born	1.34	1.02	0.83		
Using $\delta_l^{(1)}+\delta_l^{(2)}$ in (1)	1.35	1.07	0.966	(exact)	0.94

TABLE IV. Twice the total cross section in 10^{-25} cm² of proton-proton scattering at 100 Mev.

	Neutral	Charged	Symmetrical		
	Gaussian	Gaussian	Gauss.	Yukawa	Rect. hole
1st Born	2.69	2.69	0.56	0.57	0.56
Using $\delta_l^{(1)}$ in (1)	1.83	1.83	0.47		
2nd Born	3.08	0.72	0.41		
Using $\delta_l^{(1)}+\delta_l^{(2)}$ in (1)	2.39	1.24	0.43		

As the exact values of the cross sections cannot be obtained without lengthy numerical integration of the wave equation, it is not possible to make a comparison between these approximate and the exact σ's. From the values of the $\delta_l^{(1)}$ and $\delta_l^{(2)}$ in (22), one may perhaps expect the $\delta_l^{(3)}$ to be small so that (18) is satisfied. If this is the case, the value obtained by using $\delta_l^{(1)}+\delta_l^{(2)}$ in (1) should be rather close to the exact values. One can then see the improvement achieved by the Second Born approximation.

The above calculation cannot be carried out so simply for potentials other than the Gaussian one. It seems, however, that with other forms of $V(r)$ one may still estimate the correction to the first Born approximation by replacing the field $V(r)$ by an appropriate Gaussian one. The angular distribution of the scattered intensity certainly depends on the shape of the potential $V(r)$; but it seems that the total cross section is, for a given proportion of ordinary and exchange force, rather insensitive to the exact form of $V(r)$. To illustrate this, we have calculated the cross sections for the potential on the Møller-Rosenfeld meson theory, with constants deter-

[7] It is of interest to compare the phases (22) with those calculated by the method of Pais. They are as follows: from (5), $\delta_1=0.534$, $\delta_2=0.221$; from (6), $\delta_1^{(2)}=0.063$, $\delta_2^{(2)}=0.026$. For $l=0$, Pais' method is not valid, but calculation with (6) gives $\delta_0^{(2)}=-0.28$, with the sign correct.

mined by the variational method,[8] namely,

$$^3V^{\text{even}} = -3(A+B)V_0\frac{e^{-\lambda r}}{r},$$

$$^3V^{\text{odd}} = (A+B)V_0\frac{e^{-\lambda r}}{r},$$

$$^1V^{\text{even}} = (A-B)V_0\frac{e^{-\lambda r}}{r},$$

$$^1V^{\text{odd}} = -3(A-B)V_0\frac{e^{-\lambda r}}{r}, \tag{23}$$

where

$$A=-1.303, \quad B=4.606,$$
$$V=0.977\times 10^{-18}, \quad \lambda=5.655\times 10^{12}.$$

We have also calculated the cross sections on the symmetrical theory in (19), replacing the Gaussian potential by a rectanglar hole with the following constants

$$\text{range } \rho=2.80\times 10^{-13}\text{ cm}, \quad V_0=21\text{ Mev},$$
$$(1-2g)V_0=11.7\text{ Mev}. \tag{24}$$

For this potential we have also calculated the exact σ for proton-neutron scattering. The result is given in Tables III and IV. A comparison of the values obtained for the Yukawa potential in (23) and the rectangular hole potential (24) with those obtained for the Gaussian potential leads one to think that a higher approximation for these potentials probably gives approximately the same values as for the Gaussian potential.

In view of the strong dependence of the total cross section on the proportion of ordinary and exchange force and comparatively weak dependence on the exact form of the radial $V(r)$, it seems of significance to compare the total cross section calculated with Gaussian potential for various mixtures of ordinary and exchange force with the experimental values. Recently Cook *et al.*[9] reported the following total cross sections for 90 Mev:

$$\sigma_{p-n}=0.83\times 10^{-25}\text{ cm}^2,$$
$$\sigma_{p-d}=1.17\times 10^{-25}\text{ cm}^2. \tag{25}$$

It is seen that the calculated σ_{p-n} on the symmetrical theory agrees better with the observed value than the other two theories. On regarding the proton-deuteron cross section as approximately the sum of the proton-neutron and proton-proton cross sections,[10] one finds for the ratio $\sigma_{p-n}/\sigma_{p-d}$ the values 0.36, 0.51, 0.69 for the neutral, charged, and the symmetrical theory, respectively, as compared with the observed value 1.17.

This agreement, however, does not establish the symmetrical theory in the form (19). The great difference between the observed value for the ratio $\sigma(\pi)/\sigma(\pi/2)\simeq 3$ for proton-neutron and the calculated value shown in Table I seems to indicate the presence of tensor force, whose effect is to increase the scattered intensity in directions $\vartheta\simeq\pi/2$ (in the center of mass system).[1] But an exact calculation on the symmetrical theory, including tensor force, still leads to a much larger value for the ratio than the observed one.[1] It seems that both the range of the force and the proportion of central and tensor forces have to be readjusted in order to agree with the meagre data now available at 90 Mev.

The writer wishes to express his indebtedness to Professor G. E. Uhlenbeck for helpful discussions.

[8] Frohlich, Huang, and Snedden, Proc. Roy. Soc. **A191**, 61 (1947).

[9] Cook, McMillan, Peterson, and Sewell, Phys. Rev. **72**, 1264 (1947).

[10] T. Y. Wu and J. Ashkin, Phys. Rev. **73**, 986 (1948).