On a Different Form for the Interaction Between Dirac's Electron Field (ψ) and Maxwell's Electromagnetic Field (Φ, A)

FREDERICK J. BELINFANTE
Department of Physics, University of British Columbia, Vancouver, B.C., Canada
November 18, 1947

THERE is little doubt about the validity of Dirac's equation for the wave function of a free electron

$$i\hbar\partial\psi/\partial t=H_{op}{}^{0}\psi\equiv(mc^2\beta-i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla})\psi \tag{1}$$

within the limits of special relativity and present wave-mechanical principles. The recent experiments of Lamb and Retherford[1] do not shatter our believe in this equation. There is even hope that the newly discovered shift of the $2S_{\frac{1}{2}}$ level may be explained within the framework of Dirac's equation for the wave function of an electron in a Maxwell field

$$i\hbar\partial\psi/\partial t=H_{op}\psi=H_{op}{}^{0}\psi+(-e)(\Phi-\mathbf{A}\cdot\boldsymbol{\alpha})\psi, \tag{2}$$

if due attention is paid to the term with **A**, that is, to the interaction with the radiation field.[2]

It might be interesting, though, to check that no form of interaction with the Maxwell field different from (2) is capable of explaining the experimental data. A well-known attempt of introducing such an interaction within the framework of special relativity is presented, for instance, by the formula

$$\{(\boldsymbol{\nabla}-[e/i\hbar c]\mathbf{A})^2-(\partial/c\partial t+[e/i\hbar c]\Phi)^2\}\psi=(mc/\hbar)^2\psi, \tag{3}$$

which is obtained by replacing E by $E_{\text{total}}-E_{\text{pot}}$, etc., in the Gordon-Klein equation rather than in the Dirac equation (1). It can easily be shown that this is equivalent to denying the existence of the spin magnetic moment of the electron while retaining its spin mechanical moment of momentum $\frac{1}{2}\hbar$. It has been demonstrated[3] that this formula (3) leads for the hydrogen atom to levels approximately given by

$$E_{nl}\approx mc^2-\frac{R}{n^2}+R\alpha^2\left(\frac{3}{4n^4}-\frac{1}{n^3(l+\frac{1}{2})}\right)+\cdots \tag{4}$$

with $n-l-1=$ integer, so that in first approximation we obtain the correct Bohr levels, if (the orbital quantum number) l is integer. ($R=e^4m/2\hbar^2$ and $\alpha=e^2/\hbar c\approx 1/137$.)

On the other hand, Dirac's equation (2) yields

$$E_{nj}\approx mc^2-\frac{R}{n^2}+R\alpha^2\left(\frac{3}{4n^4}-\frac{1}{n^3(j+\frac{1}{2})}\right)+\cdots \tag{5}$$

with $n-j-\frac{1}{2}=$ integer, which is correct in first approximation with the quantum number of total angular momentum j half odd. As this formula (5) is in better agreement with experimental data on the fine structure than (4), one assumes that (3) cannot be correct.

Equations (2) and (3), though, are not the only two hypotheses possible for a relativistic equation for ψ in interaction with Φ and **A**. This is easily seen as follows:

Dirac's equation (2) can be obtained from a variational principle, in which

$$\rho=(-e)\psi^\dagger\psi \quad\text{and}\quad \mathbf{i}=(-e)\psi^\dagger\boldsymbol{\alpha}\psi \tag{6}$$

are considered as the charge density (in e.s.u.) and current density (in e.m.u.). A different equation now could be found, in perfect agreement with the principle of special relativity, if we would replace (6) in the variational principle by the almost identical expressions[4]

$$\left.\begin{aligned}\rho&=Re\{(e\hbar/imc^2)\psi^\dagger\beta\partial\psi/\partial t\},\\ \mathbf{i}&=Re\{(ei\hbar/mc)\psi^\dagger\beta\boldsymbol{\nabla}\psi\},\end{aligned}\right\} \tag{7}$$

where Re means the real part. The Dirac equation is then replaced by

$$i\hbar\partial\psi/\partial t=H_{op}{}^{0}\psi-i(e\hbar/mc)\beta\{\mathbf{A}\cdot\boldsymbol{\nabla}+\phi\partial/c\partial t\}\psi. \tag{8}$$

Maxwell's equations, following from the same variational principle, will ensure the validity of the continuity equation for ρ and **i**.

From (8), the corresponding levels for a hydrogen atom can be found by putting $\phi=e/r$, $\mathbf{A}=0$, and $i\hbar\partial/\partial t=E$. The result of the calculation is that $\kappa=E_{nj}/mc^2$ is a root of the equation

$$\alpha\kappa=[n'+[(j+\tfrac{1}{2})^2+\alpha^2\kappa^2]^{\frac{1}{2}}]\cdot(1-\kappa^2)^{\frac{1}{2}}, \tag{9}$$

so that, by successive approximations, we find

$$E_{nj}\approx mc^2-\frac{R}{n^2}+R\alpha^2\left(\frac{3}{4n^4}+\frac{1}{n^3(j+\frac{1}{2})}\right)+\cdots. \tag{10}$$

Here, like in (5), $n'=n-j-\frac{1}{2}=$ integer.

Comparing (10) with Sommerfeld-Dirac's formula (5) we see that again we get the correct Bohr levels, but that the fine structure given by the j-dependent term in (10) lies inverted. Trusting that spectroscopists measuring and checking formula (5) did not interchange both sides of their fine-structure spectra, we must conclude that (10) cannot fit the experimental data, so that the hypotheses (7)–(8) also cannot be correct.

[1] W. E. Lamb and R. C. Retherford, Phys. Rev. **72**, 241 (1947).
[2] H. A. Bethe, Phys. Rev. **72**, 339 (1947).
[3] See, for instance, A. Sommerfeld, *Atombau und Spektrallinien Wellenmechanischer Ergänzungsband* (F. Vieweg und Sohn, Braunschweig, 1929), Kap. I, §9E, pp. 132–138, or the English translation. S. A. Goudsmit, in his doctor's thesis (University of Leiden, 1927), states that W. Pauli was the first to derive (4) from (3).
[4] From a classical point of view, ρ and **i** from (6) represent $\psi^\dagger\psi$ times $(-e)$ and $(-e)\mathbf{v}/c$, while ρ and **i** from (7) represent $\psi^\dagger\psi$ times $(-e)(E/mc^2)(1-v^2/c^2)^{\frac{1}{2}}$ and $(-e)(\mathbf{p}/mc)(1-v^2/c^2)^{\frac{1}{2}}$.

Neutron-Proton Scattering at 100 Mev

JULIAN EISENSTEIN AND FRITZ ROHRLICH
Lyman Laboratory, Harvard Universly, Cambridge 38, Massachusetts
February 7, 1948

A VALUE $0.083\pm0.004\times10^{-24}$ cm^2 for the neutron-proton total cross section at 90 Mev has recently been reported by Cook, McMillan, Peterson, and Sewell.[1]

We have made calculations to determine whether it is possible to fit this value of the cross section if one uses existing phenomenological theories of nuclear forces. The results at 100 Mev for a rectangular well interaction potential are as follows:

1. Central, non-exchange force. Triplet range, 2.80