gamma-ray reported by Seaborg and Segrè is present at all, it is not in cascade with the 0.129 Mev converted gamma-ray of the 6-hour metastable state in Tc^{99}.

APPENDIX

Chemical Purification of Molybdenum

To about two grams of Mo_2O_3 were added 40 mg of Al_2O_3, 20 mg of CaO, and 20 mg of Ta_2O_5 as carriers for iron, calcium, and columbium impurities. The mixture was fused with Na_2CO_3, and the melt was leached with cold water. After filtering out calcium carbonate, 20 mg of Na_3PO_4 were added as a carrier for phosphorous, which was then precipitated from ammoniacal solution as ammonium magnesium phosphate. Most of the aluminum and tantalum should have accompanied this precipitate together with any iron and columbium impurities which might be present. After filtration, the filtrate was acidified with HCl, and most of the molybdenum was precipitated with H_2S as MoS_2. The precipitate was washed with dilute H_2SO_4 (1:99); washed with alcohol, and ignited to MoO_3.

For the separation of Mo from Tc, a second sample of Mo_2O_3 was purified as described above. The filtrate from the magnesium ammonium phosphate precipitation was neutralized and buffered with sodium acetate. Molybdenum was precipitated by 8-hydroxyquinoline. From the filtrate, about 0.5 N in HCl, technetium was precipitated as a sulfide. For details of the molybdenum-technetium separation, see reference 7.

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The Penetration of Gamma-Radiation through Thick Layers

I. Plane Geometry, Klein-Nishina Scattering

J. O. HIRSCHFELDER,[1] J. L. MAGEE,[2] AND M. H. HULL[3]
The Los Alamos Scientific Laboratory, Los Alamos, New Mexico

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A rough method is developed for estimating the intensity of gamma-rays which have passed through thick targets. Only Klein-Nishina scattering is considered. The treatment is exact both for those gamma-rays which are unscattered and those which are scattered once. The accuracy of the results for rays which are multiple scattered depends upon the fact that most of the scattering takes place in the forward direction with only small angles of deviation. Numerical calculations are made for energies of 2, 6, and $10m_0c^2$ and all thicknesses of target. If A is the intensity of the initial beam, and I is the intensity after passing through a target of D mean-free-paths measured in terms of the initial unscattered gamma ray, it is found that:

initial energy $=2m_0c^2$,
$$I/A=(1+0.487D+0.030D^2)e^{-D},$$
initial energy $=6m_0c^2$,
$$I/A=(1+0.400D+0.0080D^2)e^{-D},$$
initial energy $=10m_0c^2$,
$$I/A=(1+0.33D+0.0040D^2)e^{-D}.$$

The contributions to the intensity are given for the gamma-rays which have been scattered a various number of times. Extensive tables are given which are generally useful for any problems in which only Klein-Nishina scattering is assumed.

I. INTRODUCTION[4]

THE general problem of multiple scattering is exceedingly difficult. However, in some special cases it is possible to develop a rough theory which gives an accurate description of the gamma-radiation that has passed through a thick target. The method presented here is applicable to a wide homogeneous monochromatic beam of gamma-rays impinging at right angles on a flat slab of material. In subsequent papers the photoelectric effect and pair production will be considered, but only Klein-Nishina scattering is considered here. This limits the energy range in which the numerical calculations are valid to a comparatively small range which

[1] Present address: Department of Chemistry, University of Wisconsin, Madison, Wisconsin.
[2] Present address: Argonne National Laboratory, Chicago, Illinois.
[3] Present address: Department of Physics, Yale University, New Haven, Connecticut.
[4] This paper is a thorough revision of and substitution for LADC-70. The method and numerical results have been changed considerably.

fortunately is important for the gamma-rays emitted by many naturally occurring and some artificially produced radioactive substances. The treatment presented here is exact for both the unscattered and the first scattered beams. Because most of the scattered radiation makes only a small angle with the forward direction, the averaging and estimating of the angles does not greatly affect the intensity and distribution of the second, third, etc., scattered components. Then, too, the contribution to the total intensity of the multiple scattered radiation is usually quite small. If the initial beam were not perpendicular to the target, our method of averaging the angles would not be permissible. And if the multiple scattered beam were hundreds of times more intense than the unscattered beam, as in some problems where pair production is included, our approach would need modification.

There are three primary processes by which gamma-radiation is absorbed:

(a) *Photoelectric effect.* The photoelectric effect depends primarily on resonance between the energy of radiation and the energy required to remove the K-electrons from the atom. In this process, the gamma-ray photon is completely absorbed. At energies large compared to the atomic energy terms, the cross section for this process becomes negligible compared to the Compton scattering. This occurs for Al at around 0.1 Mev and for Pb at 1.0 Mev.

(b) *Compton scattering.* In the Compton effect,

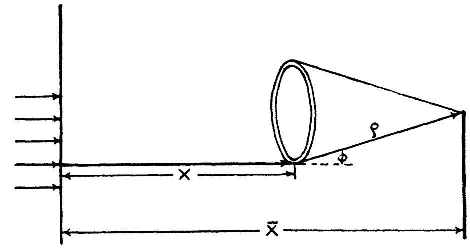

FIG. 1. Geometry of gamma-ray which is scattered through an angle ϕ after passing through the target to a distance x. Here X is the total thickness of the target. (The symbol X with a bar in the bottom of the figure should be capitalized.)

the gamma-ray photon collides with an electron. Its energy is reduced from αm_0c^2 to $\alpha' m_0c^2$, and its direction is changed through an angle ϕ. From the conservation of momentum and energy:

$$\alpha/\alpha' = 1+\alpha(1-\cos\phi). \qquad (1)$$

The cross section per electron, σ, for this process, including the relativistic corrections, is given by the Klein-Nishina formula:

$$\sigma = 2\pi r_0^2\left[\left(\frac{1+\alpha}{\alpha^2}\right)\left(\frac{2(1+\alpha)}{1+2\alpha}-\frac{1}{\alpha}\ln(1+2\alpha)\right)+\frac{1}{2\alpha}\ln(1+2\alpha)-\frac{1+3\alpha}{(1+2\alpha)^2}\right]. \qquad (2)$$

Here r_0 is the classical electron radius, $r_0 = e^2/m_0c^2$. The value of σ decreases with increasing energy. The value of σ/r_0^2 as a function of α are given in Table I.

TABLE I. Cross section as function of energy.

α	σ/r_0^2	α	σ/r_0^2	α	σ/r_0^2
0.15	6.570	1.2	3.3367	3.2	2.0622
0.2	6.1739	1.3	3.2206	3.4	1.9946
0.25	5.839	1.4	3.1149	3.6	1.9322
0.30	5.5516	1.5	3.0181	3.8	1.8743
0.35	5.3021	1.6	2.9290	4.0	1.8204
0.40	5.083	1.7	2.8465	4.2	1.7700
0.45	4.8887	1.8	2.7699	4.4	1.7229
0.50	4.7148	1.9	2.6984	4.6	1.6786
0.55	4.5579	2.0	2.6315	4.8	1.6370
0.60	4.4155	2.1	2.5688	5.0	1.5977
0.65	4.2855	2.2	2.5097	5.5	1.5084
0.70	4.1661	2.3	2.4539	6.0	1.4300
0.75	4.0558	2.4	2.4012	6.5	1.3605
0.80	3.9537	2.5	2.3513	7.0	1.2984
0.85	3.8586	2.6	2.3039	7.5	1.2424
0.90	3.7698	2.7	2.2588	8.0	1.1917
0.95	3.6866	2.8	2.2158	8.5	1.1114
1.00	3.6085	2.9	2.1748	9.0	1.1032
1.1	3.4652	3.0	2.1356	9.5	1.0643
				10.0	1.0284

(c) *Pair production.* No pair production can occur if the energy of the gamma-ray photon is less than the rest energy of an electron pair, $2m_0c^2$. The cross section for this process varies as the square of the atomic number. For Pb, this cross section becomes equal to that for Compton scattering at 5 Mev; for Al, at about 15 Mev.

The total cross section for the primary processes is given by the sum:

$$\sigma_{\text{total}} = \sigma_{\text{photoelectric}} + \sigma_{\text{Compton}} + \sigma_{\text{pair}}.$$

It goes through a minimum value at energy α_m. This occurs at $2\frac{1}{2}$ Mev for Pb and 15 Mev for Al. The nature of the transmitted radiation depends on whether the energy of the primary beam is less or greater than α_m. For convenience, we

TABLE II.

(A) $\alpha = 0.5(m_0c^2)$

μ	$2f(\mu)/r_0^2$	α'	σ_1/r_0^2
1.000	2.0000	0.5000000	4.7148
.995	1.9752	.4987531	4.7187
.990	1.9506	.4975124	4.7232
.985	1.9266	.4962779	4.7270
.980	1.9028	.4950495	4.7311
.975	1.8794	.4938272	4.7354
.970	1.8563	.4926108	4.7393
.960	1.8112	.4901961	4.7475
.950	1.7672	.4878049	4.7556
.940	1.7246	.4854369	4.7637
.930	1.6831	.4830918	4.7717
.920	1.6427	.4807692	4.7795
.910	1.6036	.4784689	4.7874
.900	1.5657	.4761905	4.7954
.880	1.4927	.4716981	4.8108
.860	1.4238	.4672897	4.8263
.840	1.3587	.4629630	4.8415
.820	1.2971	.4587156	4.8570
.800	1.2390	.4545455	4.8722
.780	1.1840	.4504505	4.8872
.760	1.1321	.4464286	4.902
.740	1.0830	.4424779	4.916
.720	1.0365	.4385965	4.931
.700	.99257	.4347826	4.945
.650	.89295	.4255319	4.981
.600	.80631	.4166667	5.016
.550	.73100	.4081633	5.050
.500	.66560	.4000000	5.083
.400	.55950	.3846154	5.148
.300	.47988	.3703704	5.210
.200	.42066	.3571429	5.269
.100	.37712	.3448276	5.326
.000	.34569	.3333333	5.381
−.100	.32364	.3225806	5.435
−.200	.30884	.3125000	5.486
−.300	.29966	.3030303	5.535
−.400	.29477	.2941176	5.583
−.500	.29320	.2857143	5.629
−.600	.29417	.2777778	5.673
−.700	.29700	.2702703	5.717
−.800	.30125	.2631579	5.757
−.900	.30652	.2564103	5.800
−1.000	.31250	.2500000	5.839

(B) $\alpha = 1(m_0c^2)$

μ	$2f(\mu)/r_0^2$	α'	σ_1/r_0^2
1.000	2.0000	1.000000	3.6085
.995	1.9604	.9950249	3.6160
.990	1.9220	.9900990	3.6236
.985	1.8843	.9852217	3.6311
.980	1.8477	.9803922	3.6386
.975	1.8119	.9756098	3.6460
.970	1.7771	.9708738	3.6534
.960	1.7096	.9615385	3.6681
.950	1.6456	.9523810	3.6828
.940	1.5844	.9433962	3.6973
.930	1.5261	.9345794	3.7117
.920	1.4704	.9259259	3.7260
.910	1.4174	.9174312	3.7402
.900	1.3667	.9090909	3.7543
.880	1.2722	.8928571	3.7822
.860	1.1858	.8771930	3.8096
.840	1.1069	.8620690	3.8366
.820	1.0346	.8474576	3.8633
.800	.96834	.8333333	3.8896
.780	.90764	.8196721	3.9155
.760	.85183	.8064516	3.9410
.740	.80049	.7936508	3.9662
.720	.75324	.7812500	3.9911
.700	.70970	.7692308	4.0157
.650	.61502	.7407407	4.0756
.600	.53728	.7142857	4.1337
.550	.47307	.6896552	4.1900
.500	.41976	.6666667	4.2446
.400	.33813	.6250000	4.3490
.300	.28052	.5882353	4.4478
.200	.23930	.5555556	4.5414
.100	.20940	.5263158	4.6303
.000	.18750	.5000000	4.7148
−.100	.17128	.4761905	4.7954
−.200	.15914	.4545455	4.8722
−.300	.14998	.4347826	4.945
−.400	.14299	.4166667	5.016
−.500	.13760	.4000000	5.083
−.600	.13340	.3846154	5.148
−.700	.13008	.3703704	5.210
−.800	.12742	.3571429	5.269
−.900	.12525	.3448276	5.326
−1.000	.12346	.3333333	5.381

(C) $\alpha = 2(m_0c^2)$

μ	$2f(\mu)/r_0^2$	α'	σ_1/r_0^2
1.000	2.0000	2.000000	2.6315
.995	1.9316	1.980198	2.6444
.990	1.8663	1.960784	2.6572
.985	1.8039	1.941748	2.6699
.980	1.7441	1.923077	2.6826
.975	1.6871	1.904762	2.6951
.970	1.6325	1.886792	2.7076
.960	1.5301	1.851852	2.7322
.950	1.4362	1.818182	2.7565
.940	1.3499	1.785714	2.7805
.930	1.2704	1.754386	2.8041
.920	1.1971	1.724138	2.8275
.910	1.1294	1.694915	2.8505
.900	1.0667	1.666667	2.8733
.880	.95505	1.612903	2.9180
.860	.85872	1.562500	2.9616
.840	.77530	1.515152	3.0042
.820	.70272	1.470588	3.0458
.800	.63932	1.428571	3.0864
.780	.58366	1.388889	3.1262
.760	.53467	1.351351	3.1652
.740	.49134	1.315789	3.2033
.720	.45290	1.282051	3.2407
.700	.41870	1.250000	3.2773
.650	.34820	1.176471	3.3658
.600	.29417	1.111111	3.4502
.550	.25205	1.052632	3.5310
.500	.21875	1.000000	3.6085
.400	.17041	.9090909	3.7543
.300	.13792	.8333333	3.8896
.200	.11519	.7692308	4.0157
.100	.098720	.7142857	4.1337
.000	.086419	.6666667	4.2446
−.100	.076981	.6250000	4.3490
−.200	.069563	.5882353	4.4478
−.300	.063610	.5555556	4.5414
−.400	.058740	.5263158	4.6303
−.500	.054688	.5000000	4.7148
−.600	.051265	.4761905	4.7954
−.700	.048334	.4545455	4.8722
−.800	.045794	.4347826	4.945
−.900	.043569	.4166667	5.016
−1.000	.041600	.4000000	5.083

TABLE II.—*Continued.*

	(D) $\alpha=6(m_0c^2)$				(E) $\alpha=10(m_0c^2)$		
μ	$2f(\mu)/r_0^2$	α'	σ_1/r_0^2	μ	$2f(\mu)/r_0^2$	α'	σ_1/r_0^2
1.000	2.0000	6.000000	1.4300	1.000	2.0000	10.000000	1.0284
.995	1.8220	5.825243	1.4563	.995	1.7211	9.523810	1.0626
.990	1.6654	5.660377	1.4822	.990	1.4945	9.090909	1.0959
.985	1.5271	5.504587	1.5077	.985	1.3083	8.695652	1.1285
.980	1.4046	5.357143	1.5327	.980	1.1538	8.333333	1.1604
.975	1.2954	5.217391	1.5574	.975	1.0243	8.000000	1.1917
.970	1.1981	5.084746	1.5817	.970	.91493	7.692308	1.2223
.960	1.0323	4.838710	1.6292	.960	.74195	7.142857	1.2818
.950	.89745	4.615385	1.6753	.950	.61310	6.666667	1.3391
.940	.78668	4.411765	1.7202	.940	.51479	6.250000	1.3943
.930	.69469	4.225352	1.7639	.930	.43825	5.882353	1.4476
.920	.61759	4.054054	1.8064	.920	.37757	5.555556	1.4992
.910	.55239	3.896104	1.8479	.910	.32868	5.263158	1.5492
.900	.49683	3.750000	1.8884	.900	.28875	5.000000	1.5977
.880	.40794	3.488372	1.9664	.880	.22811	4.545455	1.6904
.860	.34081	3.260870	2.0410	.860	.18492	4.166667	1.7782
.840	.28897	3.061224	2.1124	.840	.15306	3.846154	1.8615
.820	.24816	2.884615	2.1810	.820	.12889	3.571429	1.9408
.800	.21549	2.727273	2.2468	.800	.11012	3.333333	2.0165
.780	.18895	2.586207	2.3103	.780	.095242	3.125000	2.0890
.760	.16710	2.459016	2.3714	.760	.083241	2.941176	2.1584
.740	.14891	2.343750	2.4305	.740	.073418	2.777778	2.2252
.720	.13359	2.238806	2.4877	.720	.065272	2.631579	2.2894
.700	.12059	2.142857	2.5430	.700	.058438	2.500000	2.3513
.650	.095502	1.935484	2.6742	.650	.045484	2.222222	2.4970
.600	.077705	1.764706	2.7963	.600	.036480	2.000000	2.6315
.550	.064612	1.621622	2.9106	.550	.029957	1.818182	2.7565
.500	.054688	1.500000	3.0181	.500	.025077	1.666667	2.8733
.400	.040863	1.304348	3.2158	.400	.018376	1.428571	3.0864
.300	.031878	1.53846	3.3944	.300	.014092	1.250000	3.2773
.200	.025690	1.034483	3.5572	.200	.011181	1.111111	3.4502
.100	.021234	0.9375000	3.7069	.100	.0091100	1.000000	3.6085
.000	.017909	0.8571429	3.8455	.000	.0075815	.9090909	3.7543
−.100	.015357	0.7894737	3.9745	−.100	.0064196	.8333333	3.8895
−.200	.013352	0.7317073	4.0952	−.200	.0055152	.7692308	4.0157
−.300	.011745	0.6818182	4.2083	−.300	.0047963	.7142857	4.1337
−.400	.010434	0.6382979	4.3148	−.400	.0042154	.6666667	4.2446
−.500	.0093500	0.6000000	4.4155	−.500	.0037385	.6250000	4.3490
−.600	.0084416	0.5660377	4.5107	−.600	.0033419	.5882353	4.4478
−.700	.0076724	0.5357143	4.6012	−.700	.0030086	.5555556	4.5415
−.800	.0070146	0.5084746	4.6871	−.800	.0027253	.5263158	4.6303
−.900	.0064466	0.4838710	4.7689	−.900	.0024825	.5000000	4.7148
−1.000	.0059522	0.4615385	4.8468	−1.000	.0022728	.4761905	4.7954

shall use m_0c^2 as the unit of energy. If the primary energy, α, is less than α_m, the primary beam is more penetrating than the secondary radiation. However, if α is greater than α_m, a large fraction of the secondary radiation is more penetrating than the primary beam. After such radiation has passed through a sufficiently thick target, its energy will be degraded so that α_m becomes the maximum value of its energy spectrum. At energies of the order of 100 Mev, the problem is made considerably more complicated by the bremsstrahlung radiated from the recoil electrons.

In the remainder of this report, unless otherwise specified, only Compton scattering is considered. This limits the validity of our results to that energy range where α is sufficiently large that the photoelectric effect can be neglected and yet sufficiently small so that the cross section for pair production has not yet become appreciable. For light elements this covers a rather large range (for Al, from 0.1 to more than 10 Mev). For heavier elements the range is somewhat more restricted.

The problems of multiple scattering are still complicated for Compton scattering alone. The energy flux at any point can be broken up into a sum of components according to the number of times the photon has been scattered. If the primary beam is monochromatic and unidirectional, the first scattered beam has an anergy which is a unique function of the azimuthal

Table III. I_1/A.

$\alpha \backslash D$	1	2	5	10	20
0.5	0.1420	0.0878	0.00791	7.75×10^{-5}	4.82×10^{-9}
1.0	.1416	.0884	.00808	8.00×10^{-5}	5.01×10^{-9}
2.0	.1330	.0836	.00771	7.67×10^{-5}	4.83×10^{-9}
6.0	.1089	.0681	.00624	6.17×10^{-5}	3.86×10^{-9}
10.0	.0969	.0603	.00547	5.38×10^{-5}	3.35×10^{-9}

Table IV. $\bar{\mu}$.

$\alpha \backslash D$	1	2	5	10	20	μ_{max} ∞
0.5	0.419	0.512	0.625	0.697	0.754	0.8050
1.0	.519	.598	.694	.754	.802	.8675
2.0	.636	.694	.767	.815	.852	.9183
6.0	.803	.834	.875	.902	.922	.9675
10.0	.861	.883	.913	.932	.946	.9797

angle. For the second and higher scattered beams, the energy spectrum is a function of both the distance and the azimuthal angle. In this report, the energy flux of the first scattered beam of a monochromatic plane wave is calculated rigorously for a number of energies and distances. At energies greater than $m_0c^2/2$, most of the photons are scattered in the forward direction with only a small angle of deflection. If all of the photons were scattered through the same angle, $\bar{\phi}$, the energy flux of the first scattered beam could be expressed in terms of a simple equation, Eq. (12). The results of the rigorous calculations can be expressed in terms of appropriate values of $\bar{\phi}$ to be used in this equation. These values of $\bar{\phi}$ are only slowly varying functions of distance as well as energy.

A rough theory is developed for the penetration of gamma-radiation through a thick target. In this theory, the values of $\bar{\phi}$ determined from calculations of single scattering are used to determine the intensity of the multiple scattered beams. As long as the values of $\bar{\phi}$ are small, this theory should give reasonably good results.

The same treatment and methods which we have applied for Compton scattering can be used when either the photoelectric effect or pair production are appreciable. In these cases, the total cross sections replace the Compton cross sections. The angular dependence of the scattered radiation remains unchanged.

II. FIRST SCATTERED BEAM, PLANE GEOMETRY

Consider a plane-parallel beam of monochromatic gamma-rays which has the intensity A when it enters the scattering medium at $X=0$. In Fig. 1, the trajectory of a photon is shown which travels a distance x before it is scattered; the angle of deflection is ϕ; and it travels the additional distance ρ without being scattered again. We wish to calculate the energy flux due to all such photons which penetrate the target to a distance X after having been scattered once.

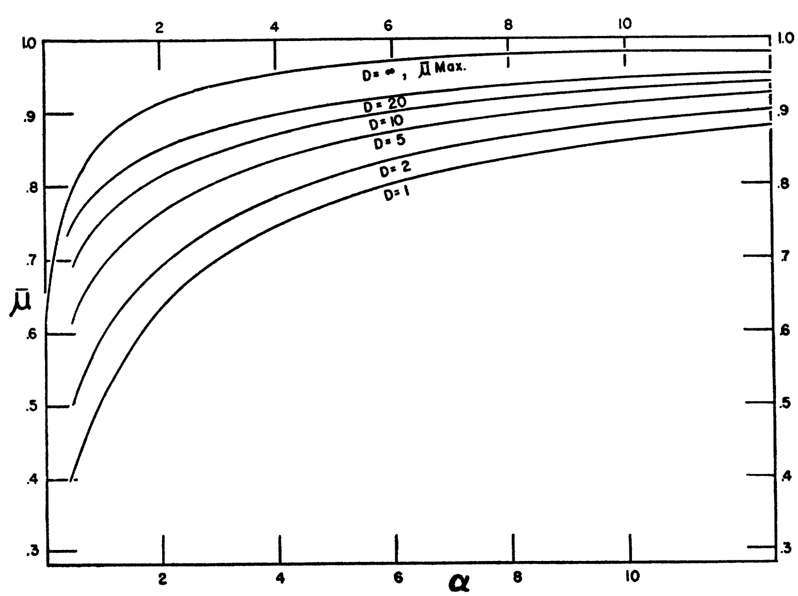

Fig. 2. The variation of $\bar{\mu}$, the cosine of the effective angle of scattering, as a function of the gamma-ray energy, αm_0c^2, for different thicknesses of target, D, measured in mean free paths for the initial gamma-rays.

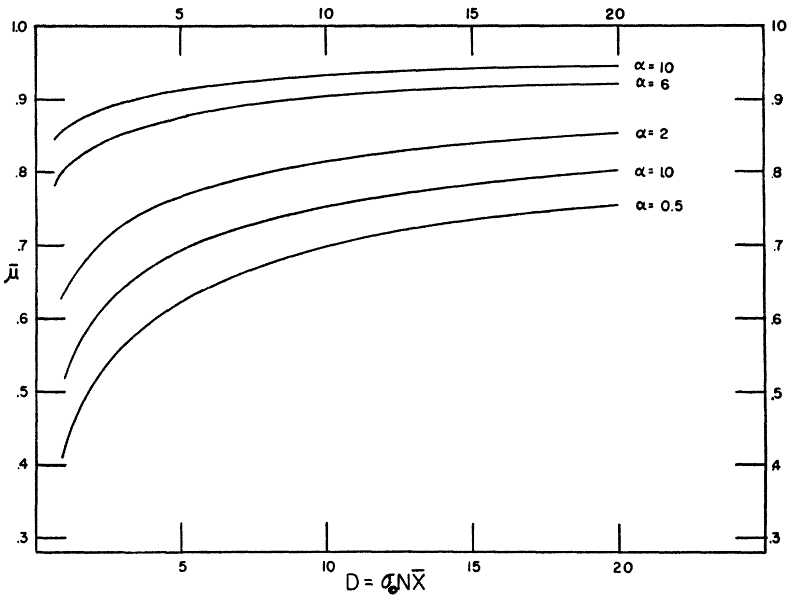

FIG. 3. The variation of $\bar{\mu}$ of the effective angle of scattering, as a function of the target thickness, D, measured in mean free paths of the initial gamma-rays. The different curves are for different values of the gamma-ray energy, αm_0c^2.

The intensity of the unscattered beam at the point, x, is

$$I_0 = A\exp(-\sigma_0 Nx) = A\exp(-\sigma_0 N(X-\rho\mu)). \quad (3)$$

Here σ_0 is the cross section for the initial beam as given in Eq. (2) and N is the density of the electrons in the target.

In order to calculate the intensity of the first scattered beam, it is necessary to introduce the function $f(\mu)$, where μ is the cosine of the angle of scattering, ϕ. Here $f(\mu)$ gives the intensity of the energy scattered in the direction corresponding to μ per unit solid angle, per unit initial intensity, and per electron. According to the Klein-Nishina formulation:

$$f(\mu) = \frac{r_0^2}{2}\left[\frac{1+\mu^2}{(1+\alpha\{1-\mu\})^3} + \frac{\alpha^2(1-\mu)^2}{(1+\alpha\{1-\mu\})^4}\right]. \quad (4)$$

The contribution to the x component of the flux due to the electrons scattered in the volume element dv is then:

$$dI_1 = I_0(Ndv)[\mu f(\mu)/\rho^2]\exp(-\sigma_1 N\rho). \quad (5)$$

Here σ_1 is the cross section given by Eq. (2) for the scattered radiation having the energy α' as given by Eq. (1). In Table II, for various values of the initial energy, α, the function $2f(\mu)/r_0^2$, together with the energy of the scattered photon α' and its total cross section are given as functions of μ, the cosine of the scattering angle. In Eq. (5) the volume element is

$$dv = 2\pi\rho^2 d\mu d\rho. \quad (6)$$

Integrating Eq. (5) gives:

$$I_1 = \int_0^1 d\mu \int_0^{X/\mu} 2\pi A N\mu f(\mu) \times\exp[-\sigma_0 N(X-\rho\mu) - \sigma_1 N\rho]d\rho \quad (7)$$

$$= [2\pi A\exp(-\sigma_0 NX)]\int_0^1 \frac{\mu f(\mu)}{\sigma_1 - \sigma_0\mu} \times[1-\exp(-\{\sigma_1-\sigma_0\mu\}NX/\mu)]d\mu. \quad (8)$$

It is convenient to let

$$D = \sigma_0 NX \quad \text{and} \quad w = (\sigma_1 - \sigma_0\mu)/\sigma_0\mu. \quad (9)$$

Then Eq. (8) may be written in the form:

$$\frac{I_1}{A} = \frac{2\pi e^{-D}}{\sigma_0}\int_0^1 \frac{f(\mu)}{w}[1-e^{-wD}]d\mu. \quad (10)$$

This expression has been numerically integrated for a number of distances and energies as given in Table III.

We would like to interpret these results in terms of an average angle of scattering, $\bar{\phi}$, whose cosine is given by $\bar{\mu}$. According to Eq. (1), the energy of the scattered photon is smaller than the initial photon by the factor, $1/(1+\alpha(1-\bar{\mu}))$.

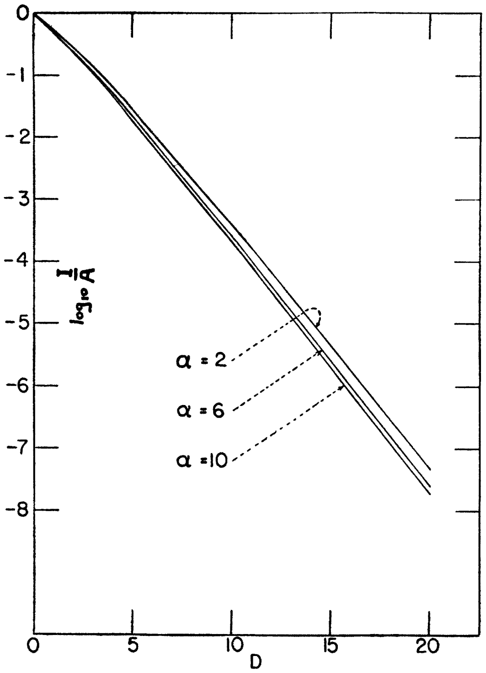

FIG. 4. The variation of the logarithm of the gamma-ray intensity with target thickness, D, measured in mean free paths of the initial gamma-rays.

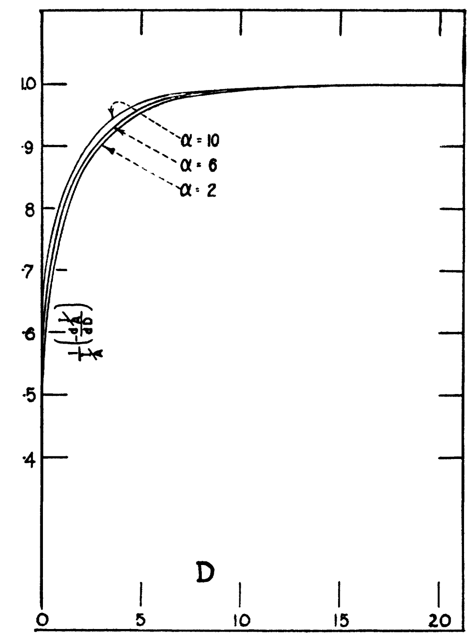

FIG. 5. The ionization produced by gamma-rays is proportional to $-(A/I)(d/dD)(I/A)$ rather than the intensity itself. After passing through a thick target, the two quantities become equal.

If all the scattered photons were deflected through the angle $\bar{\phi}$, this would correspond to having an angular dependence:

$$f(\mu)=\left(\frac{1}{2\pi}\right)\frac{\sigma_0}{1+\alpha(1-\bar{\mu})}\delta(\mu-\bar{\mu}). \qquad (11)$$

Here $\delta(\mu-\bar{\mu})$ is the Dirac delta-function. Under these conditions Eq. (10) can be integrated to give:

$$\frac{I_1}{A}=\frac{e^{-D}}{\bar{w}(1+\alpha(1-\bar{\mu}))}[1-\exp[-\bar{w}D]]. \qquad (12)$$

In this equation, $\bar{w}$ is the function defined in Eq. (9) having $\bar{\mu}$ as its argument. For each value of the energy and the distance there is some value of $\bar{\mu}$ for which Eq. (12) leads to the correct value for I_1/A. Thus, we can use the exact values of I_1/A, given in Table III, to solve for the corresponding values of $\bar{\mu}$, as shown in Table IV and Figs. 2 and 3. For distances large compared to a mean free path, $\bar{\mu}$ remains nearly constant for any given energy.

As the distance becomes very large, the value of $\bar{\mu}$ approaches μ_{max}, where μ_{max} is the value of μ for which $\mu f(\mu)$ has a maximum value. That is, μ_{max} corresponds to that angle for which the greatest amount of energy is scattered per unit angle ϕ (not per unit solid angle). The value of μ_{max} can be obtained for any energy by solving the equation:

$$(3\alpha+3\alpha^2-\alpha^3)(1-\mu_{max})^4 +(3-6\alpha-7\alpha^2+3\alpha^3)(1-\mu_{max})^3 -(9+2\alpha-5\alpha^2)(1-\mu_{max})^2 +8(\alpha+1)(1-\mu_{max})=2.$$

III. MULTIPLE SCATTERING

It would be exceedingly difficult to make an exact calculation of the multiple scattering of gamma-rays. However, the investigations of the single scattering, discussed in the previous section, suggest a method of approximation which should be fairly good. Assume that all gamma-rays of a given energy are scattered through the same angle, $\bar{\phi}$, having the cosine, $\bar{\mu}$, as given in Table IV. At each scattering, the energy of the photon is degraded by the factor $1/(1+\alpha\{1-\mu\})$. It is convenient to let the subscript n denote the number of times that the photon has been scattered. The x component of the energy flux of the nth scattered beam is designated by I_n. The attenuation of the primary unscattered beam is given by the equation:

$$dI_0/dX = -\sigma_0 N I_0 \text{ or } I_0/A = \exp(-\sigma_0 NX). \quad (13)$$

The intensity of the first scattered beam can be determined from the equation:

$$dI_1/dX = \frac{\sigma_0 N I_0}{1+\alpha_0\{1-\bar{\mu}_0\}} - \frac{\sigma_1 N I_1}{\bar{\mu}_0}. \quad (14)$$

Its solution is given by Eq. (12)

It is, of course, necessarily true that the delta-function approximation for the angular dependence of the scattering (Eq. (11)) inserted into the exact integral should lead to the same result as the assumption of a single scattering angle and the differential Eq. (14). The rate of energy loss from the primary beam is given by $\sigma_0 N I_0$, and the fraction of this which goes into radiation is $[1/1+\alpha_0(1-\bar{\mu}_0)]$; the remainder goes into the recoil electrons. The cross section for attenuation of the first scattered beam in the x direction is $(\sigma_1 N/\bar{\mu}_0)$ since it is moving at an angle $\bar{\phi}$.

A new assumption must be introduced in order to estimate the intensity of the second scattered beam. Even if all of the gamma-rays are deflected through the same angle, they make various angles, χ_2, with the x direction depending on the value of their azimuthal angle, ω:

$$\cos\chi_2 = \cos\bar{\phi}_0 \cos\bar{\phi}_1 + \sin\bar{\phi}_0 \sin\bar{\phi}_1 \cos\omega. \quad (15)$$

The rate of change of I_2 can be expressed in terms of the average value of $\cos\chi_2$ (indicated by a bar):

$$\frac{dI_2}{dX} = \frac{\sigma_1 I_1}{\bar{\mu}_0[1+\alpha_1(1-\bar{\mu}_1)]} - \frac{\sigma_2 I_2}{\langle\cos\chi_2\rangle_{\text{Av}}}. \quad (16)$$

The correct method of averaging $\cos\chi_2$ in the sense required for the intensity of the multiple

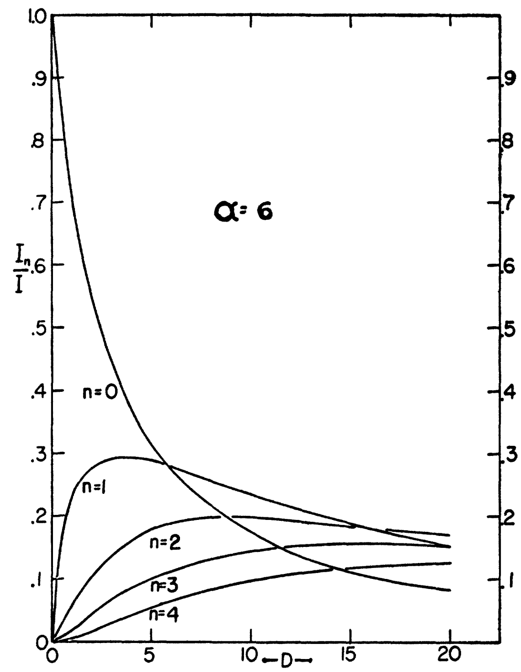

FIG. 6. For $\alpha = 6$, the relative intensity of the unscattered, singly scattered, doubly scattered, etc., beams of gamma-rays is plotted as a function of the target thickness D.

scattered radiation gives:

$$\exp[-\sigma_2 NX/\langle\cos\chi_2\rangle_{\text{Av}}] = \frac{1}{2\pi}\int_0^{2\pi} \exp[-\sigma_2 NX/\cos\chi_2]d\omega_2.$$

This integral is difficult to evaluate and the resulting expression is complicated. If the initial beam were not perpendicular to the target, this complication could not be avoided. However, in our case where the angles of scattering are small, the results are not sensitive to the method of averaging and we take

$$\langle\cos\chi_2\rangle_{\text{Av}} = \frac{1}{2\pi}\int_0^{2\pi} \cos\chi_2 d\omega_2 = \cos\bar{\phi}_0 \cos\bar{\phi}_1 = \bar{\mu}_0\bar{\mu}_1. \quad (17)$$

Equation (16) can then be integrated to give:

$$I_2/A = \left\{\frac{\sigma_0\sigma_1/\bar{\mu}_0}{[1+\alpha_0(1-\bar{\mu}_0)][1+\alpha_1(1-\bar{\mu}_1)][(\sigma_1/\bar{\mu}_0)-\sigma_0]}\right\}\left[\left(\frac{\exp[-\sigma_0 NX]-\exp[-\sigma_2 NX/\bar{\mu}_0\bar{\mu}_1]}{(\sigma_2/\bar{\mu}_0\bar{\mu}_1)-\sigma_0}\right) - \left(\frac{\exp[-\sigma_1 NX/\bar{\mu}_0]-\exp[-\sigma_2 NX/\bar{\mu}_0\bar{\mu}_1]}{(\sigma_2/\bar{\mu}_0\bar{\mu}_1)-(\sigma_1/\bar{\mu}_0)}\right)\right]. \quad (18)$$

TABLE V.

	D	1	2	5	10	20
	I/A	0.5578	0.2833	0.02817	4.028×10^{-4}	4.666×10^{-8}
	$-d(I/A)/dD$	.4240	.2424	.02687	3.983×10^{-4}	4.661×10^{-8}
	I_0/I	.6596	.4776	.2392	.1127	.0442
$\alpha_0=2$	I_1/I	.2386	.295	.2728	.1907	.1035
$D=2.6315Nr_0^2\bar{X}$	I_2/I	.0722	.136	.2064	.2002	.1448
	I_3/I	.0213	.0558	.1285	.1659	.1552
	I_4/I	.0063	.022	.0725	.1201	.1414
	I_5/I	.0018	.0085	.0389	.0802	.1159
	I/A	.5164	.2462	.02131	2.624×10^{-4}	2.516×10^{-8}
	$\frac{d(I/A)}{dD}$	.4067	.2156	.02054	2.604×10^{-4}	2.514×10^{-8}
	I_0/I	.7124	.5496	.3162	.1730	.0819
$\alpha_0=6$	I_1/I	.2109	.277	.2914	.2355	.1531
$D=1.4300Nr_0^2\bar{X}$	I_2/I	.0546	.107	.1782	.1978	.1696
	I_3/I	.016	.041	.1007	.1425	.1527
	I_4/I	.0045	.016	.0549	.0953	.1255
	I_5/I	.001	.006	.0288	.0615	.0955
	I/A	.4973	.2295	.01850	2.133×10^{-4}	1.902×10^{-8}
	$-d(I/A)/dD$	.4056	.2056	.01799	2.122×10^{-4}	1.901×10^{-8}
	I_0/I	.7398	.5895	.3642	.2128	.1084
$\alpha_0=10$	I_1/I	.195	.262	.2937	.2518	.1761
$D=1.0284Nr_0^2\bar{X}$	I_2/I	.0469	.0932	.1677	.1995	.1850
	I_3/I	.013	.034	.0851	.1301	.1526
	I_4/I	.0038	.013	.0440	.0810	.1140
	I_5/I	.001	.0048	.0226	.0504	.0824

The intensity of the higher scattered beams can be estimated in much the same way. Thus:

$$\frac{dI_n}{dX}=\frac{N\sigma_{n-1}I_{n-1}}{\langle\cos\chi_{n-1}\rangle_{\text{Av}}[1+\alpha_{n-1}(1-\bar{\mu}_{n-1})]}-\frac{N\sigma_nI_n}{\langle\cos\chi_n\rangle_{\text{Av}}}. \qquad (19)$$

Here the angle χ_n that the nth scattered beam make with the x axis is given by

$$\cos\chi_n=\cos\chi_{n-1}\cos\bar{\phi}_{n-1}+\sin\chi_{n-1}\sin\bar{\phi}_{n-1}\sin\omega_{n-1}, \qquad (20)$$

and the average value of its cosine is

$$\langle\cos\chi_n\rangle_{\text{Av}}=\frac{1}{2\pi}\int_0^{2\pi}\cos\chi_n d\omega_{n-1}=\langle\cos\chi_{n-1}\rangle_{\text{Av}}\bar{\mu}_{n-1}=\bar{\mu}_0\bar{\mu}_1\cdots\bar{\mu}_{n-1}. \qquad (21)$$

It is convenient to define:

$\sigma_0'=\sigma_0$ and for all other values of n,

$$\sigma_n'=\sigma_n/\bar{\mu}_0\bar{\mu}_1\cdots\bar{\mu}_{n-1}, \qquad (22)$$

and

$$\sigma_{sn}'=\sigma_n'/[1+\alpha_n(1-\bar{\mu}_n)].$$

The intensity of the nth beam may be written in the form:

$$I_n/A=\sum_{j=0}^{n}K_{j,n}\exp(-\sigma_j'NX). \qquad (23)$$

Substituting this expression back into Eq. (19)

$$-\sum_{j=0}^{n}\sigma_j'K_{j,n}N\exp(-\sigma_j'NX)=\sum_{j=0}^{n-1}\sigma'_{s(n-1)}K_{j,n-1}N\exp(-\sigma'NX)-\sum_{j=0}^{n-1}\sigma_n'NK_{j,n}\exp(-\sigma_j'NX). \qquad (24)$$

Since this equation must be true for all values of X, we can equate the coefficients of each of the exponentials. This leads to the recursion formula:

$$K_{j,n}=\left(\frac{\sigma'_{s(n-1)}}{\sigma_n'-\sigma_j'}\right)K_{j,n-1}, \quad (n\neq j). \qquad (25)$$

The values of $K_{n,n}$ are determined by the initial conditions. If the original beam is monochromatic, then:

$$K_{0,0}=1 \quad\text{and}\quad K_{n,n}=-\sum_{j=0}^{n-1}K_{j,n}. \qquad (26)$$

These recursion formulae are sufficient to determine all of the $K_{j,n}$.

The following procedure can be used for making numerical calculations:

1. Make a table with the column headings:

n	α_n	σ_n	$\bar{\mu}_n$	$1+\alpha_n(1-\bar{\mu}_n)$	σ_n'	σ_{sn}'

For the initial beam, $n=0$, the energy α_0 is given and the corresponding value of σ_0 can be found (e.g. from Table II). The value of $D_0=\sigma_0 N\bar{X}$ follows immediately. From Table II or Figs. 2 and 3, the value of $\bar{\mu}_0$ corresponding to α_0 and D_0 can be estimated. For the subsequent values of $\bar{\mu}_n$ we take as arguments α_n and D_0. The energy in each new row is determined by the recursion formula:

$$\alpha_n=\alpha_{n-1}/[1+\alpha_{n-1}(1-\bar{\mu}_{n-1})]. \qquad (27)$$

And the formation of the rest of the table is obvious.

2. The $K_{j,n}$ are easy to determine with the help of the above table. Starting with $K_{0,0}=1$, the recursion formula, Eq. (25), determines successively each of the $K_{0,n}$. Then by virtue of Eq. (26), $K_{1,1}=-K_{0,1}$ and Eq. (25) determines each of the $K_{1,n}$ successively in terms of $K_{1,1}$. Then $K_{2,2}=-K_{0,2}-K_{1,2}$, etc.

The most arbitrary feature of this treatment of multiple scattering is the assumption that $\bar{\mu}_n$ should be characteristic of the energy α_n and of the total thickness of the target in mean free paths of the primary energy $D_0=\sigma_0 N\bar{X}$. This assumption cannot be justified in a rigorous fashion, but rather we must argue that it cannot lead to a large error since $\bar{\mu}_n$ has only a slight dependence on distance.

IV. TOTAL INTENSITY OF GAMMA-RAYS

The total intensity of the gamma-rays reaching a given point is made up of the sum of the intensities of the primary, singly scattered, doubly scattered, etc., beams. This I, the x component of the total energy flux, may be written:

$$I/A=\sum_{j=0}^{\infty} c_j \exp(-\sigma_j' N\bar{X}), \qquad (28)$$

where

$$c_j=\sum_{n=j}^{\infty} K_{j,n}. \qquad (29)$$

From the standpoint of both the physiological effects and the measurement of radiation, the gradient of the flux rather than the flux itself is of primary interest. The gradient of the flux determines the specific ionization or roentgens per unit time. This is easily obtained by differentiating Eq. (28). Here the c_j are regarded as constants since they only depend on X very weakly through the weak dependence of $\bar{\mu}_n$ on distance. Thus:

$$-\frac{d(I/A)}{dNX}=\sum_{j=0}^{\infty} c_j\sigma_j' \exp(-\sigma_j' NX). \qquad (30)$$

Usually the initial beam has a smaller cross section than the scattered beams. Thus after the radiation has passed through a sufficient thickness of target, only the terms with $j=0$ contribute and

$$I/A=c_0 \exp(-\sigma_0 NX)=\frac{d(IA)}{d(\sigma_0 NX)}. \qquad (31)$$

TABLE VI.

D		1	2	5	10	20
$\alpha_0=2$ (1 Mev)	α_1	1.1574	1.2407	1.3643	1.4599	1.5432
	α_2	.7581	.8506	.9977	1.1184	1.2329
	α_3	.5420	.6247	.7642	.8856	1.0082
	α_4	.4138	.4843	.6081	.7212	.8409
	α_5	.3314	.3912	.4993	.6016	.7138
	α_6		.3261	.4205	.5122	.6155
	α_7		.2784	.3617	.4437	.5381
	α_8		.2422	.3164	.3901	.4761
	α_9			.2806	.3472	.4257
	α_{10}			.2517	.3123	.3842
	α_{11}			.2280	.2834	.3495
	α_{12}			.2082	.2591	.3201
	α_{13}				.2385	.2950
$\alpha_0=6$ (3 Mev)	α_1	2.7498	3.0060	3.4826	3.7783	4.0872
	α_2	1.4534	1.6810	2.0918	2.4755	2.8550
	α_3	.9072	1.0765	1.4129	1.7372	2.0865
	α_4	.6251	.7560	1.0276	1.2903	1.5996
	α_5	.4644	.5671	.7837	1.0040	1.2632
	α_6	.3646	.4467	.6214	.8054	1.0254
	α_7	.2981	.3652	.5088	.6633	.8534
	α_8	.2512	.3075	.4275	.5586	.7234
	α_9	.2166	.2647	.3670	.4795	.6230
	α_{10}			.3204	.4183	.5441
	α_{11}			.2838	.3698	.4810
$\alpha_0=10$ (5 Mev)	α_1	4.1841	4.6083	5.3476	5.9524	6.4935
	α_2	2.0064	2.3731	3.0788	3.7495	4.4149
	α_3	1.1605	1.4074	1.9165	2.4593	3.0619
	α_4	.7596	.9423	1.3200	1.7280	2.2154
	α_5	.5429	.6794	.9702	1.2931	1.6827
	α_6		.5191	.7461	1.0059	1.3307
	α_7		.4148	.5957	.8068	1.0799
	α_8		.3428	.4905	.6643	.8947
	α_9			.4140	.5594	.7550
	α_{10}			.3567	.4801	.6476
	α_{11}			.3125	.4187	.5635

This corresponds to a radiation spectrum which no longer depends on distance, an equilibrium having been reached between the primary and all of the successively scattered beams. If only the primary beam contributed, c_0 would equal to one. Instead, it usually has a value of the order of two.

Numerical calculations were made for α equal to 2, 6, and 10. The values for I/A and $[-d(I/A)/dD]/[I/A]$ as a function of the target thickness D are shown in Figs. 4 and 5, respectively. For $\alpha=6$, the contributions to the intensity of the gamma-rays which have been scattered a various number of times are plotted against D in Fig. 6. These results are shown in Table V for all three energies. In Table VI, the energies of the various scattered beams are given.

The numerical results can also be expressed in terms of the following formulae which were obtained by curve fitting:

$$\alpha=2 \qquad I/A=(1+0.487D+0.030D^2)e^{-D},$$

$$(D=2.6315Nr_0{}^2\bar{X}) \qquad -\frac{d}{dD}(I/A)=I/A-(0.527+0.073D^2)e^{-3D/2};$$

$$\alpha=6 \qquad I/A=(1+0.400D+0.0080D^2)e^{-D},$$

$$(D=1.4300Nr_0{}^2\bar{X}) \qquad -\frac{d}{dD}(I/A)=I/A-(0.438+0.021D+0.0343D^2)e^{-3D/2};$$

$$\alpha=10 \qquad I/A=(1+0.33D+0.0040D^2)e^{-D},$$

$$(D=1.0284Nr_0{}^2\bar{X}) \qquad -\frac{d}{dD}(I/A)=I/A-(0.38+0.01D+0.02D^2)e^{-3D/2}.$$

Considerable difficulty was encountered with the convergence of the contributions of the various scattered beams to the intensity when the values of $\mu_{\max}$ were used corresponding to passage through an infinitely thick target.

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