Vortices and Streams Caused by Sound Waves*

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(Received August 28, 1947)

As shown by Rayleigh, a considerable number of acoustic phenomena are known which involve the viscosity of the medium and require the solution of the hydrodynamic equations to a higher degree of approximation than is customary in elementary treatments of the theory of sound. Among these are the fluid streams that occur near intense sources of sound (e.g.: the "quartz wind").

The general equations of these second-order acoustic phenomena are developed in a systematic manner. When viscous forces are neglected, the effects are of three kinds: (1) those that can be ascribed to the inertia of acoustic energy, (2) those arising from radiation pressure, and (3) those caused by the variable compressibility of the medium. All of them result in the production of overtones of the fundamental vibration. In certain cases, this distortion can become very large, being unlimited except by the viscous forces. However, even when the average value of the gradient of the radiation pressure does not vanish, it does not, on the average, cause an acceleration of the fluid. Such gradients are balanced by the elastic rather than by the viscous forces.

When the latter are introduced into the calculation, a fourth effect appears: the irrotational motion in the sound wave generates vorticity as a second-order effect. This vortex motion will ultimately approach a steady state, being generated and resisted by forces that are independent of the time. Both generating and resisting forces are viscous, and consequently the steady motion is independent of the magnitude of the coefficient of viscosity. However, the resisting forces depend only on the shear viscosity of the medium, while the generating forces depend also on the bulk viscosity. It is suggested that the ratio of the bulk and shear coefficients of viscosity can be determined by studying these phenomena.

Calculations of the velocity of the stream generated by a beam of sound show that it is proportional (1) to $b=(4/3)+(\nu'/\nu)$, where ν' and ν are the bulk and shear viscosities, (2) to the power being radiated in the beam, (3) inversely to the square of the wave-length, and (4) inversely to ρ^2c^3, where ρ is the density and c the sound velocity of the medium. The maximum value of the steady-streaming velocity depends on the resistance offered by the walls of the vessel or room in which the experiment is performed. The time required to set up the steady state is, of course, inversely proportional to this resistance, and the flow is apt to become turbulent when the resistance is low.

INTRODUCTION

THE subject matter of this paper cannot be outlined more clearly than by quoting from the first paragraphs of Lord Rayleigh's paper[1] "On the circulation of air observed in Kundt's tubes, and on some allied acoustical problems":

> Experimenters in acoustics have discovered more than one set of phenomena, apparently depending for their explanation upon the existence of regular currents of air resulting from vibratory motion . . . such currents, involving as they do *circulation* of the fluid, could not arise in the absence of friction. . . . And even when we are prepared to include the influence of friction, we have no chance of reaching an explanation if, as usual, we limit ourselves to the supposition of infinitely small motion and neglect the squares and higher powers [of the velocity]. . . . The more important of the problems relates to the currents generated over a vibrating plate, arranged as in Chladni's experiments. It was discovered by Savart that very fine powder does not collect itself at the nodal lines, as does sand in the production of Chladni's figures, but gathers itself into a cloud which, after hovering for a time, settles itself over the places of maximum vibration. This was traced by Faraday[2] to the action of currents of air, rising from the plate at the place of maximum vibration, and falling back to it at the nodes. In a vacuum the phenomena observed by Savart do not take place, all kinds of powder collecting at the nodes. . . . [Another] problem relates to the air currents observed by Dvorak in a Kundt's tube, to which is apparently due the formation of the dust figures.

With the advant of piezoelectric generators of sound, these effects were rediscovered. Strong currents of air ("quartz wind") or liquid appear in front of the vibrating surface of the crystal. In the case of liquids, these currents are frequently great enough to disturb its free surface. Unless great care is exercised, they may vitiate intensity measurements with a Rayleigh disk. It is possible that this effect was actually discovered by Rayleigh, who performed the following experi-

* This work represents one of the results of research carried out under contract with the Bureau of Ships and Office of Naval Research, Navy Department.

[1] Lord Rayleigh, *Scientific Papers* (Cambridge University Press, Teddington, England), No. 108, p. 239; Phil. Trans. **175**, 1 (1883).

[2] Michael Faraday, Phil. Trans. 299 (1831).

ment: ". . . when the corresponding fork, strongly excited, was held to the mouth [of the Helmholtz resonator] a wind of considerable force issued from the nipple at the opposite side. This effect may rise to such intensity as to blow out a candle upon whose wick the stream is directed. . . . Closer examination revealed the fact that at the sides of the nipple the outward flowing stream was replaced by one in the opposite direction, so that a tongue of flame from a suitable placed candle appeared to enter the nipple at the same time that another candle situated immediately in front was blown away."[3]

Although Rayleigh summarized his calculations in *Theory of Sound*,[4] his results appear to be virtually unknown. At lest two different and incorrect explanations of the quartz wind can be found in recent literature, while the correct explanation appears to be virtually unknown to the experimenter. It is even possible that Rayleigh himself gave an incorrect explanation of the resonator experiment just described. Without giving adequate reasons, he says: "The two effects [flow and counter-flow] are of course in reality alternating, and only appear to be simultaneous in consequence of the inability of the eye to follow such rapid changes." It is at least possible that the streams are steady and that this is an effect similar to the others described above.

In the following pages, a systematic account of the theory of second-order acoustic effects will be developed. In the first part, acoustic radiation pressure and the inertia of acoustic energy will be considered. It will be shown that, as Rayleigh knew, these cannot cause the phenomena described above. However, the structure of this mathematical theory will be useful in the second part, where the second-order viscous forces will be considered, and in the third, where a calculation will be given of the steady flow produced by a sound beam of circular cross section. The remarkable fact will appear that the steady flow "is *independent of the value of the coefficient of viscosity*. We cannot, therefore, avoid considering this motion by supposing the coefficient of viscosity to be very small, the maintenance of the vortices becoming easier in the same proportion as the forces tending to produce the vortical motion diminish." [5]

PART I. ACOUSTIC RADIATION PRESSURE AND THE INERTIA OF ACOUSTIC ENERGY

The question is sometimes asked, why the velocity of sound does not function in acoustic theory in the same way that the velocity of light functions in relativity theory. From the standpoint of the latter, the question is foolish, for the velocity of light is both that, and also the maximum velocity with which any kind of signal or object can be transmitted. The velocity of sound is not maximal in the same sense. Still, the principle of the inertia of energy is not very directly connected with the principle of maximal velocity, but rather with the equations of motion of matter. Consequently, it would be expected that acoustic energy will display an inertia that is much greater than the inertia of electromagnetic radiation, in the inverse ratio of the squares of their velocities of propagation. It will be shown that this expectation is correct, provided the proposition be given a suitable interpretation.

The First- and Second-Order Equations of Acoustics

The general equations of hydrodynamics for a non-viscous fluid are

$$(\partial\rho/\partial t)+\nabla\cdot(\rho\boldsymbol{u})=0, \qquad (1)$$

$$[\partial(\rho\boldsymbol{u})/\partial t]+\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\boldsymbol{u}\nabla\cdot(\rho\boldsymbol{u})=-\nabla p, \qquad (2)$$

where ρ is the density, p the pressure, and $\boldsymbol{u}$ the velocity of the fluid. For the present purposes, it will be supposed that p is a function of ρ only; then

$$\nabla p=C^2\nabla\rho, \qquad (3)$$

where C is a function of ρ which has the units of a velocity. This presupposes that the motion is isentropic.

The essential idea of Rayleigh's treatment of the problems discussed in the introduction, is that some of the terms in Eqs. (1) and (2) are sometimes much less important than others. In different circumstances, different terms will be

[3] Rayleigh, *Theory of Sound* (MacMillan Company, Ltd., London, 1896), Vol. II, p. 217.

[4] Reference 3, Vol. II, p. 333.

[5] Reference 1, p. 246.

negligible and others will be important. Thus, in acoustics, the terms $\partial\rho/\partial t$ and $\nabla\rho$ are important; in hydraulics, the term $\nabla\cdot(\rho\boldsymbol{u})$ becomes more important than $\partial\rho/\partial t$, but $\nabla\rho$ retains its importance. In order to bring the relative importance of the terms clearly to the attention, it is useful to depart from the c.g.s. system of units and to introduce one that is specially adapted to the problems under consideration.

In such a system, let the unit of length be X cm; of time, T sec.; of velocity, U cm sec.$^{-1}$. The unit of density is immaterial, since the equations are essentially homogeneous in ρ. In these units the equations become

$$(\partial\rho/\partial t)+N\nabla\cdot(\rho\boldsymbol{u})=0, \tag{4}$$

$$[\partial(\rho\boldsymbol{u})/\partial t]+N[\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\boldsymbol{u}\nabla\cdot(\rho\boldsymbol{u})] = -(NC^2/U^2)\nabla\rho, \tag{5}$$

where the numeric

$$N=UT/X. \tag{6}$$

To insure that the system of units will serve its purpose, the units X and T are to be chosen so that $\partial f/\partial t$ and $\partial f/\partial x$ are of the same order of magnitude, f being any of the functions ρ, $\boldsymbol{u}$. The unit U could be chosen in any of a number of ways: so that $N=1$, or so that $C/U\sim 1$, or so that $u/U\sim 1$. The third is the choice appropriate for most problems. Having thus defined the units, acoustics may be defined as consisting of those hydrodynamic problems for which $N\ll 1$ and $NC/U\sim 1$. (Hydraulics is apparently those hydrodynamic problems for which $N\gg 1$ and $NU^2/C^2\sim 1$.)

Introducing the quantity

$$c(\rho)=NC(\rho)/U,$$

Eq. (5) becomes

$$N[\partial(\rho\boldsymbol{u})/\partial t]+N^2[\boldsymbol{u}\nabla\cdot(\rho\boldsymbol{u})+\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}] = -c^2\nabla\rho. \tag{7}$$

The numeric N is now to be treated as a perturbation parameter,** and the expansions

$$\rho=\rho_0+N\rho_1+N^2\rho_2+\cdots, \tag{4.0}$$

$$\boldsymbol{u}+\boldsymbol{u}_0+N\boldsymbol{u}_1+N^2\boldsymbol{u}_2+\cdots \tag{7.0}$$

are introduced. The zero-order equations are then

$$\partial\rho_0/\partial t=0, \tag{4.0}$$

$$c_0{}^2\nabla\rho_0=0, \tag{7.0}$$

where $c_0=c(\rho_0)$. Hence, ρ_0 is a constant, which fact may be used to simplify the first- and second-order equations:

$$(\partial\rho_1/\partial t)+\rho_0\nabla\cdot\boldsymbol{u}_0=0, \tag{4.1}$$

$$\rho_0(\partial\boldsymbol{u}_0/\partial t)=-c_0{}^2\nabla\rho_1; \tag{7.1}$$

$$(\partial\rho_2/\partial t)+\rho_0\nabla\cdot\boldsymbol{u}_1+\nabla\cdot(\rho_1\mathbf{u}_0)=0, \tag{4.2}$$

$$\rho_0(\partial\boldsymbol{u}_1/\partial t)+(\partial/\partial t)(\rho_1\boldsymbol{u}_0)+\rho_0(\boldsymbol{u}_0\cdot\nabla\boldsymbol{u}_0+\boldsymbol{u}_0\nabla\cdot\boldsymbol{u}_0) = -c_0{}^2\nabla\rho_2+c_0(dc_0/d\rho_0)\nabla\rho_1{}^2. \tag{7.2}$$

The Eqs. (4.1) and (7.1) are the equations of elementary acoustic theory, while Eqs. (4.2) and (7.2) are less familiar.

Integrals of the First-Order Equations

The equation expressing the conservation of acoustic energy is derived by multiplying Eq. (4.1) by $c_0{}^2\rho_1/\rho_0$, and Eq. (7.1) by $\boldsymbol{u}_0$, and adding. The result is

$$(\partial W/\partial t)+\nabla\cdot\boldsymbol{J}=0. \tag{8}$$

The quantity

$$W=\tfrac{1}{2}\rho_0u_0{}^2+\tfrac{1}{2}c_0{}^2\rho_1{}^2/\rho_0 \tag{9}$$

is the acoustic energy density, while

$$\boldsymbol{J}=c_0{}^2\rho_1\boldsymbol{u}_0 \tag{10}$$

is the acoustic energy flow.

The conservation law of acoustic "momentum" is obtained by multiplying Eq. (4.1) by $\boldsymbol{u}_0$, and Eq. (7.1) by ρ_1/ρ_0 and adding: the result is

$$\frac{1}{c_0{}^2}\frac{\partial\boldsymbol{J}}{\partial t}+\rho_0(\boldsymbol{u}_0\cdot\nabla\boldsymbol{u}_0+\boldsymbol{u}_0\nabla\cdot\boldsymbol{u}_0) -\nabla(W-\rho_0u_0{}^2)+\rho_0\boldsymbol{u}_0\nabla\times(\nabla\times\boldsymbol{u}_0)=0. \tag{11}$$

It will be noted that the time derivative of $\boldsymbol{J}/c_0{}^2$ appears in Eq. (11). Thus the acoustic momentum is related to acoustic energy flow in the same manner as the corresponding electromagnetic quantities. Also that, instead of the total energy density, W, Eq. (11) contains the Lagrangian difference between the potential and kinetic energy densities.

The last term in Eq. (11) usually vanishes,

** After the special units have served their purpose of providing a perturbation parameter, one may always return to the c.g.s. system, for which $N=1$, $C=c$. This will be done in the following pages.

since only solutions for which $\nabla\times\boldsymbol{u}_0=0$ are of interest. This will be assumed in the following.

Simplification of the Second-Order Equations

The second-order equations are seen to contain several combinations of terms that also appear in the conservation laws. The former can therefore be simplified by introducing a quantity ρ_{II}, defined by

$$\rho_2=\rho_{II}+W/c_0^2. \quad (12)$$

The quantity ρ_{II} obeys equations that are much simpler than those for ρ_2, and much more analogous to the first-order equations:

$$(\partial\rho_{II}/\partial t)+\rho_0\nabla\cdot\boldsymbol{u}_1=0, \quad (4\text{-II})$$

$$\rho_0(\partial\boldsymbol{u}_1/\partial t)=-c_0^2\nabla\rho_{II} -\nabla[\rho_0 u_0^2+c_0(dc_0/d\rho_0)\rho_1^2]. \quad (7\text{-II})$$

The term in $dc_0/d\rho_0$ would disappear if the medium obeyed Hooke's law. Thus, only the term in $\rho_0 u_0^2$ needs discussion. This functions like an additional (known) pressure, and may therefor be called the acoustic radiation pressure:

$$P=\rho_0\mu_0^2, \quad (13)$$

which is thus given by twice the kinetic energy density. In gases, the term

$$H=c_0(dc_0/d\rho_0)\rho_1^2 \quad (14)$$

will be of the same order of magnitude as P. In liquids, H will be much less than P.

Because of the similarity between the equations for ρ_{II} and ρ_1, physicists will find it easier to think about ρ_{II} than about ρ_2. In particular, propositions arrived at intuitively (that is, derived from experience with elementary acoustic problems) will usually apply to ρ_{II} and not to ρ_2. Fortunately, the necessary correction is simple: it is only necessary to take the "inertia" of acoustic energy into account and add W/c_0^2 to ρ_{II} in order to obtain ρ_2.

The Solution of the Second-Order Equations

Certain general conclusions can be reached about the second-order effects in simple harmonic sound fields. In these cases, neither P nor H will contain terms of the fundamental frequency, but both will contain constant terms and terms with the double frequency. The latter will result in the generation of the second harmonic in the sound field—a phenomenon that has been observed at even moderate sound intensities. In certain cases, notably that of the plane wave, the amplitude of these harmonics will increase until they are limited by the viscous forces which have thus far been omitted.

Because of interference and the divergence of sound rays, the time-constant terms in P and H will, in general, depend on position. Thus, there will be a constant pressure gradient, and one might expect that this constant gradient will produce a constant acceleration of the fluid, whose velocity would thus increase until the viscous forces balance the pressure gradient. The end result would be fluid streaming at a constant velocity, proportional to the acoustic energy-gradient and inversely proportional to the coefficient of viscosity. These might account for the phenomena described in the Introduction. Unfortunately, however, this reasoning is faulty. The constant part of the gradient of $P+H$ is balanced, not by the viscosity of the fluid, but by its elasticity. Thus neither P nor H cause streaming of the fluid.

This is very simply proven: ρ_{II} may be eliminated between Eqs. (4-II) and (7-II) by taking the gradient of the former and the time derivative of the latter; the result is

$$\rho_0(\partial^2\boldsymbol{u}_1/\partial t^2)-\rho_0 c_0^2\nabla\nabla\cdot\boldsymbol{u}_0 =-(\partial/\partial t)[\nabla(P+H)]. \quad (15)$$

Since only the time derivatives of ∇P and ∇H appear, the first-order velocity $\boldsymbol{u}_1$ is independent of the constant parts of these gradients. This result would not be essentially altered by introducing the viscous terms. On the other hand, if $\boldsymbol{u}_1$ is eliminated from the equations, the result is

$$\rho_0(\partial^2\rho_{II}/\partial t^2)-\rho_0 c_0^2\nabla^2\rho_{II}=\nabla^2(P+H). \quad (16)$$

Thus the constant parts of gradients will produce a constant part of ρ_{II}.

Consequently, neither radiation pressure nor the failure of Hooke's law can be invoked to explain the fluid streams mentioned above, and it becomes necessary to proceed to a study of the viscous forces.

PART II. THE EFFECTS OF VISCOSITY

When the viscous forces are included, the equations become (in the X, T, U system of units):

$$(\partial\rho/\partial t)+N\nabla\cdot(\rho\boldsymbol{u})=0, \qquad (17)$$

$$N[\partial(\rho\boldsymbol{u})/\partial t]+N^2[\boldsymbol{u}\nabla\cdot(\rho\boldsymbol{u})+\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}] = -c^2\nabla\rho+(NT/X^2)[((4/3)\nu+\nu')\rho\nabla\nabla\cdot\boldsymbol{u} - \nu\rho\times(\nabla\times\boldsymbol{u})]. \qquad (18)$$

The quantity $\nu\rho$ is the ordinary coefficient of shear viscosity, so that ν is measured in cm²/sec. For an ideal gas, the bulk viscosity, ν', is zero; for liquids it is presumably different from zero, although no measurements or calculations of its magnitude have been made.[6] For convenience the abbreviation

$$b=4/3+\nu'/\nu \qquad (19)$$

will be used: it is a numerical characteristic of the fluid, and its value will presumably be somewhere between 2 and 10 for liquids, and near 4/3 for gases.

For simplicity, it will be assumed that both b and the product $\nu\rho=\nu_0\rho_0$ are independent of the density of the liquid; effects due to their variation may be of importance, but will not be treated here. The coefficient of the viscous forces may be written

$$NT\nu_0/X^2=N^2/R,$$

where

$$R=UX/\nu_0$$

is the Reynold's number, calculated for the unit of velocity and the unit of length. The values of N and U have above been fixed with respect to the problem, but the value of X has not yet been specified. This freedom could be utilized to assign any desired order of magnitude to the viscous forces; however, there appear to be good physical reasons for supposing them to be of first order. This determines X by means of the equation $N=R$. The unit of length thus calculated has a simple physical significance in the case of gases: it is the mean-free path of their molecules. Since the viscous terms are introduced into the equations in order to take approximate account of molecular processes, this is a very appropriate unit.

It is then obvious that the zero-order equations will be unaffected by the viscous terms, and that the first-order equations are

$$(\partial\rho_1/\partial t)+\rho_0\nabla\cdot\boldsymbol{u}_0=0, \qquad (17.1)$$

$$\rho_0(\partial\boldsymbol{u}_0/\partial t)=-c_0^2\nabla\rho_1+\rho_0\nu_0 b\nabla\nabla\cdot\boldsymbol{u}_0 - \rho_0\nu_0\nabla\times(\nabla\times\boldsymbol{u}_0), \qquad (18.1)$$

and the second-order equations are

$$(\partial\rho_2/\partial t)+\rho_0\nabla\cdot\boldsymbol{u}_1+\nabla\cdot(\rho_1\boldsymbol{u}_0)=0, \qquad (17.2)$$

$$\rho_0(\partial\boldsymbol{u}_1/\partial t)+(\partial/\partial t)(\rho_1\boldsymbol{u}_0) +\rho_0[\boldsymbol{u}_0-\nabla\boldsymbol{u}_0+\boldsymbol{u}_0\nabla\cdot\boldsymbol{u}_0]=-c_0^2\nabla\rho_2 +\rho_0\nu_0 b\nabla\nabla\cdot\boldsymbol{u}_0-\rho_0\nu_0\nabla\times(\nabla\times\boldsymbol{u}_0). \qquad (18.2)$$

Effects due to the failure of Hooke's law have been omitted, although they may be of importance in problems involving distortion.

The conservation (or better, the dissipation) of acoustic energy is then derivable from the first-order equations as before, and results in the equation

$$(\partial W/\partial t)+\nabla\cdot\boldsymbol{J}=\rho_0\nu_0[b\boldsymbol{u}_0\cdot\nabla\boldsymbol{u}_0 -\boldsymbol{u}_0\cdot\nabla\times(\nabla\times\boldsymbol{u}_0)]. \qquad (20)$$

Similarly, the dissipation of acoustic momentum is expressed by

$$\tfrac{1}{2}c_0^2(\partial\boldsymbol{J}/\partial t)+\rho_0[\boldsymbol{u}_0\cdot\nabla\boldsymbol{u}_0+\boldsymbol{u}_0\nabla\cdot\boldsymbol{u}_0] +\nabla(W-\rho_0u_0^2)+\rho_0\boldsymbol{u}_0\times(\nabla\times\boldsymbol{u}_0) =\nu_0 b\rho_1\nabla\nabla\cdot\boldsymbol{u}_0-\nu_0\rho_1\nabla\times(\nabla\times\boldsymbol{u}_0). \qquad (21)$$

Setting $\nabla\times\boldsymbol{u}_0=0$, and introducing ρ_{II} as before, Eqs. (17.2) and (18.2) are seen to be equivalent to

$$(\partial\rho_{II}/\partial t)+\rho_0\nabla\cdot\boldsymbol{u}_1=(\nu_0 b/c_0^2)\boldsymbol{u}_0\cdot\nabla(\partial\rho_1/\partial t), \qquad (17\text{-II})$$

$$\rho_0(\partial\boldsymbol{u}_1/\partial t)=-c_0^2\nabla\rho_{II}-\nabla(\rho_0u_0^2) +\rho_0\nu_0 b\nabla\nabla\cdot\boldsymbol{u}_1-\rho_0\nu_0\nabla\times(\nabla\times\boldsymbol{u}_1) -\rho_1\nu_0 b\nabla\nabla\cdot\boldsymbol{u}_0. \qquad (18\text{-II})$$

Equations (17.2) and (17-II) show that neither ρ_2 nor ρ_{II} are conserved. In the previous part, it was noted that ρ_{II} obeyed laws that would be expected by physicists who rely on physical intuition, while ρ_2 did not. The reason that ρ_{II} is no longer conserved is clearly to be found in the dissipation of acoustic energy by the viscous forces. There appears to be no convenient way of introducing a relevant quantity that is conserved.

The Second-Order Motion of the Fluid

Elimination of ρ_{II} between the Eqs. (17-II) and (18-II) results in the equation

[6] H. Lamb, *Hydrodynamics* (Cambridge University Press, Teddington, England), sixth edition, pp. 573, 645; Reference 3, Vol II, pp. 314, 320; G. Kirchhoff, Pogg. Ann. **134**, 177 (1868).

$$\rho_0(\partial^2 \boldsymbol{u}_1/\partial t^2) - \rho_0 c_0^2 \nabla\nabla\cdot\boldsymbol{u}_1 + \rho_0\nu_0 b\nabla(\partial/\partial t)(\nabla\ \boldsymbol{u}_1) - \rho_0\nu_0(\partial/\partial t)\cdot\nabla\times(\nabla\times\boldsymbol{u}_1) = -\nabla(\partial/\partial t)(\rho_0 u_0^2) - b\nu_0(\partial/\partial t)[\rho_1\nabla\nabla\cdot\boldsymbol{u}_0] - b\nu_0\nabla[\boldsymbol{u}_0\cdot\nabla(\partial\rho_1/\partial t)]. \quad (22)$$

This equation can be simplified by introducing the divergence and rotation of the velocity

$$D = \nabla\cdot\boldsymbol{u}, \quad \boldsymbol{R} = \nabla\times\boldsymbol{u}. \quad (23)$$

Taking the divergence of Eq. (22), the rotation $\boldsymbol{R}_1$ is eliminated:

$$\partial^2 D_1/\partial t^2 - c_0^2\nabla^2 D_1 - \nu_0 b\nabla^2(\partial D_1/\partial t) = -\nabla^2(\partial u_0^2/\partial t) - (b\nu_0/\rho_0)\nabla\cdot[(\partial/\partial t)(\rho_1\nabla D_0)] - (b\nu_0/\rho_0)\nabla^2[\boldsymbol{u}_0\cdot\nabla(\partial\rho_1/\partial t)]. \quad (24)$$

Taking the rotation of Eq. (18-II) similarly eliminates D_1:

$$(\partial\boldsymbol{R}_1/\partial t) - \nu_0\nabla^2\boldsymbol{R}_1 = (b\nu_0/\rho_0^2)\nabla\rho_1\times\nabla(\partial\rho_1/\partial t). \quad (25)$$

It is worth noting in more detail than above that when the first-order quantities are simple harmonic functions of the time, the right side of Eq. (25) is independent of the time. For, let

$$p_1 = \rho_1 c_0^2 = P' \cos nt + P'' \sin nt,$$

where P and P'' are functions of the space coordinates only; then

$$c_0^2\nabla\rho_1 = \nabla P' \cos nt + \nabla P'' \sin nt,$$
$$(c_0^2/n)\nabla(\partial\rho_1/\partial t) = -\nabla P' \sin nt + \nabla P'' \cos nt,$$

whence

$$\nabla\rho_1\times\nabla(\partial\rho_1/\partial t) = (n/c_0^4)\nabla P'\times\nabla P''.$$

This is perhaps the first indication that the theory of the fluid streams generated by sound sources is governed by Eq. (25). It also indicates that the vorticity generated by a sound wave will approach a steady value after a sufficiently long time. The length of this time depends on the value of the viscosity coefficient ν_0, but the steady state itself does not: it is determined by the equation

$$-\nabla^2\boldsymbol{R}_1 = (b/\rho_0^2)\nabla\rho_1\times\nabla(\partial\rho_1/\partial t). \quad (25a)$$

The general procedure to obtain the velocity $\boldsymbol{u}_1$ will involve two steps, the first being the solution of Eqs. (24) and (25). Then $\boldsymbol{u}_1$ itself can be obtained from D_1 and $\boldsymbol{R}_1$ because of the vector identity

$$\nabla^2\boldsymbol{u}_1 = \nabla D_1 - \nabla\times\boldsymbol{R}_1.$$

This equation is most conveniently handled by introducing the scalar and vector potentials defined by

$$\nabla^2\phi_1 = -D_1, \quad (26)$$

$$\nabla^2\boldsymbol{A}_1 = -\boldsymbol{R}_1, \quad (27)$$

in terms of which

$$\boldsymbol{u}_1 = -\nabla\phi_1 + \nabla\times\boldsymbol{A}_1. \quad (28)$$

Equation (28) indicates that the irrotational and incompressible parts of the motion can be treated independently. Only the steady state of the latter will be considered further, the discussion being based on Eq. (25a).

The Diffusion of Vorticity

This equation has the general form of Fourier's equation for the conduction of heat or the diffusion of matter. Hence vorticity is generated in those regions where its right side is different from zero, and diffuses into other regions. However, the analogy to heat conduction and diffusion is not complete.

Being a partial differential equation, Eq. (25) has many solutions, and that solution appropriate to the given problem is determined by the boundary condition. In the case of heat flow, this is usually one of two: if the boundaries are thermally insulated, the normal component of the temperature gradient will be zero on them; if the boundaries are kept at a fixed temperature, this fact serves to determine the special solution.

Neither of these boundary conditions applies to vorticity. This is perhaps to be expected, since vorticity is a vector rather than a scalar. In fact, the accepted boundary condition is not formulated in terms of $\boldsymbol{R}_1$ but in terms of $\boldsymbol{u}_1$, and requires that all components of the latter vanish on the walls of the container.

For simplicity, let it be supposed that there is no free surface, so that $\boldsymbol{u}_1$ vanishes at all boundaries of the fluid. Then an easy application of Stokes' theorem shows that

$$\boldsymbol{n}\cdot\boldsymbol{R}_1 = 0, \quad (29)$$

$\boldsymbol{n}$ being the unit normal at the boundary, while the divergence theorem results in

$$\iiint \boldsymbol{R}_1 dv = 0. \quad (30)$$

These are the only conditions restricting the solutions of Eq. (25) that can be formulated in terms of $\boldsymbol{R}_1$ *alone*. They are very weak conditions, and do not suffice to determine the solution uniquely. This latter step must be postponed until $\boldsymbol{u}_1$ has been found from Eqs. (27) and (28).

Because of this difference in the boundary conditions, vorticity may be generated at the walls of the containing vessel in a manner that is quite different from the influx (or efflux) of heat in the thermal analogy. Consequently also, the steady states of vortex distribution will be quite different from those familiar in the theory of heat. If these general considerations are not kept in mind, the reader may be surprised at some of the results obtained when the above equations are applied to a special case.

PART III. THE FLOW CAUSED BY A BEAM OF SOUND

In order to simplify the calculations, consider a long tube of radius r_0, whose walls are rigid, and whose ends are closed by some material that permits an axial sound beam to enter and leave the tube without reflection. These ends prevent fluid from entering or leaving the tube; the latter is long enough so that at its center all effects due to the ends may be neglected, except that the total flow through any cross section must be zero.

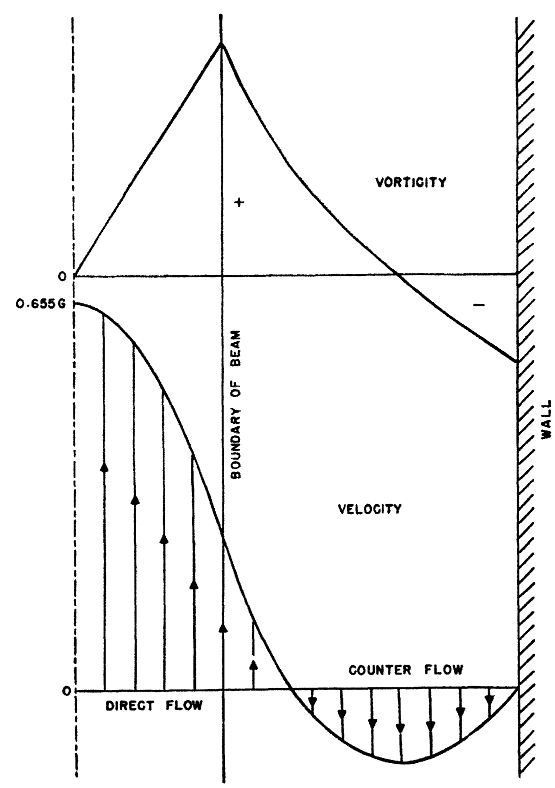

FIG. 1.

If the axis of the tube is the z axis and r the perpendicular distance from the axis, the pressure variations in the sound beam will be assumed to be

$$p_1 = \rho_1 c_0^2 = P(r)\sin(kz - nt). \tag{31}$$

This requires some justification, since it neglects both the divergence and the attenuation of the beam. The former is justified if the wave-length $(=2\pi/k)$ is very small compared to the diameter of the beam. The latter would not be justified except that only the ultimate steady state is to be investigated here, and by Eq. (25a), this is independent of the viscosity except as the latter enters into ρ_1. It is thus permissible to consider first the case of negligible attenuation, and to reserve until later the complications resulting from attenuation of the sound beam.

Using Eq. (31), Eq. (25a) reduces to

$$-\nabla^2 \boldsymbol{R}_1 = K(d^2P/dr^2)(-\mathbf{i}\sin\phi + \mathbf{j}\cos\phi), \tag{32}$$

where

$$K = bk/2\rho_0^2 c_0^3, \tag{33}$$

and ϕ is the azimuthal coordinate. This equation has the special solution

$$\boldsymbol{R}_1 = f(r)(-\mathbf{i}\sin\phi + \mathbf{j}\cos\phi), \tag{34}$$

where

$$f(r) = (K/r)\int_0^r rP^2 dr + 2\beta r + \gamma/r, \tag{35}$$

β and γ being constants of integration. The value of γ must be zero, since infinite values of the vorticity are impossible; the value of β remains indeterminate, since both Eq. (29) and Eq. (30) are satisfied for any value of β.

The calculation of the vector potential $\boldsymbol{A}_1$ can be avoided by noting that if $u_{1x} = u_{1y} = 0$, $u_{1z} = g(r)$, then

$$\boldsymbol{R}_1 = -(dg/dr)(-\mathbf{i}\sin\phi + \mathbf{j}\cos\phi),$$

so that

$$dg/dr = -f(r). \tag{36}$$

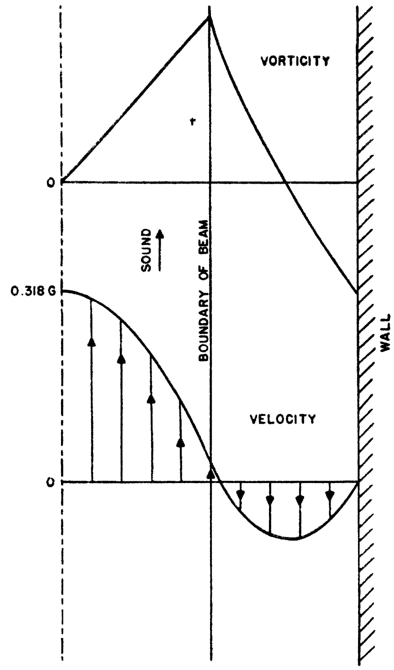

FIG. 2.

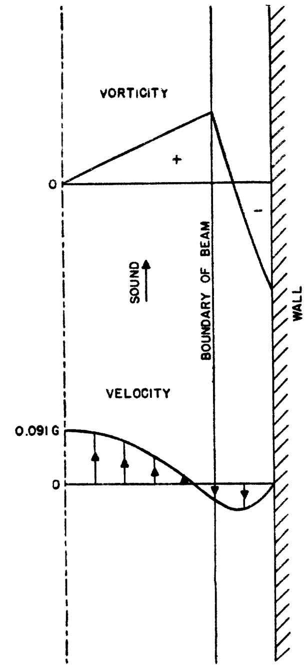

FIG. 3.

That solution of this equation is required which makes $g(r_0)=0$, since u_1 must vanish on the wall of the tube; this is given by

$$g(r)=\int_r^{r_0} f(r)dr$$

$$=K\int_0^{r_0} \Gamma(s,r)P^2(s)ds+\beta(r_0^2-r^2), \quad (37)$$

where

$$\Gamma(s,r)=s\log(r_0/r) \quad \text{when} \quad s\leqslant r, \qquad (38)$$
$$=s\log(r_0/s) \quad \text{when} \quad s\geqslant r.$$

Until this place, it has not been possible to assign a value to the constant of integration, β. The condition that there be no net flow through the tube can now be imposed:

$$\int_0^{r_0} rg(r)dr=0,$$

which is equivalent to

$$\beta=(K/r_0^4)\int_0^{r_0}(rr_0^2-r^3)P^2(r)dr. \quad (39)$$

This completes the formal solution of the problem; it remains to consider the numerical relations.

In order to carry the calculations further, suppose that the sound beam has the radius r_1, and a constant intensity throughout; then

$$P(r)=P_0, \quad r\leqslant r_1; \qquad (40)$$
$$=0, \quad r>r_1.$$

It will be found that the velocity of the stream is proportional to

$$G=\tfrac{1}{2}KP_0^2r_1^2, \quad (41)$$

so that it is convenient to calculate this quantity for several special cases.

First, suppose that the medium is water, that the beam has a radius $r_1=1.5$ cm, while $P_0=10^5\mu b$ ($=0.1$ atmos.). Since $c_0=1.5\times10^5$ cm/sec.

$$G=\tfrac{1}{6}bk^2\times10^{-5} \text{ cm/sec. (water)}.$$

If the frequency of the sound is 24 megacycles, $k=10^3$, and hence $G=1.57b$ cm/sec. It will be expected, therefore, that these streams will become appreciable only at frequencies above 1 megacycle, and have a negligible velocity at lower frequencies. This is in agreement with observation.

If the medium is air, the value of K is much larger because it is inversely proportional to the square of the density and the cube of the sound velocity. Supposing the sound beam to have the same radius and intensity, it is found that

$$G = 95.0bk^2 \text{ cm/sec. (air)}.$$

For a frequency of 1 kilocycle, $k = 0.19$, and hence $G = 3.4b$ cm/sec. Consequently, these streams should become important in air at frequencies above several hundred cycles. This is perhaps in agreement with Rayleigh's resonator experiment mentioned above, which was performed at 256 cycles per second.

While the velocity of the stream is proportional to G, its value will vary across the section of the tube which confines it. The direction of flow will coincide with that of the acoustic energy on the axis of the beam, and will be compensated by a counter-flow near the walls of the tube. The complete expression for the velocity is most conveniently written in terms of the ratios

$$x = r/r_0, \quad y = r_1/r_0,$$

and is

$$g = G\{\tfrac{1}{2}(1-x^2/y^2) - (1-\tfrac{1}{2}y^2)(1-x^2) - \log y\}, \quad 0 \leqslant x \leqslant y; \tag{42}$$
$$= -G\{(1-\tfrac{1}{2}y^2)(1-x^2) + \log x\}, \quad y \leqslant x \leqslant 1.$$

Graphs of g as a function of x, are given, for various values of y, in Figs. 1, 2, and 3.

The maximum value of g occurs on the axis and depends on the resistance offered to the flow by the confining tube. When the radius of the latter is infinite, the maximum value of g becomes logarithmically infinite. On the other hand, when the sound beam fills the whole of the tube, the flow stalls and $g = 0$ everywhere.

Experimental Determination of b

The experimental determination of the numerical constant b has eluded three generations of physicists. Stokes argued that its value must be $4/3$—i.e., that the bulk viscosity must be zero. This hypothesis has been theoretically verified for an ideal gas. Stokes' arguments do not appear to be convincing in the case of liquids, and there is a growing belief (based on discrepancies between the observed absorption of high frequency sound and that calculated on the assumption that $b = 4/3$) that it may have larger values. It is hoped that these calculations may suggest methods for its experimental measurement.

It will be noted, however, that the above discussion shows that the streaming velocity is very sensitive to the geometry of the experiment, and will readily become turbulent. Consequently, the experimental measurement of b may not be easy.