Sensitivity of Proton-Proton Scattering to Potentials at Different Distances

G. BREIT, A. A. BROYLES, AND M. H. HULL
*Yale University, New Haven, Connecticut**

(Received December 19, 1947)

Estimates are made of sensitivity of proton-proton scattering to assumed changes δV in theoretical potential energy. The energy range 0.15–9 Mev is considered for the incident protons. The changes in potential energy are taken to be small additions in small intervals at distance R. First-order perturbation theory is used. Combining the effect of δV at $R=3$ or 5 in units e^2/mc^2 with a change δD in the depth of a square potential well (width e^2/mc^2) the effects are made to neutralize at a preassigned energy E_{I} and the sensitivity of scattering to δV is estimated at a second energy E. Combination of these effects with a simultaneous change δr_0 in range of square well enables neutralization of effects at energies E_{I}, E_{II} and adjustment to arbitrary small change in scattering at a third energy E. The sensitivity of observed scattering to δV for a few choices of pairs E_{I}, E_{II} is worked out as a function of E. It is found that the region 200 kev–600 kev should be valuable in determining values of δV for preassigned R. The relative importance of the smaller energies increases with R on account of shielding of the smaller R by the Coulomb barrier. Neutralization of effects at two energies E_{I}, E_{II}, with simultaneous adjustment to changes in scattering at energies E_1, E_2, is also considered in terms of simultaneous potential energy changes at two values of R combined with δr_0, δD. An experimental accuracy of 1 percent in scattering is found to correspond to the possibility of detecting as little as 1 kev for δV through a distance e^2/mc^2 at $R=5e^2/mc^2$ and 10 kev at $R=3e^2/mc^2$.

I. INTRODUCTION

THE most accurate experimental material[1,2] on proton-proton scattering is in the energy range from about 700 to 2400 kev. There are, in addition, observations[3] at low energies (150–320 kev) and more recent material[4] at 7, 9, 10, 15 Mev. The present note contains estimates concerning the relative usefulness of measurements at different energies for the determination of the shape of the effective potential energy curve between two protons.

It is realized that the concept of potential energy as a definite function of distance is only approximate,[5,6] and that a velocity dependence of nuclear forces would vitiate conclusions regarding both the range of force and the shape of the potential energy curve. There is probably some meaning to the shape of the potential energy-distance curve, especially if the energies dealt with are confined to a region small compared with the rest mass energy of the meson. For this reason it is better to determine the effective shape from observations which do not extend over a too wide energy range. The estimates reported on below show that it would be possible to detect the presence of interaction energies acting in addition to the Coulomb energy, if they were located at distances of the order of 10^{-12} cm by means of observations in the low energy range from about 200 to about 500 kev. The development of electron multiplier and counter techniques should make it possible to make improved measurements in this energy range and to ascertain small interactions outside of the region of 3×10^{-13} cm within which the potential energy is mainly seated. It is clear intuitively that the low energy region should be good for detection of deviations from the inverse square law of potential at larger distances. This potential is small for two protons and is relatively ineffective beyond about 500 kev in keeping the protons apart. The regions of 3×10^{-13} cm and 10^{-12} cm are nearly equally accessible to protons of the higher energy. At 200 kev, however, the Coulomb potential shields the smaller distance region, and the higher distances have, therefore, a greater relative weight for giving phase shifts.

* Assisted by Contract N6ori-44, Task Order XVI, of the Office of Naval Research.

[1] R. G. Herb, D. W. Kerst, D. B. Parkinson, and G. J. Plain, Phys. Rev. 55, 998 (1939).
[2] N. P. Heydenburg, L. R. Hafstad, and M. A. Tuve, Phys. Rev. 56, 1078 (1939).
[3] G. L. Ragan, W. R. Kanne, and R. F. Taschek, Phys. Rev. 60, 628 (1941).
[4] R. R. Wilson and E. C. Creutz, Phys. Rev. 71, 339 (1947); R. R. Wilson, Phys. Rev. 71, 384 (1947); R. R. Wilson, E. J. Lofgren, J. R. Richardson, B. T. Wright, and R. S. Shankland, Phys. Rev. 71, 560 (1947); A. N. May and C. F. Powell, Proc. Roy. Soc. 190, 170 (1947).
[5] G. Breit, H. M. Thaxton, and L. Eisenbud, Phys. Rev. 55, 1018 (1939), compare p. 1062. This paper is referred to as BTE in the text.
[6] G. Breit, University of Pennsylvania Bicentennial Conference, Pamphlet N-15-16, p. 5 (1941).

This effect is only a contributing factor to the high sensitivity at 400 kev which is found below. There is besides a strong effect of interference[7] between the Coulomb wave and the expected proton-proton interaction wave which enhances the sensitivity. The total scattering at 45° at these energies is a small fraction of the scattering due to each cause taken separately. A given percentage change in either of the contributions gives, therefore, a larger effect on the percentage change in scattering. It is realized that if the scattering is very small at 45°, then the experimental determination of scattering might be impossible to carry out with the same percentage error as is normally the case. There has been no intensity difficulty reported in the published experimental work because the Rutherford scattering is large at low energies. There might be some error introduced by multiple scattering when the scattering is close to a minimum ($\sim$400 kev) at a scattering angle of 45°. On account of this possible source of error the results reported on below are given also for 35° scattering. The effects will be seen to be present, although to a smaller degree, at this angle also.

An estimate of order of magnitude of sensitivity is of interest at this stage. By first-order perturbation calculations one finds that a potential change δV, located at a distance $R=3e^2/mc^2$ and spread through a distance ΔR, gives at 0.2 Mev a change δK, in the phase shift K, of amount

$$\delta K \cong -\frac{0.618}{(0.2)^{\frac{1}{2}}}\times 0.158q = -0.218q, \qquad (1)$$

where

$$q=(\delta V_{\mathrm{Mev}})(\Delta R mc^2/e^2). \qquad (1.1)$$

Here 0.158 is the approximate value of $\mathfrak{F}^2$ in Eq. (2) below. In the notation of BTE[5] the sensitivity of the ratio to Mott's scattering expected at 0.2 Mev, for the phase shift known approximately from experiment, is

$$d\mathcal{R}/dK \cong -3.8. \qquad (1.2)$$

The expected change in $\mathcal{R}$ for 45° scattering at 200 kev is

$$3.8\times 0.58\times 0.218q = 0.47q. \qquad (1.3)$$

The value of $\mathcal{R}$ is about 0.45 and hence

$$(\delta\mathcal{R}/\mathcal{R})_{45°,\ 0.2\ \mathrm{Mev}} \cong 1.0q. \qquad (1.4)$$

Measurements to an accuracy of 1 percent at this energy would determine as small a value of q as $(e^2/mc^2)10$ kev, i.e., a lump of potential energy of 10 kev through a distance 2.8×10^{-13} cm. The factor 0.58 included in Eq. (1.3) takes into account an opposing change in the depth of the potential energy curve adjusted so as to keep the total scattering the same at 2 Mev. At 200 kev one is still far from the minimum of $\mathcal{R}$ and only the difficulty of counting low energy protons is in the way in this energy region. The sensitive condition illustrated above is still stronger at 350 kev and is especially marked between 0.3 and 0.5 Mev. One of the objects of the present note is to point out the possible value of extending previous measurements into this energy region.

It would not be satisfactory to deal only with the sensitivity of scattering to a small potential lump because the effect of the lump can be compensated at any energy by an opposing effect in the depth of the potential well. The calculations are made, therefore, on the supposition that the effect of the potential extension is compensated by a well depth change at some energy and the fractional change in $\mathcal{R}$ is estimated. The rate of change of scattering expressed in fractional amount of total scattering change is X_{I} of Eq. (13); the compensation by well depth change is supposed to take place at an energy $E=E_{\mathrm{I}}$. A change in the potential energy curve, consisting in the introduction of a potential energy lump at R, and compensated at a conveniently chosen energy E_{I} by a well depth change, can be reproduced at another energy E_{II} by a change in the range of the potential energy well (range, according to custom, means here range of force) accompanied by a suitable change in well depth. If measurements were available at only two energies there would be no way of distinguishing between a change in range, which has to do with changes of the potential energy curve from $r=0$ to $r\cong 3\times 10^{-13}$ cm, and the effect of a potential energy lump at a larger distance R. The sensitivity to range is expressed to first order by the quantity $\delta_1 K+\delta_2 K$ in Eq. (6). Graphs giving the ratio of potential lump sensitivity to

[7] G. Breit, E. U. Condon, and R. D. Present, Phys. Rev. **50**, 826 (1936).

range sensitivity are given below in terms of the quantity Y of Eq. (9). This quantity has to be multiplied by $(\delta V)\Delta R/(D\delta r_0)$ where δr_0 is the change in range in order to give the ratio of phase shift changes caused by a potential change δV through ΔR at distance R and a change δr_0 in the radius of the potential well. Both changes are supposed to be compensated by suitable well depth changes as has been explained above. The graphs show that the ratio of sensitivity depends on energy even though both changes are made so as to produce no effect at one arbitrarily chosen value of E such as 1 Mev or 2 Mev. There remains, therefore, a possibility of distinguishing between effects of change of range and effects of potential energy lumps at larger distances. The sensitivity of scattering to changes in shape of potential energy curves can be estimated by means of the graphs.

The change in scattering caused by a potential energy change at the larger distances and compensated at energy E_{I} by a suitable well depth change can be combined with a change in range of force also accompanied by a well depth change compensating its effect at energy E_{I}. The combination of the two changes can be made in such a way as to have them compensate each other at energy E_{II}. The fractional rate of change of scattering with respect to q [see Eq. (1.1)] is the quantity $Z_{\text{I, II}}(E)$ which is plotted below in a few cases. These graphs also show that the low energy region should be valuable in extending knowledge of the shape of the effective potential energy curve.

All calculations reported on are in the nature of estimates. They are subject to the following limitations and approximations. First-order perturbation theory for phase shift calculation is used. This means that the potential energy changes dealt with must be small in order that the result be applicable. The calculations have to do with additional information that can be derived from improved measurements rather than with indications from present measurements concerning the shape of potential energy curves. Radial wave functions for a "square well" of radius e^2/mc^2 and depth 10.5 Mev (without Coulomb potential inside well) have been used. This potential probably represents experiment imperfectly and an approximation is involved at this point. The values of the radial function at the larger R are determined, however, mainly by the phase shift K rather than by the potential well and this approximation is believed, therefore, to be harmless. Some of the Coulomb functions and other quantities at low energies were not computed directly but were interpolated for. No great numerical accuracy was aimed at. The methods employed are similar to some of the work of Hoisington, Share, and Breit,[8] but the present note has more to do with qualitative understanding of the relative importance of different energy regions than with an adjustment of potential energy curves to represent experimental data.

The object in dealing with compensated combinations is to enable fitting experiments at two energies by means of a potential well with orthodox range and later improving the over-all fit by adjusting q to agree with experiment at a third energy.

II. CALCULATIONS

The notation used is that of BTE. The subscript 0 is omitted for K. No confusion can result because only one K is dealt with. The first-order effect of a potential energy change, δV, is given by

$$\delta K = -\int (\delta V/E')\mathfrak{F}^2 d\rho = -\frac{0.618}{(E_{\text{Mev}})^{\frac{1}{2}}}\int \delta V_{\text{Mev}}\mathfrak{F}^2 d(rmc^2/e^2), \quad (2)$$

where $\mathfrak{F}$ is $r\times$the radial wave function and the energy of relative motion is E' so that

$$E=2E'. \quad (3)$$

For a "square well" potential the first-order phase shift caused by a change in well depth, δD, is given by

$$\delta_1 K = \rho_0\mathfrak{F}^2(\rho_0)J(z_0)\delta D/E, \quad (4)$$

where the quantity J is

$$J(z_0)=\int_0^{z_0}(\sin^2 z/\sin^2 z_0)dz = (z_0-\sin z_0\cos z_0)/(z_0\sin^2 z_0). \quad (4.1)$$

[8] L. E. Hoisington, S. S. Share, and G. Breit, Phys. Rev. 56, 884 (1939).

In these formulas the quantity ρ is the distance expressed in wave-length/2π while z is the phase of the sine curve representing the wave function inside the potential well, counting the phase as 0 at $r=0$. Subscripts 0 in these formulas indicate that quantities are evaluated at the edge of the potential energy well for which $r=r_0$. A change in width of the potential energy well gives a similar change in phase shift:

$$\delta_2K=2\rho_0\mathfrak{F}^2(\rho_0)(D/E)(\delta r_0/r_0), \tag{5}$$

where δr_0 is the change in range. The combined phase shift is given by

$$\delta_1K+\delta_2K=2\rho_0\mathfrak{F}^2(\rho_0)(D/E)(\delta r_0/r_0)[1+\lambda J(z_0)], \tag{6}$$

where

$$\lambda=(\delta D/2D)/(\delta r_0/r_0). \tag{6.1}$$

If the change in range has to be compensated by a change in depth at energy E_{I} one has to determine λ by

$$J(z_{0,\mathrm{I}})=-1/\lambda_{\mathrm{I}}. \tag{6.2}$$

A potential energy lump extending through ΔR at distance R is

$$\delta_3K=-2\mathfrak{F}^2(kR)(\delta V)k\Delta R/E. \tag{7}$$

If the effect of the potential energy lump is compensated for by a change in well depth at energy E_{I}, the remaining phase shift is given by

$$\delta_3'K=-2[\mathfrak{F}^2(kR)-a_{\mathrm{I}}J(z_0)\mathfrak{F}^2(kr_0)]k(\delta V)\Delta R/E, \tag{8}$$

where the energy independent number a_{I} is given by

$$a_{\mathrm{I}}=\mathfrak{F}^2(k_{\mathrm{I}}R)/[J(z_{0,\mathrm{I}})\mathfrak{F}^2(k_{\mathrm{I}}r_0)]. \tag{8.1}$$

The ratio of the phase shift produced by q to the phase shift produced by a change in r_0, if both are compensated at energy E_{I}, is given by

$$\frac{\delta_3'K}{\delta_1K+\delta_2K}=\frac{(\delta V)\Delta R}{D\delta r_0}Y, \tag{9}$$

where the quantity Y is

$$Y=-\frac{\mathfrak{F}^2(kR)-aJ\mathfrak{F}^2(kr_0)}{\mathfrak{F}^2(kr_0)(1+\lambda J)}. \tag{10}$$

The quantity Y has the meaning

$$Y=S'_{\mathrm{pot}}/S'_{\mathrm{range}}, \tag{11}$$

where

$$S'=\text{relative sensitivity}. \tag{11.1}$$

It follows from Eq. (2) and Eq. (8) that the fractional sensitivity of scattering to q of Eq. (1.1) is given by

$$X_{\mathrm{I}}=\frac{0.618}{(E_{\mathrm{Mev}})^{\frac{1}{2}}}\frac{\partial\mathfrak{R}}{\mathfrak{R}\partial K}[\mathfrak{F}^2(kR)-a_{\mathrm{I}}J\mathfrak{F}^2(kr_0)], \tag{12}$$

so that

$$X_{\mathrm{I}}=d_{\mathrm{I}}\mathfrak{R}/\mathfrak{R}d_{\mathrm{I}}q, \tag{13}$$

where d_{I} stands for differentiation for effect of δV when δV is compensated by δD at $E=E_{\mathrm{I}}$. If the potential lump effect is compensated for by well depth change at energy E_{I} and if, in addition, a change in range is made and also compensated for by a well depth change at energy E_{I}, and if the two changes are opposed in such a way as to compensate each other's effects at the energy E_{II}, then the fractional sensitivity to the quantity q at energy E is given by

$$Z_{\mathrm{I,II}}(E)=X_{\mathrm{I}}(E)[1-Y_{\mathrm{I}}(E_{\mathrm{II}})/Y_{\mathrm{I}}(E)]. \tag{14}$$

In this formula the term 1 in the square brackets together with the factor X_{I} represents the sensitivity to q as in Eq. (12). The remaining term in square brackets together with the factor X_{I} is, on account of Eq. (11), the compensating fractional rate of change of scattering produced by the change in range. The proportionality factor $Y_{\mathrm{I}}(E_{\mathrm{II}})$ is adjusted so that the change in range compensates for the effect of the potential lump at energy E_{II}. This means that δr_0 in Eq. (9) is suitably chosen to give compensation at E_{II}. The quantity $Z_{\mathrm{I,II}}$ is symmetrical in I, II and one verifies that

$$Z_{\mathrm{I,II}}=Z_{\mathrm{II,I}}=0.618(E_{\mathrm{Mev}})^{-\frac{1}{2}}\frac{\partial\mathfrak{R}}{\mathfrak{R}\partial K}\times\left\{\mathfrak{F}^2(kR)+\frac{\mathfrak{F}^2(kr_0)}{J_{\mathrm{I}}-J_{\mathrm{II}}}\left[\frac{\mathfrak{F}^2(k_{\mathrm{II}}R)}{\mathfrak{F}^2(k_{\mathrm{II}}r_0)}(J-J_{\mathrm{I}})+\frac{\mathfrak{F}^2(k_{\mathrm{I}}R)}{\mathfrak{F}^2(k_{\mathrm{I}}r_0)}(J_{\mathrm{II}}-J)\right]\right\}, \tag{15}$$

and $Z_{\mathrm{I,II}}$ = fractional sensitivity of scattering to potential change (in Mev, and e^2/mc^2) compensated by potential depth and range changes at E_{I} and E_{II}. The calculation of $\partial\mathfrak{R}/\partial K$ was made by means of

$$\frac{\partial\mathfrak{R}}{\partial K}=-\frac{2X}{\eta\mathfrak{M}}\cos 2K+\left(\frac{4}{\eta^2\mathfrak{M}}+\frac{2Y}{\eta\mathfrak{M}}\right)\sin 2K. \quad (16)$$

Other formulas and tables in BTE were used. Additional Coulomb functions for higher energies were obtained from the paper by Thaxton and Hoisington[9] and a few additional Coulomb functions were computed directly.

III. RESULTS AND DISCUSSION

In Fig. 1 the quantity X is plotted against the energy for $R=5e^2/mc^2$, $E_{\text{I}}=1.994$, 1.001 Mev for the scattering angle $\Theta=45°$, 35°, 25°. The quantity X, it will be remembered, is the fractional sensitivity to the quantity q when the potential is put at distance R; it is understood that the effect of the potential lump is compensated by a suitable change in well depth. For $E_{\text{I}}=1.00$ Mev, the curve for $\Theta=45°$ crosses through 0 once. If, however, $E_{\text{I}}=2.00$ Mev, the curve for $\Theta=45°$ crosses the axis of $X=0$ not only at 2.00 Mev but also between 3 and 4 Mev. The second crossing is caused by a node of $\mathfrak{F}$ going through $r=5e^2/mc^2$. On account of the second crossing the much higher sensitivity of scattering to the exterior part of the potential is especially marked in this case. The sensitivity to q is seen to be higher at $\Theta=45°$ than at other angles. The effect is still pronounced at $\Theta=35°$. At $\Theta=45°$ the scattering as a function of angle is nearly a minimum, and sufficiently small angular apertures or receiving slits in a scattering chamber should be attainable for approximating the more favorable conditions close to $\Theta=45°$. In Fig. 2 the quantity X is plotted against E for $R=3e^2/mc^2$, $E_{\text{I}}=4.00$, 2.07 Mev, and $\Theta=45°$, 35°, 25°. Comparison of Fig. 2 with Fig. 1 shows that the sensitivity to q is smaller for $R=3e^2/mc^2$ than for $R=5e^2/mc^2$ at 1 Mev. The ratio is roughly represented by a factor of 2 or 3. In other respects the two values of R compared in Figs. 1 and 2 give similar behavior of X on E.

In Fig. 3 the quantity Z for $R=5e^2/mc^2$ is plotted against E. Here $E_{\text{I}}=1$ Mev while E_{II} has been given the values 2 Mev, 4 Mev, respectively, for the two sets of graphs. In this comparison the value of observations in the high energy range 5 Mev–9 Mev shows up to better advantage than in Figs. 1, 2. Nevertheless, it is seen that simultaneous compensation of change of range and addition of potential energy at $R=5e^2/mc^2$ at both 1 Mev and 2 Mev makes observations at 0.5 Mev and 0.3 Mev of about equal interest with those at 9 Mev. The required accelerating equipment is more modest for the lower energy range, although there are technical difficulties in counting low energy protons which are absent at the higher energies. For these there is only a slight difference in the effectiveness of scattering angles 45°, 35°, 25°, while for lower energies 45° is by far the more effective angle in detecting changes in scattering produced by q. As is well known, the scattering approaches spherical symmetry in the center of mass system of the two protons at the higher energies. There is, accordingly, no decrease of sensitivity towards the smaller angles. This advantage is present only if *absolute* measurements are made, i.e., if the yield of scattered protons per incident proton is determined. On the other hand, absolute measurements are usually more difficult than the determination of angular distributions. Checks on absolute measurements are also difficult, requiring repetition of experiments with different geometries. It is felt, therefore, that the apparent advantage of the availability of a wide angular range at high energies for testing the potential extension at R is partly offset by the necessity of drawing conclusions from absolute determinations of the scattered proton yield. There is also the added uncertainty of the effects of phase shifts corresponding to high angular momenta which enters at high energies more seriously than at the low ones. For the latter the consistency of ratios of observed scattering with expectation for a given K at a given energy is itself a test of the correctness of absolute yield determinations, as is well known.[5,7] The wave scattered by the Coulombian potential can be used as a reference standard for the wave scattered on account of the change in the s wave produced by the potential well. If one were to assume that phase shifts for angular momenta

[9] H. M. Thaxton and L. E. Hoisington, Phys. Rev. 56, 1194 (1939).

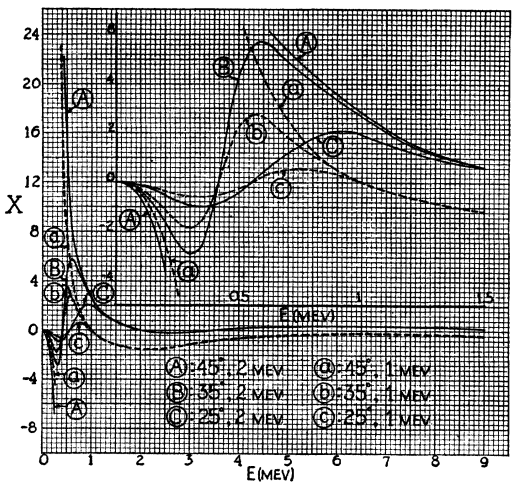

FIG. 1. X of Eqs. (12) and (13) as a function of energy for $R=5e^2/mc^2$. Here X is the fractional sensitivity of scattering to quantity q of Eq. (1.1) when the potential change is placed at R. Curves A, B, C are for compensation by potential well depth change at 2 Mev; curves a, b, c are for the same at 1 Mev.

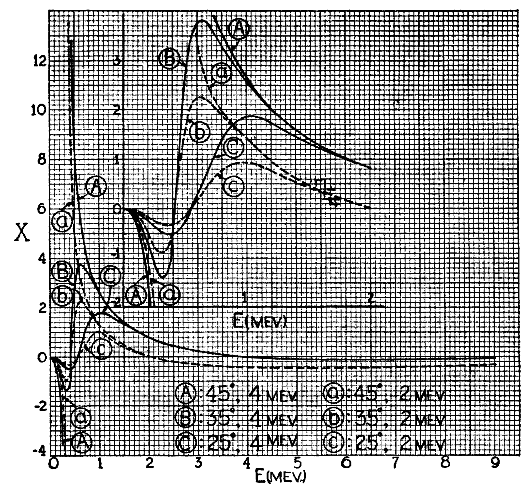

FIG. 2. X of Eqs. (12) and (13) as a function of energy for $R=3e^2/mc^2$. Curves A, B, C are for compensation by potential well depth change at 4 Mev, and curves a, b, c are for the same at 2 Mev.

higher than 0 are absent one could do away with absolute yield determinations in this region. It is seen, therefore, that the drop in sensitivity at $\Theta=35°$, 25° is not a disadvantage but can even be used. If compensation of potential extension, range, and well depth is made at 1 Mev and 4 Mev, the lower energy range shows up somewhat better in comparison with that between 7 and 9 Mev. In this case $\Theta=35°$ gives a peak sensitivity for differentiation between the two types of potential energy changes which is higher than that at 9 Mev. In Fig. 4 the quantity Z is plotted against E for $R=3e^2/mc^2$ with compensation at 1 Mev and 4 Mev. The higher energy range shows more effectiveness in giving changes in scattering than in the previous figures. This is to be expected because the potential lump is put at a smaller distance. Here also the usefulness of observations in the low energy range is seen to be definite.

Estimates of potential energy$\times$distance which are detectable by experiments of given fractional accuracy can be made as follows. Both X and Z are fractional rates of change of scattering with respect to q of Eq. (1.1). For X this relation is described by Eq. (12). For Z one has

$$Z_{\mathrm{I,II}}=d_{\mathrm{I,II}}\mathcal{R}/\mathcal{R}d_{\mathrm{I,II}}q, \tag{17}$$

where $d_{\mathrm{I,II}}$ stands for differentiation with respect to q subject to the requirement of compensating the effect of q at energies E_{I}, E_{II} by changes in potential depth D and range of force r_0. If either Z or X has the absolute value 1 and if the experimental accuracy is 1 percent so that $\delta\mathcal{R}/\mathcal{R}=1/100$ then, according to Eqs. (12), (17), the detectable δq is 0,01 Mev=10 kev. For $r=5e^2/mc^2$, $E_{\mathrm{I}}=1$ Mev, $E_{\mathrm{II}}=4$ Mev, Fig. 3 shows values of Z at $\Theta=45°$ exceeding 10 which corresponds to the possibility of detecting with a 1 percent accuracy in the measured scattering of an interaction potential of less than 1 kev through a distance e^2/mc^2. This very high sensitivity is perhaps not fully attainable because it is mainly the result of the smallness of scattering at 45° when the interference between the Coulomb and non-Coulomb waves becomes pronounced. But even for $\Theta=35°$ for the distance R and compensation energies just mentioned Z takes on the value 2 and more corresponding to the possibility of detecting less than 5 kev through a distance e^2/mc^2. For $R=3e^2/mc^2$ the sensitivity to q is appreciably smaller. In order to detect 5 kev through e^2/mc^2 one needs, in this case, according to Fig. 4, measurements at $\Theta=45°$ close to 400 kev while $\Theta=35°$ gives at best the possibility of detecting 30 kev through a distance e^2/mc^2. The ordinates in Figs. 1 and 2 correspond to still higher sensitivities. Thus, according to Fig. 2, even for $R=3e^2/mc^2$ one can detect $q=0.01$ Mev (e^2/mc^2). The possibility of doing so presupposes, however, that the range of the main part of the potential well is known in some other way. It is probable that Figs. 3 and 4 are more representative of what is attainable.

In the comparisons of sensitivity made so far it was supposed that the shape of the potential energy curve is known or decided on in some way except for the hypothetical addition to the potential described by q and the depth and range of the main part of the potential well. Such a procedure corresponds to fitting the experimental material by three parameters: depth and range of the potential well (for a preassigned shape of well), and in addition the quantity q for a preassigned R which adds essentially a tail to the potential well. Three energies are required in principle to determine the three parameters and, again in principle, a fourth energy could be used to determine a fourth parameter. In practice, however, the experimental accuracy will make it necessary to be more modest in the number of parameters derived from experiment.

If it is desired to distinguish between two alternative theories containing the three parameters in such a way as to have a short-range potential of unknown depth and width superposed on a long-range potential of known range but unknown depth or height the above estimates or their obvious extensions can be used for obtaining an idea of the possibility of determining the magnitude of the long-range part. The question arises as to what can be attempted to determine the range of the long-range part of the potential. To do this one has to adjust theory to experiment at 4 values of E.

In such a determination experimental information regarding the phase shifts for higher angular momenta may give valuable clues because these phase shifts are more easily produced by interaction potentials at larger distances. In principle, however, it is hard to exclude nearly unrelated potential energy curves for different angular momenta and it is desirable to be able to say something definite regarding sensitivity independently of any clues from other phase shifts than that of the s wave. The possibilities can be seen by comparing the values of Z for compensation at 1 Mev and 4 Mev for $R=5e^2/mc^2$ and $3e^2/mc^2$, respectively. The ratio of the sensitivities has the approximate values 1 at 9 Mev, 2 at 5 Mev, 5 at 2 Mev, 7.5 at 0.6 Mev, 8 at 0.35 Mev, 9.5 at 0.2 Mev. Accordingly, fitting at 0.2 Mev, 1 Mev, 4 Mev will not distinguish between a value q for $R=5e^2/mc^2$ and $q/10$ at $R=3e^2/mc^2$. On the other hand, an additional fit at say 9 Mev will distinguish between these possibilities determining essentially $q(3e^2/mc^2)+q(5e^2/mc^2)$, while the 0.2 Mev region determines approximately $q(3e^2/mc^2)+10q(5e^2/mc^2)$. The error of $q(3e^2/mc^2)$ is thus not decreased by

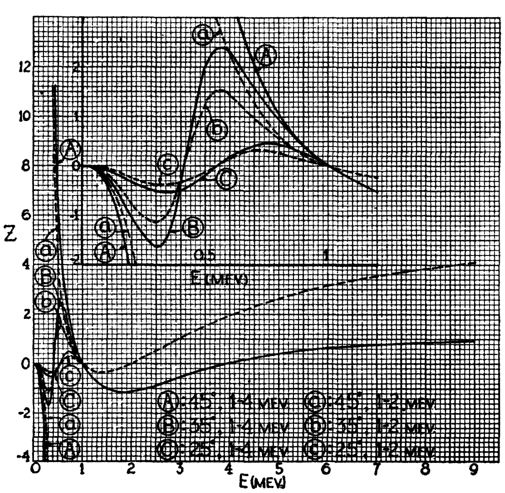

FIG. 3. Z of Eqs. (14) and (15) as a function of energy for $R=5e^2/mc^2$. Here Z is the fractional sensitivity of scattering to potential change at R compensated by potential well depth and range changes at energies E_{I} and E_{II}. For curves A, B, C, $E_{\mathrm{I}}=1$ Mev, $E_{\mathrm{II}}=4$ Mev. For curves a, b, c, $E_{\mathrm{I}}=1$ Mev, $E_{\mathrm{II}}=2$ Mev.

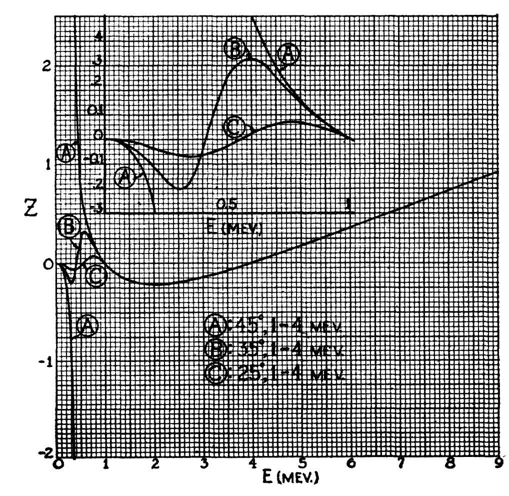

FIG. 4. Graph of Z of Eqs. (14) and (15) against energy for $R=3e^2/mc^2$. In this case, $E_{\mathrm{I}}=1$ Mev, $E_{\mathrm{II}}=4$ Mev.

the inclusion, in this case, of data at 0.2 Mev. The relations between errors in a fit of data at 4 energies can be expressed as follows. The quantity $Z_{\mathrm{I,II}}$ is a function of energy E and of the distance R at which the potential change is made. For fixed E_{I}, E_{II}, the symbol ζ will be used for $Z_{\mathrm{I,II}}$. The quantity ζ will be denoted by

$$\zeta_1(a),\quad \zeta_1(b),\quad \zeta_2(a),\quad \zeta_2(b),$$

for the pairs of values (E_1, R_a), (E_1, R_b), (E_2, R_a), (E_2, R_b) of E and R, respectively. It will be supposed that after the potential well has been adjusted there remain fractional deviations δ_1, δ_2 of experiment from theory at the energies E_1, E_2, respectively. The values of q at R_a, R_b will be denoted by q_a, q_b, respectively. There are then the equations

$$\begin{aligned}\zeta_1(a)q_a+\zeta_1(b)q_b&=\delta_1,\\ \zeta_2(a)q_a+\zeta_2(b)q_b&=\delta_2,\end{aligned}\tag{18}$$

that determine q_a, q_b as

$$\begin{aligned}q_a&=\frac{\delta_1/\zeta_1(b)-\delta_2/\zeta_2(b)}{\zeta_1(a)/\zeta_1(b)-\zeta_2(a)/\zeta_2(b)},\\ q_b&=\frac{\delta_1/\zeta_1(a)-\delta_2/\zeta_2(a)}{\zeta_1(b)/\zeta_1(a)-\zeta_2(b)/\zeta_2(a)}.\end{aligned}\tag{19}$$

The accuracy of the determination of q_a, q_b depends on the uncertainty in the values of δ_1 and δ_2. If through the use of the low energy region one makes $\zeta_1(a)$ and $\zeta_1(b)$ very large, one only renders the terms in δ_1 negligible and makes the terms in δ_2 the only important ones. For an extreme condition of entire suppression of δ_1 and for $\zeta_1(b)\gg\zeta_1(a)$ which corresponds to $R_b=5e^2/mc^2$, $R_a=3e^2/mc^2$, one has the following limiting form of the above equations

$$\begin{aligned}q_a&\cong+\delta_2/\zeta_2(a),\\ q_b&\cong-\zeta_1(a)\delta_2/[\zeta_1(b)\zeta_2(a)].\end{aligned}\tag{20}$$

The largeness of both $\zeta_1(a)$ and $\zeta_1(b)$ is supposed here to be a stronger condition than $\zeta_1(b)\gg\zeta_1(a)$. The accuracy in the knowledge of q_a is thus not helped at all by the high values of $\zeta_1(a)$, $\zeta_1(b)$ in this instance. The accuracy in the knowledge of q_b is, however, appreciably improved by having $\zeta_1(b)\gg\zeta_1(a)$. In fitting data at four energies, the usefulness of the high sensitivity in the low energy region is seen to be more limited. The weakest link in the chain counts most, and there is only a limited value in having one region of high sensitivity. Since, however, by increasing the energy beyond 9 Mev one may hope to obtain an improved sensitivity at the high energy end, the low energy region in the vicinity of 0.4 Mev should be a good one to combine with the high energy end since by doing so both ζ_1 and ζ_2 can be enlarged.

The calculations reported on here show that it may be just as important to improve counting techniques for measurements in the energy range 0.2 to 0.5 Mev in order to obtain additional information regarding the shape of the proton-proton potential energy curves as to be extending observations into the 10-Mev region. Questions of phase shifts for higher angular momenta are less troublesome and may be perhaps non-existent in the low energy range. Questions of velocity dependence of the interaction potential are reduced in importance and relatively modest proton accelerating equipment is required.

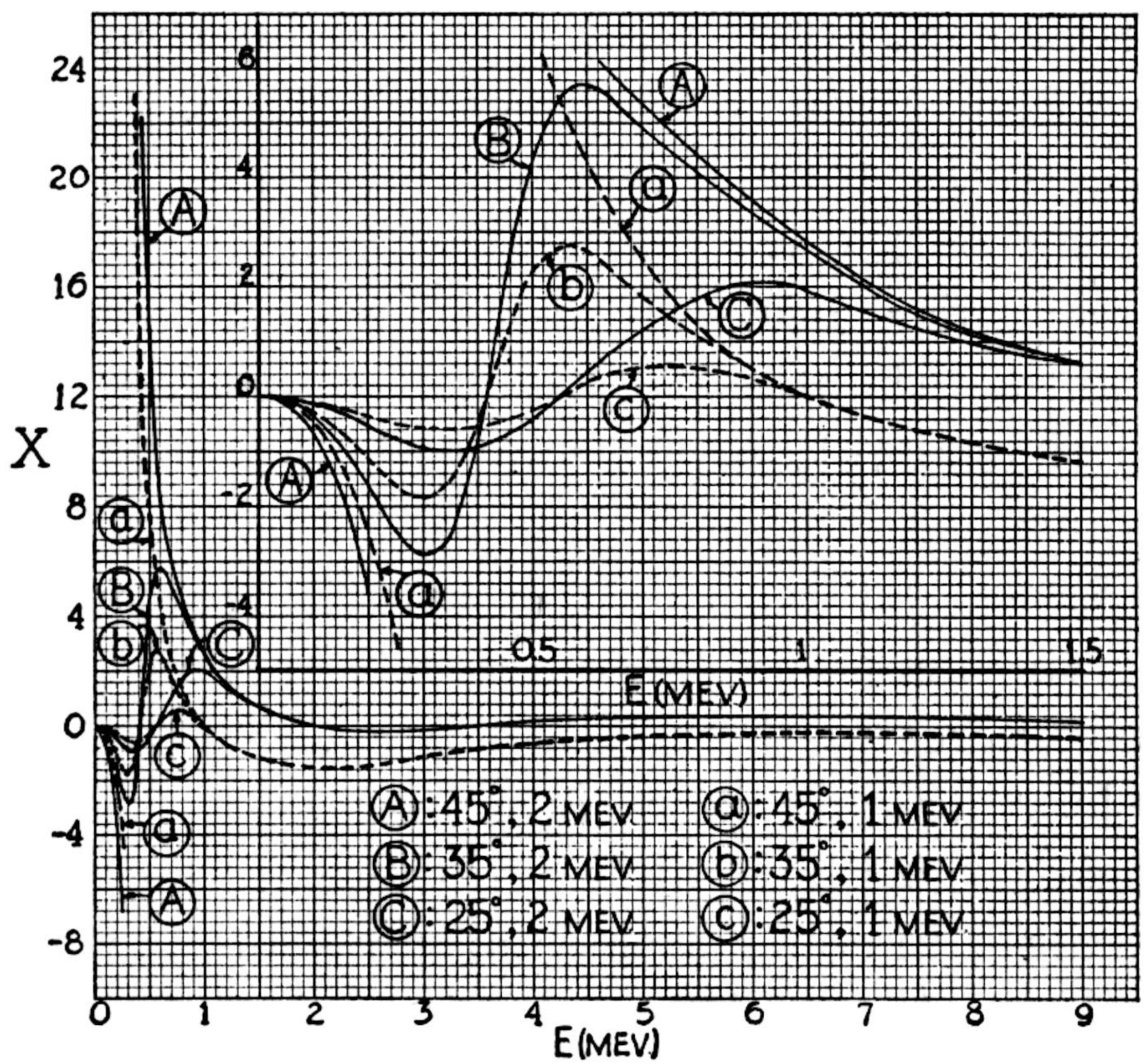

FIG. 1. X of Eqs. (12) and (13) as a function of energy for $R=5e^2/mc^2$. Here X is the fractional sensitivity of scattering to quantity q of Eq. (1.1) when the potential change is placed at R. Curves A, B, C are for compensation by potential well depth change at 2 Mev; curves a, b, c are for the same at 1 Mev.

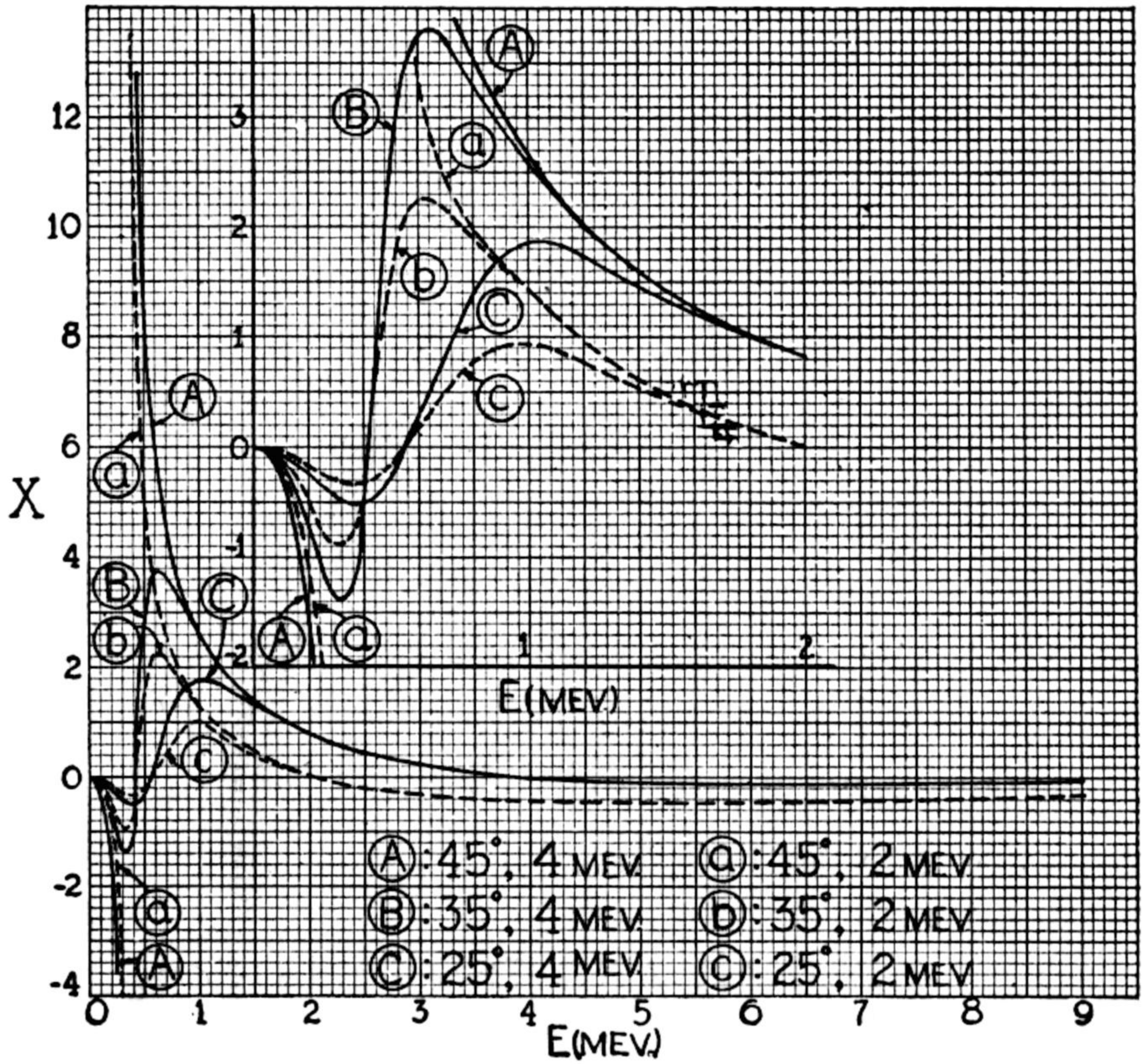

FIG. 2. X of Eqs. (12) and (13) as a function of energy for $R=3e^2/mc^2$. Curves A, B, C are for compensation by potential well depth change at 4 Mev, and curves a, b, c are for the same at 2 Mev.

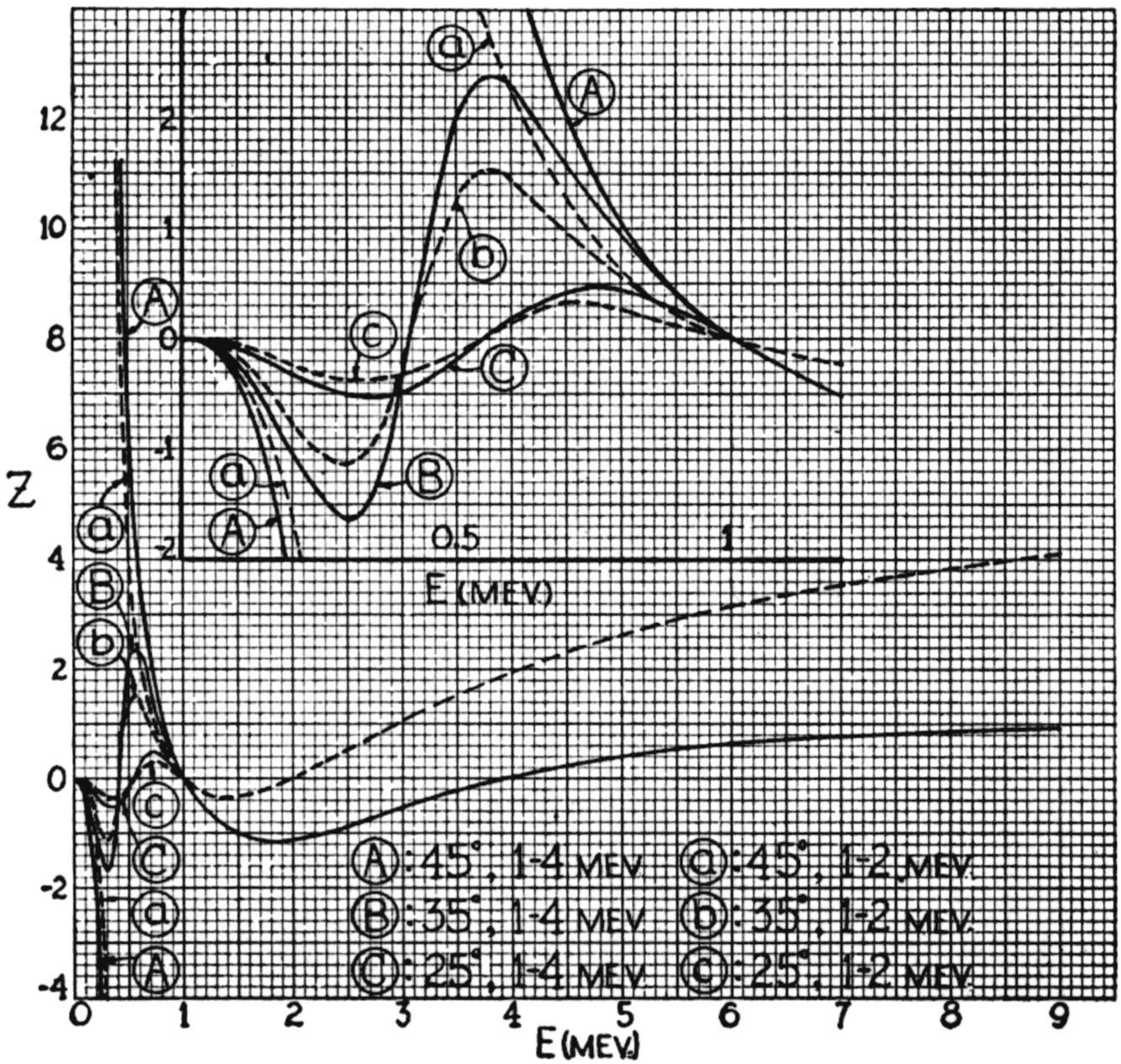

FIG. 3. Z of Eqs. (14) and (15) as a function of energy for $R=5e^2/mc^2$. Here Z is the fractional sensitivity of scattering to potential change at R compensated by potential well depth and range changes at energies E_{I} and E_{II}. For curves A, B, C, $E_{\mathrm{I}}=1$ Mev, $E_{\mathrm{II}}=4$ Mev. For curves a, b, c, $E_{\mathrm{I}}=1$ Mev, $E_{\mathrm{II}}=2$ Mev.

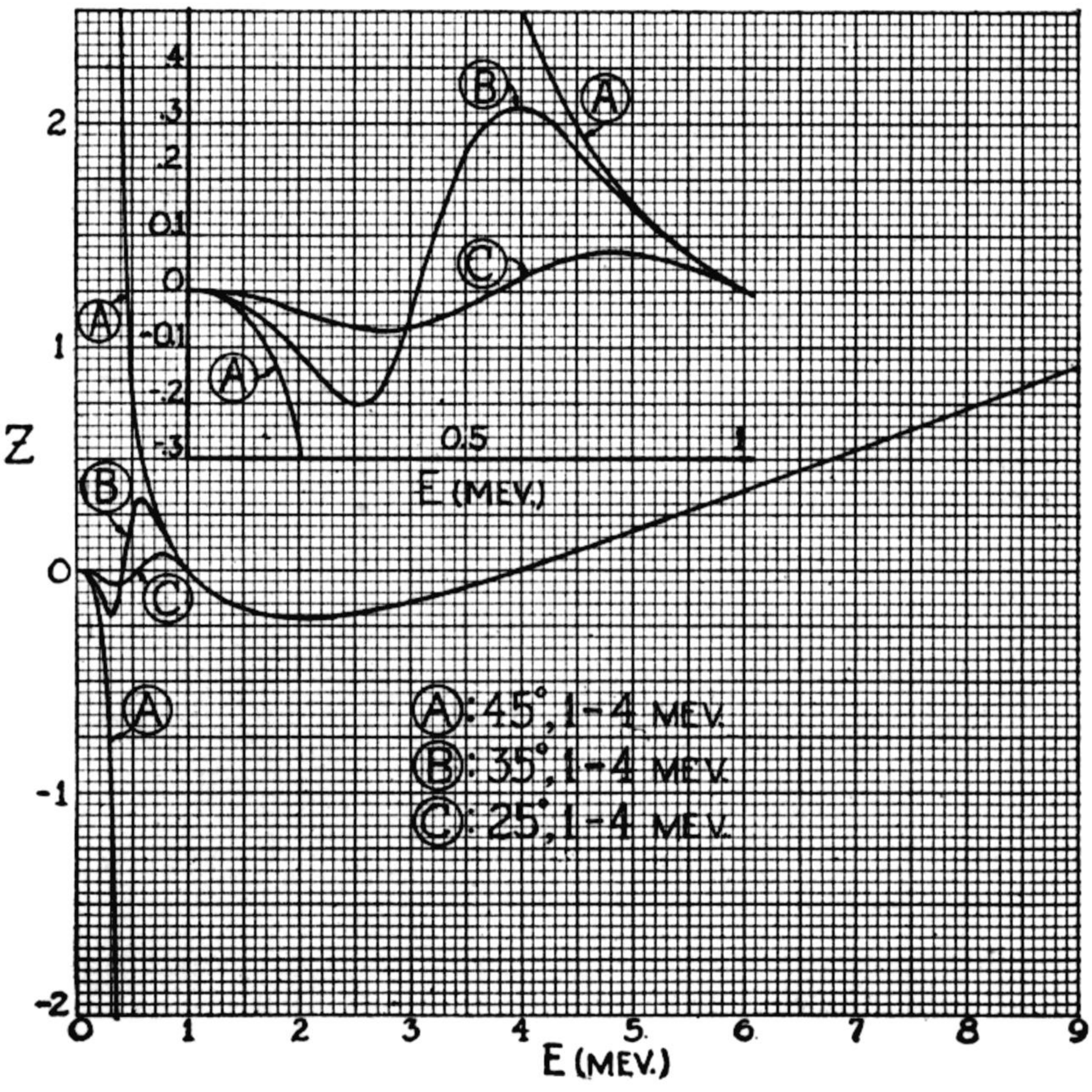

FIG. 4. Graph of Z of Eqs. (14) and (15) against energy for $R=3e^2/mc^2$. In this case, $E_{\rm I}=1$ Mev, $E_{\rm II}=4$ Mev.