what less than the cathode jet, which has a mean radius of curvature of about a centimeter, implying a net charge to mass ratio for the jet of about −40 e.m.u./gram.

SECTION IX

Conclusion

The existence of mercury vapor jets in sparks, which have been shown to be arcs of short duration, appears to have been proven beyond reasonable doubt. The velocity of the jets is such that neither the positive nor negative ions in the arc column can reach the electrodes since the ion velocities are considerably less than the jet velocities and are oppositely directed. From consideration of the literature of vapor velocities in both sparks and arcs, it is certain that these vapor jets occur with a wide variety of electrode materials. An obvious test of the theory of the mechanism of jet production would be to obtain the initial jet velocity as a function of the atomic weight of the electrode material. Since the program of peacetime research prevents further work in this organization, it is hoped that the study will be carried on elsewhere.

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On the Calculation of Self-Energies in Quantum Theory by Analytic Continuation

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Riesz's method of solving hyperbolic differential equations by analytic continuation has been used by Gustafson to eliminate infinities in quantum theory. Treating the one-electron case, he found finite values of the self-energy integrals in the second approximation, also for those integrals for which the λ-limiting process fails (without the further assumption of negative-energy photons). In the present paper it is shown that the general result of Gustafson's procedure is to remove all divergences normally appearing in self-energy expressions, except logarithmic divergences. Thus the total self-energy of the electron, to the second approximation, is found to be zero on the one-electron theory if calculated by this method, whereas in the hole theory the logarithmically divergent expression of Weisskopf is retained. A proposal by Pauli to alter the commutation rules in a certain way gives essentially the same results.

I. INTRODUCTION

WE are going to investigate the general effect of evaluating self-energies in quantum theory by the method used by Gustafson in the case of the second approximation.[4–8*] Before entering on the problem we set down a few notations to be employed in the following.

A point in space-time is denoted by $x=x^\nu$ ($\nu=0, 1, 2, 3$; $x^0=ct$), and a metrical tensor with $-g_{00}=g_{11}=g_{22}=g_{33}=1$ is used. The length $r(x)$ of a vector x is defined by

$$-r(\mathbf{x})^2=g_{\mu\nu}x^\mu x^\nu=x_\nu x^\nu=(\mathbf{x},\mathbf{x}),$$

so that r^2 is positive for a time-like vector. Further, we write $\Box$ for the wave operator $-\partial^2/\partial x^\nu\partial x_\nu$, and $\vec{x}$ for a vector (x^1, x^2, x^3) in ordinary space. Then $(x,y)=x_\nu y^\nu=\vec{x}\vec{y}-x^0y^0$.

Now, to form an expression for the self-energy of an electron or of a nucleon, one may start with the equations giving the interaction with the electromagnetic field and the meson field, respectively. Here we use the formulation of the theory in which the dynamic variables are operators satisfying field equations analogous to those of the classical theory. The specific state of the system is then characterized by a normalized and time-independent state vector C. A variable represented by an operator F has the expectation value $\langle F\rangle=C^*FC$.

* Numbered references will be found at the end of the text.

To fix our ideas, consider the case of quantum electrodynamics. The interaction between electrons, of mass m and charge $-e$, and the electromagnetic field is described by the equations[a]

$$\left(-i\alpha^{\nu}\frac{\partial}{\partial x^{\nu}}+\beta m\right)\psi=-eA_{\nu}\alpha^{\nu}\psi,$$
$$\Box A^{\nu}=-4\pi e s^{\nu}(=-4\pi e\psi^{*}\alpha^{\nu}\psi), \tag{1.1}$$

where the expression in brackets holds in the one-electron theory, and where the vector A and spinor ψ fulfill well-known commutation and anticommutation relations. α^{ν} and β are Dirac matrices ($\alpha^0=1$).

Using perturbation theory we expand A and ψ, and hence s, in power series in e,

$$\mathbf{A}=\mathbf{A}^{(0)}+e\mathbf{A}^{(1)}+\cdots,$$

etc. The second approximation of the self-energy of the electron is then given by

$$W=\frac{e^2}{2}\int(dx)^3\langle(\mathbf{s}^{(0)},\mathbf{A}^{(1)})+(\mathbf{s}^{(1)},\mathbf{A}^{(0)})\rangle$$
$$=(W_{\text{st}}+W_{\text{sp}})+W_{\text{fluct}}. \tag{1.2}$$

(see Weisskopf[14]). Here $A^{(1)}$ satisfies

$$\Box\mathbf{A}^{(1)}=-4\pi\mathbf{s}^{(0)}, \tag{1.3}$$

so that the solution of the inhomogeneous wave equation is of importance for the first part of W, that of the Dirac equation playing a corresponding role for the second part (in the case of a nucleon the meson equation takes the place of the wave equation). As is well known, simple insertion of these solutions into 1.2 as in the ordinary treatment gives rise to divergent integrals. In Gustafson's papers use is made of the method of solving normal hyperbolic differential equations given by Riesz[11,12] and employed by him and Fremberg[2,3] to eliminate divergences in the classical theory of point electrons.

II. RIESZ'S SOLUTION OF HYPERBOLIC EQUATIONS

A. To solve the wave equation

$$\Box u(\mathbf{x})=f(\mathbf{x}) \tag{2.1}$$

by analytic continuation, we form with Riesz[11,12] an expression u^{α} (the "α-solution") depending analytically on a parameter α. In the following we are not interested in special boundary conditions[b] and can then take u^{α} as

$$u^{\alpha}(\mathbf{x})=I^{\alpha}f(\mathbf{x})=\int_{D^{\mathbf{x}}}V^{\alpha}(\mathbf{x}-\mathbf{y})f(\mathbf{y})(dy)^4, \tag{2.2}$$

where

$$V^{\alpha}(\mathbf{x})=\frac{r(\mathbf{x})^{\alpha-4}}{H(\alpha)},\quad H(\alpha)=\pi 2^{\alpha-1}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha-2}{2}\right).$$

The integration element is $(dy)^4=dy^0dy^1dy^2dy^3$, and the domain of integration $D^{\mathbf{x}}$ is the whole interior of the retrograde light-cone $r(x-y)^2=0$ with its vertex at the point x. Under certain conditions on f the integral $I^{\alpha}f$ converges for $\alpha>2$ or in some interval $A>\alpha>2$; for other values of α it has to be defined by analytic continuation.

We can now assert that $u^{(2)}$ is a solution of the wave equation 2.1. That a solution is indeed obtained for $\alpha=2$ is seen from the relation

$$\Box V^{\alpha}=V^{\alpha-2},\quad\text{whence}\quad\Box I^{\alpha}f=I^{\alpha-2}f,$$

(which is an immediate consequence of the definitions above) together with the fact that I^0 is an identity operator,[c]

$$I^0f=f.$$

B. The solution of the "meson equation" (Klein-Gordon's equation),

$$(\Box+\kappa^2)u(\mathbf{x})=f(\mathbf{x}), \tag{2.3}$$

can be obtained in an exactly analogous way. We have only to replace the integral operator I^{α} in (2.2) by a similar operator I_{κ}^{α} in which the new kernel V_{κ}^{α} has to satisfy

$$(\Box+\kappa^2)V_{\kappa}^{\alpha}=V_{\kappa}^{\alpha-2}.$$

The further condition that $V_{\kappa}^{\alpha}\rightarrow V^{\alpha}$ as $\kappa\rightarrow 0$ now determines V^{α} completely:

$$V_{\kappa}^{\alpha}=\sum_{k=0}^{\infty}\binom{-\alpha/2}{k}\kappa^{2k}V^{\alpha+2k}=\frac{(r/\kappa)^{(\alpha/2)-2}J_{(\alpha/2)-2}(\kappa r)}{\pi 2^{(\alpha/2)+1}\Gamma(\alpha/2)},$$

where J is a Bessel function.[12,3]

We may note that, owing to the form of $H(\alpha)$, the contributions to the integral from the

[a] In natural units, $\hbar=c=1$.

[b] The solution of the general Cauchy problem in any number of dimensions is to be found in references 3 and 12.

[c] For proofs see references 1 and 3.

interior of the cone are cancelled in I^2f (Huygens's principle: retarded potentials), but not in $I_\kappa^2 f$ for $\kappa \neq 0$.

C. The Dirac equation may be written

$$(\nabla + im)\psi(\mathbf{x}) = g(\mathbf{x}), \tag{2.4}$$

where $\nabla = \gamma^\nu(\partial/\partial x^\nu)$ ($\gamma^\nu = \beta\alpha^\nu$, $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2g^{\mu\nu}$). Multiplying on the left by $\nabla - im$ we get

$$(\Box + m^2)\psi = (\nabla - im)g,$$

which is solved, as in the case of (2.3) above, by $I_m^2(\nabla - im)g$.[d]

D. As a special case, which will be of importance in the applications below, let us choose for $f(x)$ a plane wave of the form $e^{i(K,x)}$. Then f does not vanish for $x^0 = -\infty$, and the integrals $I^\alpha f$ and $I_\kappa{}^\alpha f$ over D^x will not converge for any value of α. However, for $\alpha > 2$ they are oscillating and can be defined in the usual way by first letting iK_0 have a real positive part, which is then made to tend to zero. In this way we get, assuming K to be a time-like vector and choosing a suitable set of coordinate axes,

$$\int_{D^x} r(\mathbf{x}-\mathbf{y})^{\alpha-4} \exp[iK_0 y^0](dy)^4$$
$$= \exp[iK_0 x^0] \int_0^\infty dz^0 \exp[-iK_0 z^0] \times \int_0^{z^0} 4\pi\rho^2 d\rho((z^0)^2 - \rho^2)^{(\alpha-4)/2}$$
$$= \exp[iK_0 x^0](iK_0)^{-\alpha} H(\alpha),$$

(where $z = x - y$, $|\vec{z}| = \rho$), whence generally,

$$\int_{D^x} V^\alpha(\mathbf{x}-\mathbf{y}) \exp[i(\mathbf{K},\mathbf{y})](dy)^4 = \exp[i(\mathbf{K},\mathbf{x})] r(i\mathbf{K})^{-\alpha}.$$

From the expansion of $V_\kappa{}^\alpha$ we now obtain for $I_\kappa{}^\alpha \exp[i(\mathbf{K},\mathbf{x})]$ a binomial series giving

$$\int_{D^x} V_\kappa{}^\alpha(x-y) \exp[i(\mathbf{K},\mathbf{y})](dy)^4 = \frac{\exp[i(\mathbf{K},\mathbf{x})]}{(K_\nu K^\nu + \kappa^2)^{\alpha/2}}, \tag{2.5}$$

which holds, provided the denominator does not vanish, whether K is time-like or not (also for $\kappa = 0$).

With this form of $f(x)$, and with $g(x) = f(x)u$ (u a spinor independent of x), (2.5) is the α-solution of (2.3) or of (2.1) ($\kappa = 0$), that of the Dirac equation (2.4) being given by

$$\psi^\alpha(\mathbf{x}) = \frac{i(K_\nu\gamma^\nu - m)u}{(K_\nu K^\nu + m^2)^{\alpha/2}} \exp[i(\mathbf{K},\mathbf{x})]. \tag{2.6}$$

The actual solutions are obtained for $\alpha = 2$.

III. FORMATION OF SELF-ENERGY INTEGRALS GENERAL DISCUSSION

A. Returning to the self-energy (1.2) with a view to establishing what types of integral to expect in such expressions, we can content ourselves for the moment with studying one typical term, e.g., the electrostatic energy

$$W_{st} = -\frac{e^2}{2}\int \langle s_0^{(0)} A_0^{(1)} \rangle (dx)^3. \tag{3.1}$$

We introduce the α-solution for $A_0^{(1)}$ (from (1.3) and (2.2)),

$$A_0^{(1)\alpha} = -4\pi I^\alpha s_0^{(0)}(\mathbf{x}).$$

In evaluating the integral we write $\psi^{(0)}$ as a Fourier series,

$$\psi^{(0)} = V^{-\frac{1}{2}} \textstyle\sum_n a_n u_n \exp[i(\mathbf{p}_n, \mathbf{x})],$$

where V is a periodicity volume, p_n is the momentum four-vector of an electron in state n, u_n is a certain spinor, and the a_n's are Jordan-Wigner matrices.

The one-electron theory (cf. (1.1)) will then give for an electron with the momentum vector q,

$$\langle s_0^{(0)} A_0^{(1)\alpha} \rangle = -4\pi V^{-2} \textstyle\sum_n u_q{}^* u_n u_n{}^* u_q \times \exp[i(\mathbf{p}_n - \mathbf{q}, \mathbf{x})] I^\alpha \exp[-i(\mathbf{p}_n - \mathbf{q}, \mathbf{x})].$$

If this is inserted into W_{st} and the sum over the momenta is written as an integral, the resulting expression reads (with $p = |\vec{p}|$, $z = y - x$)

$$W_{st}{}^\alpha = -\frac{e^2}{2}\int \langle s_0^{(0)} A_0^{(1)\alpha} \rangle (dx)^3$$
$$= \frac{e^2}{\pi} \oint \frac{d\omega}{4\pi} \int_0^\infty p^2 dp \sum_{(u)} \left\{ u_q{}^* u u^* u_q \times \int_D V^\alpha(-\mathbf{z}) \exp[-i(\mathbf{p}-\mathbf{q},\mathbf{z})](dz)^4 \right\}. \tag{3.2}$$

[d] This remark is sufficient for our present purpose. A more detailed treatment of the Dirac equation, also taking certain boundary conditions into consideration, is given by Gustafson (reference 8).

The summation here is over the four states belonging to a fixed momentum $\vec{p}$, $d\omega$ is a solid-angle element, and D the retrograde light-cone from the origin.

So far everything is straightforward and no new hypotheses have been made, the usual infinite value of W_{st} being obtained by putting $\alpha=2$ after the integration over D has been performed. However, in the form given above, W_{st}^{α} can be computed also for other values of α and is not necessarily infinite everywhere. In the domain of convergence it is then, as $A_0^{(1)\alpha}$, an analytic function of α, which can be continued analytically to $\alpha=2$ (just as, e.g., the Eulerian integral for $\Gamma(x)$ converges only for $x>0$, the function having to be defined by analytic continuation for $x<0$).

Gustafson's procedure consists in taking this W_{st}^2 as the significant value of W_{st}, corresponding definitions being used for other quantities. A certain justification for defining physical quantities by means of analytic continuation can be given in the classical case, where the whole theory, as shown by Fremberg,[2,3] can be built up with general α-quantities in place of the ordinary ones, and satisfying the same equations. Such a procedure does not seem adapted to the quantum theory. Instead, we must there lay down as a rule that all the equations should be solved by Riesz's method, then all physical quantities are first to be defined as functions of the parameters entering into these solutions, and afterwards calculated by analytic continuation. In this way an unambiguous scheme is obtained. It means a redefinition, but if we want to have some means of eliminating divergences without changing the fundamental equations, it is a rather natural one.

B. In connection with the expression (3.2) for W_{st}^{α} we can now make some general observations. Using Fourier analysis, it is evident from (2.5) and (2.6) that self-energies involving a solution of the wave or meson equation, or of the Dirac equation, can always be expressed as a sum of terms of the same general form, *viz.*

$$\text{const.}\oint d\omega\int_0^{\infty} f_1(p)dp \times\int_D V_\kappa^{\alpha}(-\mathbf{z})\exp[-i(\mathbf{K},\mathbf{z})](dz)^4$$

($\kappa=0$ for the wave equation), where $\boldsymbol{K}$ is linear in $\boldsymbol{p}$. The integral over D equals

$$(K_\nu K^\nu+\kappa^2)^{-\alpha/2}=f_2(p)^{-1-\beta}, \qquad (3.3)$$

where we have introduced $\beta=(\alpha/2)-1$, so that $\beta\rightarrow 0$ as $\alpha\rightarrow 2$. Thus, apart from an integration over the directions[e] and from a constant factor, the terms of interest are of the form

$$Q^{\beta}=\int_0^{\infty}\frac{f_1(p)}{f_2(p)^{1+\beta}}dp. \qquad (3.4)$$

The functions f_1 and f_2 are rational in p or may contain square roots. For large values of p $f_1/f_2^{1+\beta}$ can then be supposed to behave essentially as $p^{n-r\beta}$, where r is the degree of f_2 (1 or 2).

The ordinary result is obtained by setting $\beta=0$ in the integrand. A possible infinity is presupposed in the following to arise only from the upper limit of the integral (i.e., if $n\geqslant -1$), so that we have no convergence difficulties at the origin for $\beta=0$. It may happen, however, that new singularities are in some cases introduced here for other values of β. To exclude this contingency we may agree always to let the lower limit of the integrals over the momenta be some small positive value ϵ, which is allowed to approach zero only after the analytic continuation of the integral has been performed. Thus

$$\underset{\beta\rightarrow 0}{\text{anal. cont.}}\int_{\epsilon}^{\infty} p^{n-\beta}dp\rightarrow 0 \text{ as } \epsilon\rightarrow 0\ (n>-1) \qquad (3.5)$$

(we may also choose $\epsilon=\text{const.}\ \beta$). However, we will continue to write 0 for the lower limit, understanding it to be reached only in the final result if this is necessary for the convergence.

To calculate Q^0 by analytic continuation assuming that the integral Q^{β} does exist for some β's, we split it up into

$$Q^{\beta}=\left(\int_0^{P}+\int_P^{\infty}\right)\frac{f_1}{f_2^{1+\beta}}dp =Q^{\beta}(0,P)+Q^{\beta}(P,\infty). \qquad (3.6)$$

The two parts are both analytic in β with some domain of convergence in common and, consequently, Q^0 is obtained by performing the

[e] This integration does not give rise to any new infinities, so that we can pass to $\beta=0$ before carrying it out.

continuation of each integral separately. In fact, the same procedure can always be used for a general Q^β if we have first a lower integration limit $\epsilon>0$ as above.

If P is chosen large enough, $f_1/f_2^{1+\beta}$ can be expanded into a series of falling powers of p in the interval (P,∞), so that for $\beta>(n+1)/r$

$$Q^\beta(P,\infty)=\int_P^\infty p^{n-r\beta}\sum_{m=0}^\infty a_{n-m}(\beta)p^{-m}$$
$$=-\sum_{m=0}^\infty a_{n-m}(\beta)\frac{P^{n-m+1-r\beta}}{n-m+1-r\beta},$$

and as $\beta\rightarrow 0$ by analytic continuation

$$Q^0(P,\infty)=-\sum_{n-m\neq-1}a_{n-m}\frac{P^{n-m+1}}{n-m+1}$$
$$+\left[a_{-1}\frac{P^{-r\beta}}{r\beta}+\frac{a_{-1}(\beta)-a_{-1}}{r\beta}\right]_{\beta=0},$$

where $a_{n-m}=a_{n-m}(0)$.

In the first integral of (3.6), with the limits 0 and P, β may be set equal to zero under the integration sign. We determine the dependence upon P by expanding f_1/f_2 as before and making an indefinite integration:

$$Q^0(0,P)=\sum_{n-m\neq-1}a_{n-m}\frac{P^{n-m+1}}{n-m+1}$$
$$+a_{-1}\log P+\bar{Q}, \quad (3.7)$$

where $\bar{Q}$ does not depend on P.

Thus we get by addition

$$Q^0=\bar{Q}+\frac{1}{r}\frac{d}{d\beta}a_{-1}(0)+a_{-1}\left[\log P+\frac{P^{-r\beta}}{r\beta}\right]_{\beta=0}$$

which is, as it must be, independent of P, the last term equaling $a_{-1}(1/r\beta)_{\beta=0}$. If we let P tend to infinity in this term before $\beta\rightarrow 0$, the result may be written as

$$Q^0=\bar{Q}+\frac{1}{r}\frac{da_{-1}}{d\beta}+\lim_{P\rightarrow\infty}a_{-1}\log P. \quad (3.8)$$

(Nothing is changed in this formula if Q^β is a sum of terms of the form (3.4).)

So, if $a_{-1}\neq 0$ the value is logarithmically infinite in P, and otherwise finite. The general result, then, of defining self-energy terms by means of analytic continuation as above is that logarithmic divergences are retained, whereas all other types are eliminated. For the analytic function Q^β this means that in the presence of logarithmically divergent integrals Q^β has, and otherwise has not, a pole at $\beta=0$.

The result is an improvement on the λ-limiting process of Wentzel and Dirac, which is only able to remove the singularities of uneven orders if it is not combined with Dirac's assumption of negative-energy photons.**

IV. EXAMPLES—SELF-ENERGY OF THE ELECTRON

The calculation of the self-energy (1.2) of the electron by means of analytic continuation has been carried out by Gustafson[4,8] when neglecting the retardation in the solution of the equations (i.e., putting $y^0=x^0$ in $f(y)$ in (2.2), etc.). As an application of the formalism developed above, the continuation of the exact integrals is given here.

A. The electrostatic energy is expressed in the one-electron theory by (3.2).[f] We make the calculation also for a $\kappa\neq 0$ (meson theory) and get for the integral over D, in the case of a particle at rest: $\mathbf{q}=(m,0,0,0)$,

$$[p^2-(p^0-m)^2+\kappa^2]^{-\alpha/2}$$
$$=\left[2m\left(p^0-m+\frac{\kappa^2}{2m}\right)\right]^{-\alpha/2},$$

where $p^0=\pm(p^2+m^2)^{\frac{1}{2}}$. If we let p_0 stand for the positive square root, the result of performing the

** In the classical case, the λ-limiting process and the Riesz method have been shown by Ma[15] to be equivalent.

[f] This is not the usual expression for the electrostatic energy, E_{st} (obtained, e.g., by eliminating the longitudinal waves). Indeed, E_{st} can be obtained from our W_{st} by neglecting the retardation, which is equivalent to letting the time component of the vector $\mathbf{p}-\mathbf{q}$ in (3.2) be zero. Then we get, instead of (4.1) and (4.2),

$$E_{st}{}^\alpha=e^2/\pi\int_0^\infty[p^2dp/(p^2+\kappa^2)^{1+\beta}]\rightarrow-e^2\kappa/2$$

(cf., reference 5). So our separation into 'W_{st}' and 'W_{sp}' is not identical with that in reference 14.

summation over u in (3.2) can then be written

$$W_{st}{}^{\alpha}=\frac{e^2}{4\pi m}\frac{1}{(2m)^{\beta}}\int_0^{\infty}\frac{p^2dp}{p_0}\left\{\frac{p_0+m}{\left(p_0-m+\frac{\kappa^2}{2m}\right)^{1+\beta}}-\frac{p_0-m}{\left(-p_0-m+\frac{\kappa^2}{2m}\right)^{1+\beta}}\right\}=\frac{e^2}{\pi}Q^{\beta} \quad (4.1)$$

($\beta=\alpha/2-1$). The integral converges for $\beta>2$. As in the last section the analytic continuation to $\beta=0$ is conveniently carried out by dividing the integration interval into two parts. After putting $\beta=0$ directly in the first part and simplifying the integrand, we obtain

$$Q^0(0,P)=b\int_0^P\frac{p^2dp}{p^2+\kappa^2b}=bP-\kappa b^{\frac{3}{2}}\left(\frac{\pi}{2}-\arctan\frac{\kappa b^{\frac{1}{2}}}{P}\right),$$

with $b=1-\kappa^2/4m^2$. The last term can be expanded in descending powers of P. Comparing with (3.7) we have $a_{-1}=0$, and since $da_{-1}/d\beta$ is found to be a pure imaginary and so has no physical significance, (3.8) gives

$$Q^0=\bar{Q}=-(\pi/2)\kappa b^{\frac{3}{2}},$$

i.e.,

$$W_{st}=-\frac{e^2\kappa}{2}\left(1-\frac{\kappa^2}{4m^2}\right)^{\frac{3}{2}}, \quad (4.2)$$

which is zero for $\kappa=0$ (electrodynamics).[g]

The effect of assuming the hole theory is to substitute a plus sign for the minus sign between the terms in (4.1); the result is then logarithmically divergent, *viz.* essentially (for $\kappa=0$)

$$W_{st}=\frac{9e^2m}{8\pi}+\lim_{P\to\infty}\frac{3e^2m}{4\pi}\log\frac{2P}{m}.^{\text{h}} \quad (4.3)$$

B. The α-expression for the "spin energy" W_{sp} (obtained from the product $\vec{s}^{(0)}\vec{A}^{(1)}$ in (1.2)) is found to be ($\kappa=0$)

$$W_{sp}{}^{\alpha}=-\frac{3e^2}{4\pi m}\frac{1}{(2m)^{\beta}}\int^{\infty}\frac{p^2dp}{p_0}\times[(p_0-m)^{-\beta}\mp(-p_0-m)^{-\beta}],$$

where the upper and lower signs refer to the one-electron theory and hole theory, respectively. In the former case we get the result zero immediately by setting $\beta=0$, while the hole theory gives, by analytic continuation,

$$W_{sp}=-\frac{3e^2m}{8\pi}+\lim_{P\to\infty}\frac{3e^2m}{4\pi}\log\frac{2P}{m}.$$

C. The last energy term W_{fluct} in (1.2) contains the solution of Dirac's equation for $\psi^{(1)}$. If $A^{(0)}$ and $\psi^{(0)}$ are written as Fourier series, this solution is expressed as a sum of terms of the form (2.6) with $\boldsymbol{K}=\boldsymbol{q}-\boldsymbol{k}$, $\boldsymbol{q}$ being the momentum vector of the electron and $\boldsymbol{k}$ that of an emitted photon. For an electron at rest $\boldsymbol{q}=(m, 0, 0, 0)$, and then

$$K_{\nu}K^{\nu}+m^2=2mk, \quad (4.4)$$

where $k=|\vec{k}|=k^0$. Comparing with (3.3), (3.4), and with the ordinary result,[13] we see that we must have

$$W_{\text{fluct}}{}^{\alpha}=\frac{e^2}{\pi m}\frac{1}{(2m)^{\beta}}\int_0^{\infty}k^{1-\beta}dk.$$

(Waller's result is obtained for $\beta=0$; this holds also in the hole theory.[14]) If the integral is interpreted as in (3.5) the continuation to $\beta=0$ gives the value zero.[i]

D. Collecting our results, we obtain for the second approximation of the total self-energy of the electron,

$$W=W_{\text{st}}+W_{\text{sp}}+W_{\text{fluct}}(=E_{\text{st}}+E_{\text{dyn}}),$$

in the one-electron theory

$$W=0,$$

and in the hole theory

$$W=\frac{3e^2m}{4\pi}+\lim_{P\to\infty}\frac{3e^2m}{2\pi}\log\frac{2P}{m}.^{\text{j}}$$

---

[g] If $\kappa=0$ at the outset we must use an $\epsilon>0$, as in (3.5), in order to make the integral (4.1) convergent (for $\beta>2$). For another device serving the same purpose see reference 4.

[h] For E_{st} Weisskopf's result is obtained, i.e., it is only the logarithmic term in (4.3) that is retained, with the factor $\frac{3}{4}$ replaced by unity.[9]

[i] Neglect of the retardation results in changing the denominator $2mk$ as given by (4.4) into k^2+m^2, the numerical value then becoming $-e^2m$.[8]

[j] The value of the finite term is really immaterial.

Consequently, the singularities have been removed in the one-electron case[k] by using analytic continuation, but the situation in the hole theory is not improved as compared with the usual treatment.[14]

V. PAULI'S MODIFICATION—CONCLUSION

An alternative formulation of quantum electrodynamics which leads to the elimination of the same divergences as above in a simplified manner, by likewise making use of analytic continuation, has been proposed by Pauli.[1] In this method the commutation rules for the Fourier components of the electromagnetic potentials are changed by introducing a factor $\varphi_\beta(k)$ (corresponding to the factor $\cos(\boldsymbol{k},\boldsymbol{\lambda})$ in the λ-limiting process, see, e.g., reference 10, p. 193), chosen in such a way that $\varphi_\beta(k)\rightarrow 1$ as $\beta\rightarrow 0$, and that

$$\underset{\beta\rightarrow 0}{\text{anal. cont.}}\int_0^\infty k^n\varphi_\beta(k)dk=0 \quad \text{for } n>-1.$$

From (3.5) it is seen that we may choose

$$\varphi_\beta(k)=k^{-\beta}.$$

To avoid having to introduce a lower integration limit $\epsilon>0$ as in (3.5), we may define φ_β to be constant, equal to $l^{-\beta}$, for $k\leqslant l$.

In this modification of the theory[m] the function φ_β will enter as a factor under the integration sign in all self-energy terms (cf. the corresponding expressions obtained with the λ-process). Then the considerations of Section III can be repeated, again giving the result (3.8), with the difference that the term containing $da_{-1}/d\beta$ is no longer present. Consequently, the same divergences are removed as when Riesz's solution is used, though there may in some cases be a difference in the numerical results.

To sum up, we have seen that by admitting analytic continuation and not only ordinary limit transition it is possible in quantum theory to extend the class of divergent integrals that can be made finite by formal mathematical methods, though the difficulties of the logarithmic divergences cannot be overcome in this way. It is true, however, that if Pauli's method is formulated with several parameters (as done for the λ-process by Pomeranchuk), these divergences can be removed too, but the result can then be made wholly arbitrary.

In conclusion, I want to thank Professor T. Gustafson cordially for suggesting an investigation of problems within this sphere and for valuable discussions and suggestions. My thanks are also due to Professor W. Pauli for instructive criticism of the procedure and especially for permission to mention his alternative method.

[k] However, in the higher approximations logarithmic singularities appear also in the one-electron theory. The calculations by analytic continuation (using Pauli's modification below) have been made in the e^4 approximation, and will be published before long (the result is infinite).

[l] I am indebted to Professor Pauli for permission to mention his method here.

[m] The theory is consistent for a finite β, but is seen to be relativistically invariant only in the limit $\beta=0$.

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