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Orbits of Charged Particles in Constant Fields

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Solutions of the relativistic equations of motion of a charged particle in a constant electromagnetic field are obtained in terms of a Lorentz transformation determined by the tensor describing the field. Some results of O. Veblen, J. W. Givens, and the author on Lorentz transformations are summarized in Appendix A and are used to obtain explicit expressions for the orbits. The methods used may be applied to the orbit of a charged particle in the field of a plane wave. The latter orbits are discussed in detail in Appendix B.

1. INTRODUCTION

It is the purpose of this paper to classify and discuss some properties of the solutions of the differential equations determining the orbits of charged particles in uniform external electromagnetic fields when the interaction between the field of the particle and the external field is neglected. In case the external field varies in space and/or time, the results given here may be used as a zeroth approximation to start an iterative procedure for determining the orbit.

The discussion will be based on the observation that the four-dimensional antisymmetric tensor, $f_{\mu\nu}$, which describes the external field determines a family of Lorentz matrices, L, of which it is an infinitesimal generator. It is a consequence of the equations of motion that the four-dimensional velocity vector at any point of the orbit is related to its initial value by means of this family of Lorentz transformations.

Some hitherto unpublished results of O. Veblen, J. W. Givens, and the author give a complete classification of these Lorentz transformations in terms of the tensor $f_{\mu\nu}$ as well as closed expressions for L. These results are summarized Appendix A. They are used to give a classification of the various cases which may arise and to obtain various properties of the orbits.

We shall formulate our problems in terms of the four-dimensional Minkowski space with the metric

$$d\sigma^2 = -ds^2 = g_{\mu\nu} dx^\mu dx^\nu = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2, \quad (1.1)$$
$$x^1 = x, \quad x^2 = y, \quad x^3 = z, \quad x^4 = ct.$$

The four-dimensional velocity vector will then be denoted by

$$V^\mu = \frac{dx^\mu}{ds}, \quad (1.2)$$

and as a consequence of (1.1) will satisfy

$$g_{\mu\nu} V^\mu V^\nu = V^\mu V_\mu = -1. \quad (1.3)$$

If the three-dimensional components of velocity are v^i $(i = 1, 2, 3)$ and if

$$v^2 = \sum_{i=1}^{3} (v^i)^2,$$

then

$$V^i = \frac{v^i/c}{(1 - v^2/c^2)^{\frac{1}{2}}} \quad V^4 = \frac{1}{(1 - v^2/c^2)^{\frac{1}{2}}}.$$

In terms of the tensor $f^{\mu\nu} = -f^{\nu\mu}$,

$$\|f^{\mu\nu}\| = \left\| \begin{matrix} 0 & H_3 & -H_2 & -E_1 \\ -H_3 & 0 & H_1 & -E_2 \\ H_2 & -H_1 & 0 & -E_3 \\ E_1 & E_2 & E_3 & 0 \end{matrix} \right\|, \quad (1.4)$$

where E_i and H_i are the components of the electric and magnetic field strengths, respectively, in the direction of x^i, Maxwell's equations are

$$\frac{\partial f^{\mu\nu}}{\partial x^\nu} = j^\mu, \quad \frac{\partial f_{\mu\nu}}{\partial x^\sigma} + \frac{\partial f_{\nu\sigma}}{\partial x^\mu} + \frac{\partial f_{\sigma\mu}}{\partial x^\nu} = 0, \quad (1.5)$$

where j^μ is the four-dimensional current vector,

$$f_{\mu\nu} = g_{\mu\sigma} f^{\sigma\tau} g_{\tau\nu},$$

and, hence,

$$\|f_{\mu\nu}\| = \left\| \begin{matrix} 0 & H_3 & -H_2 & E_1 \\ -H_3 & 0 & H_1 & E_2 \\ H_2 & -H_1 & 0 & E_3 \\ -E_1 & -E_2 & -E_3 & 0 \end{matrix} \right\|. \quad (1.6)$$

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The Lorentz ponderomotive force equations are then

$$\frac{d^2x^\mu}{ds^2}=\lambda f^\mu{}_\nu\frac{dx^\nu}{ds}, \tag{1.7}$$

where

$$\|f^\mu{}_\nu\|=\begin{Vmatrix} 0 & H_3 & -H_2 & E_1 \\ -H_3 & 0 & H_1 & E_2 \\ H_2 & -H_1 & 0 & E_3 \\ E_1 & E_2 & E_3 & 0 \end{Vmatrix}=F, \tag{1.8}$$

and

$$\lambda=e/m_0c^2. \tag{1.9}$$

It is our purpose to discuss the solutions of Eqs. (1.7). They may be written in matrix form as

$$\frac{dV}{ds}=\lambda FV, \tag{1.10}$$

where $V=\|V^\mu\|$ and is a one column matrix.

In case the tensor $f_\nu{}^\mu$ is constant, that is, independent of x^μ or s, the solution of (1.10) may br written as

$$V=L(s)V_0, \tag{1.11}$$

where V_0 is the one column matrix representing a constant vector, the initial four-dimensional velocity vector, and where

$$L(s)=e^{\lambda sF}=1+\lambda sF+\frac{\lambda^2s^2}{2!}F^2+\cdots. \tag{1.12}$$

$L(s)$ is a proper Lorentz matrix as follows from the antisymmetry of $f_{\sigma\tau}$. That is

$$g_{\sigma\tau}f_\rho{}^\tau=-g_{\rho\tau}f_\sigma{}^\tau,$$

or

$$(GF)'=-GF, \tag{1.13}$$

where

$$G=\|g_{\sigma\tau}\|=\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix},$$

and the prime denotes the transposed matrix.

The proof of this statement follows from the fact that

$$L'(-s)=GL(s)G^{-1}$$

as a consequence of (1.13). Since

$$L(-s)L(s)=1,$$

we then have

$$G=L'(s)GL(s),$$

which is the condition that $L(s)$ be an extended Lorentz matrix. Now $L(\frac{1}{4}s)$ also satisfies this condition. Hence $L(\frac{1}{2}s)=L^2(\frac{1}{4}s)$ has determinant one and either satisfies $L^4{}_4>0$ in the coordinate system in which (1.1) holds or $L^4{}_4<0$. In any case $L(s)=L^2(\frac{1}{2}s)$ must satisfy $L^4{}_4>0$. Thus $L(s)$ is a proper Lorentz matrix. In fact every such matrix may be expressed in the form (1.12).

When $L(s)$ is determined, the orbit may be obtained from (1.11) by a quadrature.

It follows from Eqs. (1.10) and (1.11) that the particle undergoes constant acceleration in the sense of special relativity. That is, at every point of the orbit, the acceleration, as measured by an observer instantaneously at rest with respect to the particle, is constant. If the constant Lorentz transformation which carries the initial velocity vector $V_0{}^\mu$ into $V_1{}^\mu=\delta_1{}^\mu$ is denoted by M, then the Lorentz transformation which transforms the particle to rest at any point of its orbit is $M^{-1}L^{-1}(s)$. The acceleration in this coordinate system is given by the spatial components of the vector

$$b^*=M^{-1}L^{-1}\frac{dV}{ds}=\lambda M^{-1}L^{-1}FLMV_1=\lambda M^{-1}FMV_1.$$

Hence b^* is a constant vector. The magnitude of the acceleration is given by

$$b^{*\sigma}b_\sigma{}^*=\lambda^2f_{\lambda\beta}f_\gamma{}^\beta V_0{}^\lambda V_0{}^\gamma. \tag{1.14}$$

2. THE MATRIX $L(s)$

Since the matrix F must satisfy its reduced characteristic equation, the matrix $L(s)$ is linearly expressible in terms of F, F^2, F^3, and F^4, the coefficients of these matrices being functions of s and certain scalars determined by the tensor $f_\nu{}^\mu$. We proceed to the determination of $L(s)$ in these terms after introducing some auxillary quantities.

We define

$$f^{\dagger\sigma\tau}=\tfrac{1}{2}E^{\sigma\tau\lambda\mu}f_{\lambda\mu}=g^{\sigma\rho}g^{\tau\nu}f^\dagger{}_{\rho\nu}=g^{\sigma\rho}g^{\tau\nu}\tfrac{1}{2}E_{\rho\nu\lambda\mu}f^{\lambda\mu}, \tag{2.1}$$

where

$$E^{\sigma\tau\lambda\mu}=\frac{1}{(g)^{\frac{1}{2}}}\epsilon^{\sigma\tau\lambda\mu},\quad E_{\sigma\tau\lambda\mu}=(g)^{\frac{1}{2}}\epsilon_{\sigma\tau\lambda\mu}, \tag{2.2}$$

g is the determinant of the matrix G, and

$$\epsilon_{\sigma\tau\lambda\mu}=\epsilon^{\sigma\tau\lambda\mu}=\begin{cases}1 \text{ if } \sigma\tau\lambda\mu \text{ is an even permutation of } 1, 2, 3, 4,\\ -1 \text{ if } \sigma\tau\lambda\mu \text{ is an odd permutation of } 1, 2, 3, 4,\\ 0 \text{ otherwise.}\end{cases} \tag{2.3}$$

The ϵ's are tensor densities which have the same value in each coordinate system. In the coordinate system in which (1.1) holds we have

$$\|f^{\dagger\sigma\tau}\|=\begin{Vmatrix} 0 & -iE_3 & +iE_2 & -iH_1\\ iE_3 & 0 & -iE_1 & -iH_2\\ -E_2 & iE_1 & 0 & -iH_3\\ iH_1 & iH_2 & iH_3 & 0\end{Vmatrix}. \tag{2.4}$$

The tensor $f^{\dagger\sigma\tau}$ will be referred to as the dual tensor of $f^{\sigma\tau}$. It may readily be verified that the dual operation is of period two, that is,

$$(f^\dagger)^{\dagger\sigma\tau}=f^{\sigma\tau}.$$

We define

$$F^\dagger=\|f^{\dagger\sigma}{}_\tau\|=\|f^{\dagger\sigma\tau}\|G,$$

and in virtue of (2.4) we have

$$F^\dagger=\begin{Vmatrix} 0 & -iE_3 & iE_2 & iH_1\\ iE_3 & 0 & -iE_3 & iH_2\\ -iE_2 & iE_1 & 0 & iH_3\\ H_1 & iH_2 & iH_3 & 0\end{Vmatrix}. \tag{2.5}$$

From (1.4) and (2.5) it may readily be verified that

$$FF^\dagger=F^\dagger F=ib1, \tag{2.6}$$

where 1 is the four-dimensional unit matrix

$$ib=i\sum_{i=1}^{3}E_iH_i=-\tfrac{1}{4}f^{\sigma\tau}f^\dagger{}_{\sigma\tau}, \tag{2.7}$$

$$\det F=(ib)^2=(\tfrac{1}{4}f^{\sigma\tau}f^\dagger{}_{\sigma\tau})^2, \tag{2.8}$$

and

$$F^2+F^{\dagger 2}=a1, \tag{2.9}$$

where

$$a=\sum_{i=1}^{3}(E_i{}^2-H_i{}^2)=-\tfrac{1}{2}f^{\sigma\tau}f_{\sigma\tau}=-\tfrac{1}{2}f^{\dagger\sigma\tau}f^\dagger{}_{\sigma\tau}. \tag{2.10}$$

Equations (2.6) and (2.9) enable us to express any polynomial in the matrices F, F^2, F^3, and F^4 as one in terms of F, $F^\dagger$, F^2, and $F^{\dagger 2}$.

In case $a=b=0$, which we shall call the parabolic case, it follows from (2.6) and (2.7) that

$$F^3=-F^\dagger(F^\dagger F)=0=F^n,\quad n\geqslant 3.$$

Hence, in this case

$$L(s)=1+\lambda sF+\frac{\lambda^2s^2}{2!}F^2. \tag{2.11}$$

Before discussing the orbit in the parabolic case we obtain the expression analogous to (2.11) when a and b do not both vanish, the so-called non-parabolic case.

It is somewhat more convenient to deal with two tensors $S^{\sigma\tau}$ and $S^{\dagger\sigma\tau}$ defined in terms of $f^{\sigma\tau}$ and $f^{\dagger\sigma\tau}$ as follows:

$$S=\alpha F+i\beta F^\dagger,\quad S^\dagger=i\beta F+\alpha F^\dagger, \tag{2.12}$$

where α and β are related to a and b by

$$\alpha+i\beta=\frac{1}{(a^2+4b^2)^{\frac{1}{2}}}[(\tfrac{1}{2}((a^2+4b^2)^{\frac{1}{2}}+a))^{\frac{1}{2}} -i(\tfrac{1}{2}((a^2+4b^2)^{\frac{1}{2}}-a))^{\frac{1}{2}}]=\frac{1}{(a+2ib)^{\frac{1}{2}}} \tag{2.13}$$

the positive square root of a^2+4b^2 is to be taken, and the signs of α and β are to be taken such that $\alpha\beta>0$. The matrix S is thus determined up to sign.

Equations (2.12) may be inverted to give

$$F=\nu S+i\theta S^\dagger,\quad F^\dagger=i\theta S+\nu S^\dagger, \tag{2.14}$$

where

$$\nu=\frac{\alpha}{\alpha^2+\beta^2}=(\tfrac{1}{2}((a^2+4b^2)^{\frac{1}{2}}+a))^{\frac{1}{2}}$$

$$\theta=\frac{-\beta}{\alpha^2+\beta^2}=(\tfrac{1}{2}((a^2+4b^2)^{\frac{1}{2}}-a))^{\frac{1}{2}}. \tag{2.15}$$

It follows from Eqs. (2.6), (2.9), (2.12), and (2.13) that

$$S^2+S^{\dagger 2}=1,\quad \text{trace}S^2=\text{trace}S^{\dagger 2}=2,\quad SS^\dagger=0, \tag{2.16}$$

and, hence,

$$\begin{aligned}S^3&=S^2S=(1-S^{\dagger 2})S=S,\\ S^{\dagger 3}&=S^{\dagger 2}S^\dagger=(1-S^2)S^\dagger=S^\dagger.\end{aligned} \tag{2.17}$$

Hence, substituting the first of (2.14) into (1.12), we obtain

$$L(s)=\sinh\nu\lambda sS+\cosh\nu\lambda sS^2 +i\sin\theta\lambda sS^\dagger+\cos\theta\lambda sS^{\dagger 2}. \tag{2.18}$$

or, alternatively,

$$L(s)=\frac{1}{(a^2+4b^2)^{\frac{1}{2}}}[(\nu\sinh\nu\lambda s+\theta\sin\theta\lambda s)F-(\theta\sinh\nu\lambda s-\nu\sin\lambda\theta s)F^{\dagger}]+\frac{1}{a^2+4b^2}[\nu^2\cosh\nu\lambda s-\theta^2\cos\theta\lambda s)F^2-(\theta^2\cosh\nu\lambda s-\nu^2\cos\theta\lambda s)F^{\dagger 2}]+\frac{4b^2}{a^2+4b^2}1. \quad (2.19)$$

Equations (2.11) and (2.18) (or (2.19)) enable one to give an explicit expression for the orbit for any fields described by F. For, if these expressions are substituted into Eq. (1.11), the resulting expression may be integrated to give $x^\mu(s)$. It is evident from the latter equation that the proper vector of the matrix L will play an important role in the discussion of the various possible types of orbits. For example, if the time-like vector V_0^μ could coincide with a proper vector of the matrix L, then the orbit would be particularly simple.

In Appendix A, the proper vectors of L are determined from the coefficients of the matrix F. It follows from that work that all possible Lorentz matrices none of which is the identity fall into one of the following four classes

(1) $E^2=H^2$, $E\cdot H=0$ $(a=b=0)$,

(2) $H^2>E^2$, $E\cdot H=0$ $(a<0,\ b=0;\ \nu=0)$,

(3) $E^2>H^2$, $E\cdot H=0$ $(a>0,\ b=0;\ \theta=0)$,

(4) $E^2-H^2\neq 0$, $E\cdot H\neq 0$ $(a\neq 0,\ b\neq 0)$.

The Lorentz matrices of the first class have been called parabolic.

The classes differ in the range allowed for the proper values of L and the nature of the corresponding proper vectors. By choosing the coordinate system of the Minkowski space appropriately (so that the invariant vectors associated with L have prescribed values), any Lorentz matrix of a given class may be reduced to a canonical form for that class. Hence it suffices to discuss Eq. (1.11) when L is assumed to have its canonical form. An equivalent procedure is to express all vectors as linear combinations of a set of four linearly independent invariant vectors and determine how the coefficients of this expansion vary with the parameter s. We shall find it convenient to use both methods below.

It follows from the material in the appendix that a Lorentz transformation, not the identity, can have a time-like vector as a proper vector if and only if it belongs to class (2). Hence only in this case may we expect the orbits to have particularly simple properties for a particle whose initial velocity vector is chosen to be the proper vector of L.

3. ORBITS IN THE PARABOLIC CASE

In this case $E^2=H^2$, $E\cdot H=0$ and we may always choose our coordinate system so that

$$E_i=+H\delta_i^{\,1},\quad H_i=H\delta_i^{\,2}.$$

Then

$$F=\|f^\sigma{}_\tau\|=H\begin{Vmatrix}0&0&-1&+1\\0&0&0&0\\1&0&0&0\\+1&0&0&0\end{Vmatrix}, \quad (3.1)$$

$$F^2=H^2\begin{Vmatrix}0&0&0&0\\0&0&0&0\\0&0&-1&1\\0&0&-1&1\end{Vmatrix}.$$

Equation (1.11), may be written as

$$V=\left(1+\lambda sF+\frac{\lambda^2s^2}{2}F^2\right)V_0 \quad (3.2)$$

in virtue of (2.11). The integral of this is

$$x=x_0+sV_0+\frac{\lambda}{2}s^2FV_0+\frac{\lambda^2}{3!}s^3F^2V_0, \quad (3.3)$$

where x_0 is the matrix of one column $x_0=\|x_0^\mu\|$ and x_0^μ are the constant space-time coordinates of the position of the particle when $s=0$.

In the coordinate system where (3.1) holds we have

$$x^1-x_0^1=s\left(V_0^1+\frac{\lambda s}{2}H(V_0^4-V_0^3)\right),$$

$$x^2-x_0^2=sV_0^2,$$

$$x^3-x_0^3=s\left(\frac{\lambda s}{2}HV_0^1+V_0^3+\frac{\lambda^2s^2}{6}H^2(V_0^4-V_0^3)\right),$$

$$x^4-x_0^4=s\left(+\frac{\lambda s}{2}HV_0^1+\frac{\lambda^2s^2H^2}{6}(V_0^4-V_0^3)+V_0^4\right).$$

From these equations it is evident that $|x^i - x_0{}^i|$ $(i=1, 2, 3)$ increases as s increases regardless of the choice of $V_0{}^\sigma$. Hence there cannot be a spatially "closed" orbit. We also have

$$V^1 = V_0{}^1 + \lambda s H(V_0{}^4 - V_0{}^3),$$

$$V^2 = V_0{}^2,$$

$$V^3 = \lambda H s V_0{}^1 + \frac{\lambda^2 s^2}{2} H^2 (V_0{}^4 - V_0{}^3) + V_0{}^3,$$

$$V^4 = +\lambda H s V_0{}^1 + \frac{\lambda^2 s^2}{2} H^2 (V_0{}^4 - V_0{}^3) + V_0{}^4.$$

If $V_0{}^1 = V_0{}^2 = 0$, $x_0{}^\sigma = 0$,

$$x^1 = +\frac{s^2 \lambda H}{2}(V_0{}^4 - V_0{}^3), \quad x^2 = 0,$$

$$x^3 = V_0{}^3 s + \frac{\lambda^2 H^2}{6}(V_0{}^4 - V_0{}^3) s^3,$$

$$x^4 = V_0{}^4 s + \frac{\lambda^2 H^2}{6}(V_0{}^4 - V_0{}^3) s^3.$$

That is the particle orbit is in the x^1, x^3 plane (perpendicular to H) and is given by

$$x^2 = 0,$$

$$x^3 = \left(\frac{2x^1}{\lambda H (V_0{}^4 - V_0{}^3)}\right)^{\frac{1}{2}} \left[V_0{}^3 + \frac{\lambda H}{3} x^1 \right] = \left(\frac{2x^1}{\lambda H (1-v_0)(1-v_0{}^2)^{\frac{1}{2}}}\right)^{\frac{1}{2}} \left[v_0 + \frac{eH(1-v_0{}^2)^{\frac{1}{2}}}{3m_0 c^2} x' \right],$$

$$x^4 = \left(\frac{2x^1}{\lambda H (V_0{}^4 - V_0{}^3}\right)^{\frac{1}{2}} \left[V_0{}^4 + \frac{\lambda H}{3} x^1 \right] = \left(\frac{2x^1}{\lambda H (1-v_0)(1-v_0{}^2)^{\frac{1}{2}}}\right)^{\frac{1}{2}} \left[v_0 + \frac{eH(1-v_0{}^2)^{\frac{1}{2}}}{3m_0 c^2} x^1 \right]$$

where v_0 is the magnitude of the initial three-dimensional velocity.

4. ORBIT FOR PLANE WAVE

In case H of Eqs. (3.1) is a function of the space-time coordinates, it follows from Maxwell's equations that

$$H = H(z - ct) = H(x^3 - x^4). \tag{4.1}$$

Thus, in this case Eqs. (3.1) describe the electromagnetic field of a plane wave progressing in the x^3 direction. We may readily solve Eqs. (1.10) in this case also by the methods used above, since it is a consequence of Eqs. (1.10), (3.1), and (4.1) that

$$\frac{d^2 x^3}{ds^2} - \frac{d^2 x^4}{ds^2} = 0.$$

Hence,

$$\frac{dx^3}{ds} - \frac{dx^4}{ds} = V_0{}^3 - V_0{}^4 = \text{constant},$$

and

$$x^3 - x^4 = (V_0{}^3 - V_0{}^4)s + x_0{}^3 - x_0{}^4.$$

Therefore, our original differential equation is of the form

$$\frac{dV}{ds} = \lambda H(s) F_0 V,$$

where F_0 is a constant matrix and its solution is given by

$$V = e^{\lambda h(s) F_0} V_0,$$

where

$$h(s) = \int_0^s H(s) ds.$$

Applying the methods discussed above, we obtain the following parametric equations for the orbit

$$x^1 - x_0{}^1 = V_0{}^1 s + \lambda (V_0{}^4 - V_0{}^3) \int_0^s h(s) ds,$$

$$x^2 - x_0{}^2 = V_0{}^2 s,$$

$$x^3 - x_0{}^3 = V_0{}^3 s + \lambda \int_0^s \left(h(s) V_0{}^1 + \frac{\lambda}{2} h^2(s) (V_0{}^4 - V_0{}^3) \right) ds, \tag{4.2}$$

$$x^4 - x_0{}^4 = V_0{}^4 s + \lambda \int_0^s \left(h(s) V_0{}^1 + \frac{\lambda}{2} h^2(s) (V_0{}^4 - V_0{}^3) \right) ds.$$

The nature of these orbits is discussed in detail in Appendix B for the special case where $H(s) = \sin 2\pi f s$.

5. ORBITS IN THE NON-PARABOLIC CASE

A particularly simple form for the parametric equations of the orbit may be obtained in terms of the proper vectors of the non-parabolic Lorentz transformation discussed in Appendix A. These are called P^σ, Q^σ, N^σ, and $\bar{N}^\sigma$, where the bar denotes the complex conjugate, and they may be determined in terms of the components of $f\sigma\tau$ as in Appendix A. In terms of these we may write

$$L^\sigma{}_\rho = \frac{1}{P^\tau Q_\tau}(e^{\lambda s\nu}P^\sigma Q_\rho + e^{-\lambda s\nu}Q^\sigma P_\rho - e^{-i\lambda s\theta}\bar{N}^\sigma N_\rho + e^{i\lambda s\theta}N^\sigma \bar{N}_\rho).$$

The initial four-dimensional velocity vector of the particle may be written as

$$V_0{}^\sigma = p_0 P^\sigma + q_0 Q^\sigma + n_0 N^\sigma + \bar{n}_0 \bar{N}^\sigma,$$

where

$$p_0 = \frac{V_0{}^\sigma Q_\sigma}{P^\sigma Q_\sigma}, \quad q_0 = \frac{V_0{}^\sigma P_\sigma}{P^\sigma Q_\sigma}, \quad n_0 = \frac{-V_0{}^\sigma \bar{N}_\sigma}{P_\sigma Q^\sigma}.$$

Neither p_0 nor q_0 can vanish for

$$V_0{}^\sigma V_{0\sigma} = 2(p_0 q_0 - n_0 \bar{n}_0)P^\sigma Q_\sigma = -1.$$

Since $P^\sigma Q_\sigma = -2 < 0$ we must have

$$p_0 q_0 = \tfrac{1}{4} + n_0 \bar{n}_0 > 0.$$

The velocity vector at any point along the orbit is then given by

$$V^\sigma(s) = L_\tau{}^\sigma(s)\, V_0{}^\tau = pP^\sigma + qQ^\sigma + nN^\sigma + \bar{n}\bar{N}^\sigma,$$

where $p \cdot q$, and n are functions of s given by

$$p(s) = e^{\lambda\nu s}p_0, \quad q(s) = e^{-\lambda\nu s}q_0, \quad n(s) = e^{i\lambda\theta s}n_0. \tag{5.1}$$

The parametric equations of the orbit are

$$x^\sigma - x_0{}^\sigma = \frac{1}{\lambda}\left(\left(\frac{e^{\lambda\nu s}-1}{\nu}\right)p_0 P^\sigma - \frac{e^{-\lambda\nu s}-1}{\nu}q_0 Q^\sigma - i\frac{(e^{i\lambda\theta s}-1)}{\theta}n_0 N^\sigma + i\frac{(e^{-i\lambda\theta s}-1)}{\theta}\bar{n}_0 \bar{N}^\sigma\right). \tag{5.2}$$

These formulas hold when ν and θ are different from zero. In case either is zero the orbit may be obtained by taking the limit of the above expression as ν or θ goes to zero. This equation may be written as

$$x^\sigma - x_0{}^\sigma + c^\sigma = \frac{1}{\lambda}\left(\frac{e^{\lambda s\nu}}{\nu}p_0 P^\sigma - \frac{e^{-\lambda s\nu}}{\nu}q_0 Q^\sigma - \frac{ie^{i\lambda s\theta}}{\theta}n_0 N^\sigma + \frac{ie^{-i\lambda\theta s}}{\theta}\bar{n}_0 \bar{N}^\sigma\right),$$

where

$$c^\sigma = \frac{1}{\lambda\nu}p_0 P^\sigma - \frac{1}{\lambda\nu}q_0 Q^\sigma - \frac{i}{\theta}n_0 N^\sigma + \frac{i}{\theta}\bar{n}_0 \bar{N}^\sigma.$$

From this it follows that

$$(x^\sigma - x_0{}^\sigma + c^\sigma)(x_\sigma - x_{0\sigma} + c_\sigma) = -\frac{2}{\lambda^2}\left(\frac{p_0 q_0}{\nu^2} - \frac{n_0 \bar{n}_0}{\theta^2}\right)P^\sigma Q_\sigma = \frac{4}{\lambda^2}\left(\frac{p_0 q_0}{\nu^2} - \frac{n_0 \bar{n}_0}{\theta^2}\right). \tag{5.3}$$

That is, the orbit lies on a hyperboloid in space-time and is thus analogous to the path of a particle undergoing constant one-dimensional acceleration in the sense of special relativity. It has already been shown that the particle is undergoing constant acceleration, and the magnitude of this acceleration may be computed from (1.14) to be

$$c^4 b^2 = b^{*\sigma} b_\sigma{}^* = +4\lambda^2(\nu^2 p_0 q_0 + \theta^2 n\bar{n}_0). \tag{5.4}$$

Equation (5.3) is a generalization of the result that a charged particle in a constant electric field undergoes constant four-dimensional acceleration in the direction of the field.

It is evident from Eqs. (5.1) that if $\nu \neq 0$, no time-like vector $V_0{}^\sigma$ can be a proper vector of $L(s)$. However, if the particles initial velocity is such that $n_0 = 0$, that is, the particle is initially moving in the space-time plane determined by the vectors

$$T^\sigma = \tfrac{1}{2}(P^\sigma + Q^\sigma), \quad Z^\sigma = \tfrac{1}{2}(P^\sigma - Q^\sigma), \tag{5.5}$$

then it remains in this plane. This statement is independent of the value of θ. If $\theta = 0$, then the orbit is essentially that of a particle in a constant electric field. Thus for properly chosen initial velocity vectors V_0^μ ($n_0=0$) the orbits of particles in electromagnetic fields with $E^2>H^2$ and $E\cdot H\neq 0$ are qualitatively the same as those for $E^2>H^2$ and $E\cdot H=0$. Both are essentially the same as that for a particle in a constant electric field, that is, $H=0$.

In case $\theta=0$, b of (5.4) becomes

$$b=\lambda\nu=\frac{e}{m_0}(\tfrac{1}{2}((a^2+4b^2)^{\frac{1}{2}}+a)^{\frac{1}{2}}).$$

6. PERIODIC ORBITS

If $\nu=0$ then the time-like vector T^σ given by the first of (5.5) is a proper vector of $L(s)$ corresponding to the proper value 1. We may therefore expect in this case that the orbit of the particle is particularly simple for a properly chosen initial velocity vector V_0^μ. We shall now discuss this case. We first show: *the condition $\nu=0$ is the necessary and sufficient condition that the orbit be periodic in the sense that there exist a constant σ such that*

$$V^\mu(s+\sigma)\equiv V^\mu(s).$$

From Eqs. (5.1) this condition is equivalent to the set of identities

$$e^{\lambda\nu(s+\sigma)}p_0\equiv e^{\lambda\nu s}p_0,$$
$$e^{-\lambda\nu(s+\sigma)}q_0\equiv e^{\lambda\nu s}q_0,$$
$$e^{i\lambda\theta(s+\sigma)}n_0\equiv e^{i\lambda\theta s}n_0.$$

The numbers p_0 and q_0 cannot both vanish, since V_0^σ must be time-like and hence these identities can be satisfied if and only if $\nu=0$, that is,

$$a+(a^2+4b^2)^{\frac{1}{2}}=0.$$

This in turn implies that

$$E\cdot H=0,\quad H^2>E^2. \tag{6.1}$$

If these equations are satisfied we have

$$\sigma=\frac{2\pi}{\lambda\theta}=\frac{2\pi m_0c^2}{e(H^2-E^2)^{\frac{1}{2}}}.$$

In case $E=0$ this reduces to

$$\sigma=\frac{2\pi m_0c^2}{eH}\quad\text{and}\quad\frac{2\pi c}{\sigma}=\frac{eHc}{m_0c^2}$$

is the well-known proper circular frequency of the orbit of a particle in a constant magnetic field H.

As follows from (6.1), the orbits in case $\nu=0$ are those of a particle in crossed electric and magnetic fields where the magnetic field is larger than the electric one. The cycloidal nature of the orbits is well known. However, to illustrate the methods used we briefly discuss this case in detail.

In case (6.1) holds, $\alpha=0$ and Eqs. (2.24) become

$$S=\frac{i}{(H^2-E^2)^{\frac{1}{2}}}F^\dagger,\quad S^\dagger=\frac{i}{(H^2-E^2)^{\frac{1}{2}}}F.$$

If we choose the coordinate system so that $H_i=H\delta_i^{\ 3}$, $E_i=E\delta_i^{\ 2}$, then

$$F=\begin{Vmatrix}0 & H & 0 & 0\\ -H & 0 & 0 & E\\ 0 & 0 & 0 & 0\\ 0 & E & 0 & 0\end{Vmatrix}, \tag{6.2}$$

$$F^\dagger=\begin{Vmatrix}0 & 0 & iE & 0\\ 0 & 0 & 0 & 0\\ -iE & 0 & 0 & iH\\ 0 & 0 & iH & 0\end{Vmatrix}.$$

The orthogonal ennuple of vectors associated with this field is then given by

$$\begin{aligned}X^\sigma&=\left(\frac{H}{(H^2-E^2)^{\frac{1}{2}}},\,0,\,0,\,\frac{E}{(H^2-E^2)^{\frac{1}{2}}}\right)=\tfrac{1}{2}(N^\sigma+\bar{N}^\sigma),\\ Y^\sigma&=(0,1,0,0)=-\frac{i}{2}(N^\sigma-\bar{N}^\sigma),\\ Z^\sigma&=(0,0,1,0)=\tfrac{1}{2}(P^\sigma-Q^\sigma),\\ T^\sigma&=\left(\frac{E}{(H^2-E^2)^{\frac{1}{2}}},\,0,\,0,\,\frac{H}{(H^2-E^2)^{\frac{1}{2}}}\right)=\tfrac{1}{2}(P^\sigma+Q^\sigma).\end{aligned} \tag{6.3}$$

The Lorentz transformation

$$\|L^\sigma{}_\tau\| = \left\| \begin{matrix} \dfrac{H}{(H^2-E^2)^{\frac{1}{2}}} & 0 & 0 & \dfrac{-E}{(H^2-E^2)^{\frac{1}{2}}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \dfrac{-E}{(H^2-E^2)^2} & 0 & 0 & \dfrac{H}{(H^2-E^2)^{\frac{1}{2}}} \end{matrix} \right\|$$

carries the four vectors X^σ, Y^σ, Z^σ, T^σ into $X^{*\sigma}=\delta_1{}^\sigma$, $Y^{*\sigma}=\delta_2{}^\sigma Z^{*\sigma}=\delta_3{}^\sigma$, and $T^{*\sigma}=\delta_4{}^\sigma$, respectively, and it transforms F into

$$F^* = LFL^{-1} = \left\| \begin{matrix} 0 & (H^2-E^2)^{\frac{1}{2}} & 0 & 0 \\ -(H^2-E^2)^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \right\|.$$

Thus, as is well known, for an observer traveling with a velocity

$$v = c\frac{E}{H}.$$

In the direction perpendicular to both E and H, the electromagnetic field given by (6.2) appears as a pure magnetic one in the direction of H and of magnitude $(H^2-E^2)^{\frac{1}{2}}$. The orbits may be computed in the new coordinate system quite readily and then transformed into the original one. An alternative procedure is to refer the orbits to the vectors X^σ, Y^σ, Z^σ, and T^σ in a general coordinate system.

That is, we write

$$V_0{}^\sigma = \xi_0 X^\sigma + \eta_0 Y^\sigma + \zeta_0 Z^\sigma + \tau_0 T^\sigma,$$

where

$$\xi_0 = V_0{}^\sigma X_\sigma, \quad \eta_0 = V_0{}^\sigma Y_\sigma, \quad \zeta_0 = V_0{}^\sigma Z_\sigma,$$

and

$$\tau_0 = V_0{}^\sigma T_\sigma.$$

Since $V_0{}^\sigma V_{0\sigma} = -1$, we must have

$$\xi_0{}^2 + \eta_0{}^2 + \zeta_0{}^2 - \tau_0{}^2 = -1.$$

It follows from (4.8), (4.18), and (4.22) that

$$V^\sigma = L^\sigma{}_\tau(s)\, V_0{}^\tau = (\cos\lambda\theta s\xi_0 + \sin\lambda\theta s\eta_0)X^\sigma + (-\sin\lambda\theta s\xi_0 + \cos\lambda\theta s\eta_0)Y^\sigma + \zeta_0 Z^\sigma + \tau_0 T^\sigma,$$

and hence

$$\lambda\theta(x^\sigma - x_0{}^\sigma) = (\sin\lambda\theta s\xi_0 - (\cos\lambda\theta s - 1)\eta_0)X^\sigma + (\cos\lambda\theta s - 1)\xi_0 + \sin\lambda\theta s\eta_0)Y^\sigma + \lambda\theta\zeta_0 s Z^\sigma + \lambda\theta\tau_0 s T^\sigma, \quad (6.4)$$

where

$$\theta = (H^2 - E^2)^{\frac{1}{2}}.$$

It follows from (6.4) that if $V_0{}^\mu = T^\mu$, then the orbit is given by the simple formula.

$$x^\sigma - x_0{}^\sigma = sT^\sigma.$$

That is, if the initial velocity of the particle is Ec/H in the direction perpendicular to both E and H, then this velocity is always unaltered and the particle always moves along this direction.

From (6.4) it follows that the projection of the orbit on the plane of the vectors X^σ, Y^σ is a circle of radius

$$r = \frac{1}{\lambda\theta}(\xi_0{}^2 + \eta_0{}^2)^{\frac{1}{2}} = \frac{c}{e(H^2-E^2)^{\frac{1}{2}}}((p_\sigma X^\sigma)^2 + (p_\sigma Y^\sigma)^2)^{\frac{1}{2}},$$

where p_μ is the four-dimensional momentum vector defined by

$$p_\mu = m_0 c V_\mu.$$

In the coordinate system in which (6.3) holds the parametric equations of the orbit are

$$\begin{aligned} x^1 &= x_0{}^1 + \frac{H}{\lambda^2\theta}\left(\tau_0\lambda\theta s\frac{E}{H} + \xi_0 \sin\lambda\theta s - \eta_0 \cos\lambda\theta s + \eta_0\right), \\ x^2 &= x_0{}^2 + \frac{1}{\lambda\theta}(\xi_0 \cos\lambda\theta s + \eta_0 \sin\lambda\theta s - \xi_0), \\ x^3 &= x_0{}^3 + \zeta_0 s, \\ x^4 &= x_0{}^4 + \frac{H}{\lambda\theta^2}\left(\tau_0\lambda\theta s + \frac{E}{H}[\xi_0 \sin\lambda\theta s - \eta_0 \cos\lambda\theta s + \eta_0]\right). \end{aligned} \quad (6.5)$$

From these equations it is evident that the spatial coordinates $x^i (i=1, 2, 3)$ will be periodic functions of s if and only if $E=0$ and $\zeta_0=0$, and then the period will be σ.

The first two of Eqs. (6.5) may be written as

$$X=(1-E^2/H^2)^{\frac{1}{2}}(x^1-x_0^1)$$
$$=R\mu+r(\sin(\mu-\omega)+\sin\omega), \quad (6.6)$$
$$Y=(x^2-x_0^2)=r(\cos(\mu-\omega)-\cos\omega),$$

where

$$\mu=\lambda\theta s,$$

$$R=\frac{\tau_0 E}{\lambda H^2(1-E^2/H^2)^{\frac{1}{2}}}$$
$$=\frac{Em_0c^2}{e(H^2-E^2)(1-v^2/c^2)^{\frac{1}{2}}}\left(\frac{E}{H}\frac{v_x}{c}-1\right),$$

$$r=\frac{(\xi_0{}^2+\eta_0{}^2)^{\frac{1}{2}}}{\lambda O}=\frac{m_0c^2}{eH(1-E^2/H^2)^{\frac{1}{2}}}$$
$$\times\left[\left(\frac{v_x}{c}-\frac{E}{H}\right)^2+\frac{v_y{}^2}{c^2}\right]^{\frac{1}{2}}\frac{1}{(1-v^2/c^2)^{\frac{1}{2}}}, \quad (6.7)$$
$$\tan\omega=\eta_0/\xi_0,$$

v_x and v_y are the x and y components of the initial velocity, respectively, and v is the magnitude of the initial velocity.

From Eqs. (6.6) the following known results are readily obtained. The motion in the X, Y plane is a cycloid generated by rolling a circle of radius R on the X axis with the generating point a distance r from the center of the circle. From Eq. (6.7), it follows that the radius of the rolling circle depends on the initial velocity of the particle. Hence particles entering the crossed fields at the same point with different initial velocities will not have orbits intersecting at the same point. If, however, $(E/H)^2$, $(v/c)^2$, (Ev/Hc) are negligible with respect to one, this "defocusing" will disappear.

APPENDIX A

Proper Values of Lorentz Matrices

(a) *Parabolic Case*

In this appendix we determine a set of invariant vectors associated with the Lorentz matrix $L(s)$ given by Eqs. (2.11) and (2.18) and express L in terms of this set. It is evident that these are to be determined from those of F. Instead of determining the proper vectors of F by solving equations of the form

$$Fx=\lambda x,$$

we achieve our purpose by forming simple combinations of the components of the tensor $f_{\sigma\tau}$. We begin with the parabolic case.

In this case the matrix F is of rank 2 since we assume that it is not identically zero, its determinant vanishes, and $f_{\sigma\tau}=-f_{\tau\sigma}$. From the fact that

$$f_{\sigma\tau}f^{\dagger\sigma\tau}=\frac{1}{(g)^{\frac{1}{2}}}(f_{12}f_{34}+f_{13}f_{42}+f_{14}f_{23})=0,$$

it may readily be verified that if $f_{14}\neq 0$ we may write

$$f_{\sigma\tau}=\frac{1}{f_{14}}(f_{1\sigma}f_{4\tau}-f_{1\tau}f_{4\sigma}).$$

That is,

$$f_{\sigma\tau}=\tfrac{1}{2}(X_\sigma Y_\tau-Y_\sigma X_\tau),$$

where X_σ and Y_τ are vectors which in a fixed Gallileau coordinate system are, respectively, proportional to two independent rows of the matrix $f_{\sigma\tau}$. Then

$$f_{\sigma\tau}f^{\sigma\tau}=\tfrac{1}{2}[(X^\sigma Y_\sigma)^2-(X_\sigma X_\sigma)(Y_\sigma Y_\sigma)]=0.$$

The scalars $X^\sigma X_\sigma$ and $Y^\sigma Y_\sigma$ cannot both vanish for then it would follow that $X^\sigma Y_\sigma=0$, and since both are null-vectors this would imply $X^\sigma=\rho Y^\sigma$ which is contrary to the assumption that $f_{\sigma\tau}$ is not identically zero. If $X^\sigma X_\sigma\neq 0$, we define

$$P^\sigma=\frac{X^\rho Y_\rho}{X^\tau X_\tau}X^\sigma-Y^\sigma, \quad U^\sigma=X^\sigma.$$

It then follows that

$$P^\sigma P_\sigma=0, \quad P^\sigma U_\sigma=0, \quad (a1)$$
$$f_{\sigma\tau}=\tfrac{1}{2}(P_\sigma U_\tau-U_\sigma P_\tau). \quad (a2)$$

If $X^\sigma X_\sigma=0$ we define

$$P^\sigma=X^\sigma, \quad U^\sigma=Y^\sigma.$$

We then have the result that any antisymmetric tensor $f_{\sigma\tau}$ for which $f^{\sigma\tau}f_{\sigma\tau}=f^{\sigma\tau}f^\dagger{}_{\sigma\tau}=0$, defines a null-vector P_σ and a space-like vector U^σ orthogonal to it. The vectors P^σ and U^σ may be replaced by ρP^σ and $(1/\rho)U^\sigma$, respectively, without affecting $(a1)$ and $(a2)$. We may determine ρ up to a sign by the requirement that

$$U^\sigma U_\sigma = 1. \qquad (a3)$$

The sign of ρ may then be determined so that $P^4>0$. Thus the Eqs. ($a1$) to ($a3$) and this condition on P^σ determine P^σ and U^σ uniquely.

We then have

$$F^2 = \|f^{\sigma\tau}f^\tau{}_\rho\| = -\tfrac{1}{4}\|P^\sigma P_\rho\|.$$

Equation (2.11) then becomes

$$L^\sigma{}_\tau(s) = \delta_\tau{}^\sigma + \frac{\lambda s}{2}(P^\sigma U_\tau - U^\sigma P_\tau) - \frac{1}{\delta}\lambda^2 s^2 P^\sigma P_\tau. \qquad (a4)$$

From a similar argument it follows that we may write

$$if^{\dagger\sigma\tau} = -\tfrac{1}{2}(P_1{}^\sigma W^\tau - W^\sigma P_1{}^\tau),$$

where $P_1{}^\sigma$ and W^σ are vectors satisfying

$$P_1{}^\sigma W_\sigma = P_1{}^\sigma P_{1\sigma} = 0$$

and are determined up to a common factor. Equation (2.6) then may be written as

$$f^{\sigma\tau}f^\dagger{}_{\tau\rho} = (P_1{}^\sigma P_\sigma)W^\sigma W_\rho - (P_1{}^\tau U_\tau)W^\sigma P_\rho - (W^\tau P_\tau)P_1{}^\sigma U_\rho + (W^\tau U_\tau)P_1{}^\sigma P_\rho = 0.$$

On multiplying this equation by U^ρ and summing we obtain

$$(P_1{}^\sigma P_\sigma)W^\tau - (W^\sigma P_\sigma)P_1{}^\tau = 0,$$

and hence

$$P_1{}^\sigma P_\sigma = W^\sigma P_\sigma = 0.$$

Therefore,

$$P_1{}^\sigma = \rho P^\sigma.$$

We may then choose $\rho=1$ by redefining W^σ if necessary. Equation (2.9) then gives for this new choice of

$$W^\sigma W_\sigma = U^\sigma U_\sigma = 1, \qquad (a5)$$

and

$$if^{\dagger\sigma\tau} = -\tfrac{1}{2}(P^\sigma W^\tau - W^\sigma P^\tau). \qquad (a6)$$

If we define the space-like vector Z^σ as one orthogonal to the two space-like vectors U^σ and W^σ and the time-like vector T^σ (with $T^4>0$) as that orthogonal to these three space-like vectors, the four vectors U^σ, W^σ, Z^σ, and T^σ form an orthogonal ennuple in terms of which any vector may be expressed. In fact, we have

$$P^\sigma = \mu(T^\sigma - Z^\sigma), \qquad (a7)$$

and

$$P_\sigma Z^\sigma = P_\sigma T^\sigma = -\mu \neq 0. \qquad (a8)$$

It follows from (3.4) and the properties of the vectors involved that

$$L^\sigma{}_\tau(s)P^\tau = P^\sigma,$$

$$L^\sigma{}_\tau(s)U^\tau = U^\sigma + \frac{\lambda s}{2}P^\sigma,$$

$$L^\sigma{}_\tau(s)W^\tau = W^\sigma,$$

$$L^\sigma{}_\tau(s)Z^\tau = Z^\sigma + \frac{\lambda s}{2}\mu U^\sigma + \frac{\lambda^2 s^2}{8}\mu P^\sigma.$$

From these equations it is evident that the vectors of the form $aP^\sigma + bW^\sigma$ are carried into themselves under $L(s)$ and a and b unchanged. Moreover, no direction in space-time is left invariant under $L(s)$ unless it is in the plane of the vectors P^σ and W^σ. In particular, no time-like vector can be a proper vector of $L(s)$ given by ($a4$).

(b) *Non-Parabolic Case*

In this case it is convenient to work with $S^{\sigma\tau}$ defined by Eqs. (2.12) in terms of $f^{\sigma\tau}$. The argument given at the beginning of the preceeding section enables us to write

$$S_{\sigma\tau} = A_\sigma B_\tau - B_\sigma A_\tau, \qquad (b1)$$

where A_σ and B_σ are vectors which in a particular coordinate system are proportional to two independent rows of $\|S_{\sigma\tau}\|$. Thus if $S_{12}\neq 0$ we may take

$$A_\sigma = \frac{1}{S_{12}}S_{1\sigma}, \quad B_\sigma = S_{2\sigma}. \qquad (b2)$$

If $A^\sigma A_\sigma$ and $B^\sigma B_\sigma$ are not both zero it is sufficient to consider the case $A^\sigma A_\sigma \neq 0$ and $A^\sigma B_\sigma = 0$, for we may always interchange A^σ and $-B^\sigma$ without altering the form of ($b1$). Moreover, if $A^\sigma B_\sigma \neq 0$, we may replace A^σ and B^σ by

$$A_1{}^\sigma = A^\sigma, \quad B_1{}^\sigma = B^\sigma - \frac{A^\tau B_\tau}{A^\tau A_\tau}A^\sigma,$$

and obtain

$$S_{\sigma\tau} = A_{1\sigma}B_{1\tau} - B_{1\sigma}A_{1\tau}.$$

From the second of Eqs. (2.16) we have

$$\tfrac{1}{2}S^{\sigma}{}_{\tau}S^{\tau}{}_{\sigma}=[(A^{\sigma}B_{\sigma})^2-(A^{\sigma}A_{\sigma})(B^{\sigma}B_{\sigma})]$$
$$=-(A_1{}^{\sigma}A_{1\sigma})(B_1{}^{\sigma}B_{1\sigma})=1. \quad (b3)$$

Hence $A_1{}^{\sigma}A_{1\sigma}$ and $B_1{}^{\sigma}B_{1\sigma}$ are of opposite sign. If $A^{\sigma}A_{\sigma}>0$ we write

$$Z^{\sigma}=\frac{A^{\sigma}}{(A^{\tau}A_{\tau})^{\frac{1}{2}}},\quad T^{\sigma}=(A^{\tau}A_{\tau})^{\frac{1}{2}}\left(B^{\sigma}-\frac{A^{\tau}B_{\tau}}{A^{\tau}A_{\tau}}A^{\sigma}\right), \quad (b4)$$

and if $A^{\sigma}A_{\sigma}<0$ we write

$$Z^{\sigma}=-(-A^{\tau}A_{\tau})^{\frac{1}{2}}B_1{}^{\sigma},\quad T^{\sigma}=\frac{A^{\sigma}}{(-A^{\sigma}A_{\sigma})^{\frac{1}{2}}}. \quad (b5)$$

The sign of the radical occurring in (b4) and (b5) is determined by the condition that $T^4>0$. Thus, in either case we have

$$S_{\sigma\tau}=Z_{\sigma}T_{\tau}-T_{\sigma}Z_{\tau}, \quad (b6)$$

where

$$Z^{\sigma}Z_{\sigma}=-T^{\sigma}T_{\sigma}=1, \quad (b7)$$

that is, Z^{σ} and T^{σ} are uniquely determined orthogonal unit space-like and time-like vectors, respectively. We shall have occasion to use the null-vectors

$$P^{\sigma}=(T^{\sigma}-Z^{\sigma}),\quad Q^{\sigma}=(T^{\sigma}+Z^{\sigma}), \quad (b8)$$

satisfying

$$P^{\sigma}Q_{\sigma}=-2,\quad P^{\sigma}P_{\sigma}=Q^{\sigma}Q_{\sigma}=0. \quad (b9)$$

In terms of these vectors we may write

$$S_{\sigma\tau}=\frac{1}{P^{\rho}Q_{\rho}}(P_{\sigma}Q_{\tau}-Q_{\sigma}P_{\tau}). \quad (b10)$$

In case $A^{\tau}A_{\tau}$ and $B^{\tau}B_{\tau}$ both vanish, it follows from (b3) that $A^{\sigma}B_{\sigma}=\pm 1$ and hence by a relabeling if necessary we may always write $S_{\sigma\tau}$ in the form given by (b10) with the second of Eqs. (b9) satisfied. T^{σ} and Z^{σ} may then be defined by the inverses of Eqs. (b8).

Similarly, we may write for the real tensor $iS^{\dagger}{}_{\sigma\tau}$

$$iS^{\dagger}{}_{\sigma\tau}=C_{\sigma}D_{\tau}-C_{\tau}D_{\sigma}, \quad (b11)$$

where C_{σ} and D_{τ} are vectors which in a particular coordinate system are proportional to two independent rows of $\|iS^{\dagger}{}_{\sigma\tau}\|$. Thus if $iS^{\dagger}{}_{34}\neq 0$ (as will be the case of $S_{12}\neq 0$) we may take

$$C_{\sigma}=\frac{1}{iS^{\dagger}{}_{34}}iS^{\dagger}{}_{3\sigma},\quad D_{\sigma}=iS^{\dagger}{}_{4\sigma}. \quad (b12)$$

From Eq. (2.16) we have

$$\tfrac{1}{2}iS^{\dagger}{}_{\tau}{}^{\sigma}iS^{\dagger}{}_{\sigma}{}^{\tau}=[(C^{\sigma}D_{\sigma})^2-(C^{\sigma}C_{\sigma})(D^{\sigma}D_{\sigma})]=-1. \quad (b13)$$

It is evident from this equation and the fact that C^{σ} and D^{σ} are real that $C^{\sigma}C_{\sigma}$ and $D^{\sigma}D_{\sigma}$ cannot both vanish. It is sufficient to consider the case $C^{\sigma}C_{\sigma}\neq 0$. We define

$$C_1{}^{\sigma}=C^{\sigma},\quad D_1{}^{\sigma}=D^{\sigma}-\frac{C^{\tau}D_{\tau}}{C^{\tau}C_{\tau}}C^{\sigma}.$$

Then

$$iS^{\dagger}{}_{\sigma\tau}=C_{1\sigma}D_{1\tau}-D_{1\sigma}C_{1\tau},$$

and (b13) becomes

$$(C_1{}^{\sigma}C_{1\sigma})(D_1{}^{\sigma}D_{1\sigma})=1.$$

Hence $C_1{}^{\sigma}C_{1\sigma}$ and $D_1{}^{\sigma}D_{1\sigma}$ are of the same sign and hence positive. That is, $C^{1\sigma}D_{1\sigma}$ and every vector $aC_1{}^{\sigma}+bD_1{}^{\sigma}$ with real a and b must be space-like.

We now define

$$X^{\sigma}=\frac{C_1{}^{\sigma}}{(C_1{}^{\sigma}C_{1\sigma})^{\frac{1}{2}}}=\frac{C^{\sigma}}{(C^{\sigma}C_{\sigma})^{\frac{1}{2}}},\quad Y^{\sigma}=(C^{\sigma}C_{\sigma})^{\frac{1}{2}}\left(D^{\sigma}-\frac{C^{\tau}D_{\tau}}{C^{\sigma}C_{\sigma}}C^{\sigma}\right). \quad (b14)$$

Then we have

$$iS^{\dagger}{}_{\sigma\tau}=X_{\sigma}Y_{\tau}-Y_{\sigma}X_{\tau}, \quad (b15)$$

with

$$X^{\sigma}X_{\sigma}=Y^{\sigma}Y_{\sigma}=1,\quad X^{\sigma}Y_{\sigma}=0. \quad (b16)$$

It follows on substituting (b15) and (b6) in the equation

$$iS^{\dagger\sigma}{}_{\tau}S^{\tau}{}_{\rho}=0,$$

and making use of (b7) and (b16) that

$$X_{\tau}Z^{\tau}=Y_{\tau}Z^{\tau}=X_{\tau}T^{\tau}=Y_{\tau}T^{\tau}=0. \quad (b17)$$

These equations together with (b7) and (b16) state that the four vectors X^{σ}, Y^{σ}, Z^{σ}, T^{σ} form an orthogonal ennuple of unit vectors, the first three being space-like and the fourth time-like.

It is sometimes convenient to introduce the complex vector N^σ, and its complex conjugate $\bar{N}^\sigma$ where

$$N^\sigma = \frac{(-P^\sigma Q_\sigma)^{\frac{1}{2}}}{2}(X^\sigma + iY^\sigma),\quad \bar{N}^\sigma = \frac{(-P^\sigma Q_\sigma)^{\frac{1}{2}}}{2}(X^\sigma - iY^\sigma). \tag{b18}$$

Then

$$S^\dagger{}_{\sigma\tau} = \frac{1}{P^\sigma Q_\sigma}(\bar{N}_\sigma N_\tau - N_\sigma \bar{N}_\tau). \tag{b19}$$

The factor under the radical in (b18) is one when the first of (b9) holds. However, even when this normalization is not made Eqs. (b19) and (b10) hold. The equations satisfied by P^σ, Q^σ, N^σ are

$$\begin{aligned} &P^\sigma P_\sigma = Q^\sigma Q_\sigma = N^\sigma N_\sigma = \bar{N}^\sigma \bar{N}_\sigma = 0,\\ &P^\sigma N_\sigma = P^\sigma \bar{N}_\sigma = Q^\sigma N_\sigma = Q^\sigma \bar{N}_\sigma = 0,\\ &\bar{P}^\sigma = P^\sigma,\quad \bar{Q}^\sigma = Q^\sigma,\quad -P^\sigma Q_\sigma = N^\sigma \bar{N}_\sigma. \end{aligned} \tag{b20}$$

It follows readily from these equations, (b10) and (b19), that

$$S^\sigma{}_\tau S^\tau{}_\rho = \frac{1}{P^\tau Q_\tau}(P^\sigma Q_\rho + Q^\sigma P_\rho),$$

$$S^{\dagger\sigma}{}_\tau S^{\dagger\tau}{}_\rho = \frac{-1}{P^\tau Q_\tau}(\bar{N}^\sigma N_\rho + N^\sigma \bar{N}_\rho).$$

Hence Eq. (2.28) may be written as

$$L^\sigma{}_\rho(s) = \frac{1}{P^\tau Q_\tau}(e^{\lambda s\nu} P^\sigma Q_\rho + e^{-\lambda\nu s} Q^\sigma P_\rho - e^{-i\lambda\theta s} \bar{N}^\sigma N_\rho - e^{i\lambda\theta s} N^\sigma \bar{N}_\rho). \tag{b21}$$

Using (b20), we obtain

$$\begin{aligned} L^\sigma{}_\rho P^\rho &= e^{\lambda\nu s} P^\sigma,\\ L^\sigma{}_\rho Q^\rho &= e^{-\lambda\nu s} Q^\sigma,\\ L^\sigma{}_\rho N^\rho &= e^{i\lambda s\theta} N^\sigma,\\ L^\sigma{}_\rho \bar{N}^\rho &= e^{-i\lambda s\theta} \bar{N}^\sigma. \end{aligned} \tag{b22}$$

That is, P^σ, Q^σ, N^σ, and $\bar{N}^\sigma$ are proper vectors of $L(s)$ corresponding to the proper values $e^{\lambda\nu s}$, $e^{-\lambda\nu s}$, $e^{i\lambda\theta s}$, and $e^{-i\lambda\theta s}$, respectively. The quantities ν and θ are given in terms of the components of E and H by Eqs. (2.15), (2.10), and (2.7).

APPENDIX B

Orbit for a Plane Wave of Frequency f

In this appendix we discuss Eqs. (4.2) in the special case where

$$H = H_0 \cos 2\pi f(ct - z).$$

We choose the origin in space-time so that at $s=0$

$$x_0 = y_0 = z_0 = t_0 = 0.$$

Then Eqs. (4.2) becomes

$$x = V_0{}^1 s - \frac{\lambda H(V_0{}^4 - V_0{}^3)}{\omega^2}[\cos\omega s - 1], \tag{B 1}$$

$$y = V_0{}^2 s,$$

$$z = V_0{}^3 s + \frac{\lambda H_0 V_0{}^1}{\omega^2}(1-\cos\omega s) + \frac{\lambda^2 H_0{}^2}{8\omega^3}(V_0{}^4 - V_0{}^3)[2\omega s - \sin 2\omega s],$$

$$ct = V_0{}^4 s + \frac{\lambda H_0 V_0{}^1}{\omega^2}(1-\cos\omega s) + \frac{\lambda^2 H_0{}^2}{8\omega^3}(V_0{}^4 - V_0{}^3)[2\omega s - \sin 2\omega s],$$

with

$$\omega = 2\pi f(V_0{}^4 - V_0{}^3).$$

If the particle initially moves in the z direction, ($V_0{}^1 = V_0{}^2 = 0$), then its subsequent motion will be in the x, z plane, the plane determined by the direction of propagation of the wave and the direction of the electric field. Suppose $V_0{}^1 = V_0{}^2 = 0$ and write

$$V_0{}^3 = \frac{v_0/c}{(1-v_0{}^2/c^2)^{\frac{1}{2}}},\quad V_0{}^4 = \frac{1}{(1-v_0{}^2/c^2)^{\frac{1}{2}}},$$

$$\omega = \frac{2\pi f(1-v_0/c)}{(1-v_0{}^2/c^2)^{\frac{1}{2}}},$$

where v_0 is the magnitude of the initial velocity.

Then the equations given above become

$$x=\frac{\lambda H_0}{(2\pi f)\omega}[1-\cos\omega s],$$

$$y=0,$$

$$z=\left(\frac{v_0/c}{\omega(1-v_0^2/c^2)^{\frac{1}{2}}}+\frac{\lambda^2H_0^2}{16\pi f\omega^2}\right)\omega s-\frac{\lambda^2H_0^2}{32\pi f\omega^2}\sin 2\omega s,$$

$$ct=\left(\frac{1}{\omega(1-v_0^2/c^2)^{\frac{1}{2}}}+\frac{\lambda^2H_0^2}{16\pi f\omega^2}\right)\omega s-\frac{\lambda^2H_0^2}{32\pi f\omega^2}\sin 2\omega s.$$

It is evident from these equations that the projection of the orbit in the x, z plane is a distortion of $x=\sin^2 z$ such that the particle stays one side of the z axis, returning to the z axis when z changes by

$$2\pi\left(\frac{v_0/c}{\omega(1-v_0^2/c^2)^{\frac{1}{2}}}+\frac{\lambda^2H_0^2}{16\pi f\omega^2}\right).$$

The maximum displacement from the z axis occurs when

$$s=\pi/\omega,$$

and is given by

$$x_{\max}=\frac{2\lambda H_0}{2\pi f\omega}.$$

Here the particle has maximum energy given by

$$m_0c^2V^4=\frac{m_0c^2}{(1-v_0^2/c^2)^{\frac{1}{2}}}+\frac{\lambda^2H_0^2}{8\pi f\omega}.$$

In case the function H has a phase angle α, that is,

$$H=H_0\cos[2\pi f(ct-z)+\alpha],$$

the orbit of the particle no longer returns to the z axis when $V_0^1=V_0^2=0$. The projection of the motion on the x axis has in addition to the motion described above a constant velocity proportional to $\sin\alpha$.

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The Threshold for the Positive Pre-Onset Burst Pulse Corona and the Production of Ionizing Photons in Air at Atmospheric Pressure

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AN analysis of the mechanism of the pre-onset burst pulse corona indicated that use might be made of the theory to throw some light on the photoelectric ionization processes responsible for various other breakdown mechanisms in gases. In view of the work currently being done on this subject, it is of importance to report the conclusions arrived at. The discussion will be limited to the positive point to plane corona in air at atmospheric pressure, fields X being expressed in volts per cm and pressure p in mm of Hg.

Currents from a corona point of radius r below the onset potential V_g of the intermittent Geiger regime consist of a field intensification of the negative ion currents produced in the volume of the corona gap by an external source. Field intensification can only begin when the potential of the point reaches a value V_f such that the ratio X_f/p at the point surfaces exceeds 90. At this field the negative O_2^- ions produced in lower field regions by external agencies begin to shed

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