Wave Transmission and Reflection Phenomena in Liquid Helium II*

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A method is formulated for treating acoustical transmission and boundary value problems in liquid helium II. According to present concepts, He II is a mixture of two fluids obeying a special system of complex hydrodynamics. In particular, this is known to result in two virtually independent modes of sound propagation. Therefore, a reformulation of the intrinsic (or characteristic) acoustical impedance concept is required for which a matrix representation is applicable. By similarly associating a matrix form of impedance with plane-reflecting surfaces, boundary conditions may be imposed. The classical requirements (continuity of pressure and particle velocity at the boundary) are generalized to apply individually to each fluid component in He II. Expressions are obtained for the reflective properties of various types of surfaces. In particular, materials which present unlike boundary conditions to the two fluid components are shown capable of partially converting one mode of sound to the other upon reflection. For example, surfaces of highly porous substances exert unequal viscous forces and should therefore act as such converters (with possible application for extending present frequency ranges of second sound). These properties of reflectors are expressed in terms of *reflectivity arrays*. The array gives direct reflective factors for both types of sound, plus coupling factors between types. Examples are given for several special cases, and a form of reciprocity is shown to exist for the coupling process. The boundary condition is derived for still another type of coupling, due to heat transfer, which occurs at a liquid-vapor interface; a modified form is applicable to the resonance type (Yale) experiment.

I. INTRODUCTION

THE macroscopic hydrodynamic equations of liquid helium II have been developed by Tisza[1,2] and Landau.[3,3a] Whereas these investigators started from different molecular assumptions, most of their macroscopic results were identical. Actually Tisza[2] has recently shown that these results can be obtained from very general assumptions leaving the molecular interpretation open to a large extent. Whatever differences do exist between the two theories are irrelevent for the problems discussed here.

The essence of the complex hydrodynamics of helium II is the presence of two interpenetrating liquids ("normal" and "superfluid") of different densities

$$\rho=\rho_n+\rho_s, \quad (1)$$

velocity fields v_n, v_s, and correspondingly two modes of longitudinal sound propagation. In the first sound (pressure waves) the two liquids move in phase; in the second (temperature waves) the two velocities are out of phase so as to give no net transfer of matter. The first sound can be generated by an ordinary transducer, the second by periodical heating as demonstrated by Peshkov.

The subject of the present paper is to develop a scheme for the solution of boundary value problems in this complex hydrodynamics (reflection, transmission). The interest of this problem lies in the fact that particular boundaries may affect the two fields in a different manner, producing thereby an unbalance, or coupling between the two sound modes.

Generally speaking, there are two reasons for this: (1) heat absorption or rejection by He II is accomplished by the transition $\rho_s\rightarrow\rho_n$, or *vice versa*, (2) the boundary conditions are different for v_n and v_s since the superfluid liquid can slip along solid walls.

(1) has been used to transform second sound generated in the liquid into ordinary sound in the equilibrium vapor phase detectable by a microphone (Yale).[4] (2) could be used to generate second sound mechanically which would be advantageous for high frequencies. The practical application of this principle is not so obvious, since longitudinal waves involve particle motions perpendicular to the plane of a radiating surface, whereas the difference in boundary conditions exists only for the tangential velocity

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[1] L. Tisza, J. de phys. et rad. (8) 1, 165 and 350 (1940).
[2] L. Tisza, Phys. Rev. 72, 838 (1947).
[3] L. Landau, J. Phys. U.S.S.R. 5, 71 (1941).
[3a] L. Landau, J. Phys. U.S.S.R. 8, 1 (1944).

[4] C. Lane, H. A. Fairbank, and W. M. Fairbank, Phys. Rev. 71, 600 (1947).

components. An artifice to circumvent this difficulty consists in using surfaces of porous materials, thereby creating a region of helium where the direction *perpendicular* to the radiating surface proper can be considered—from the microscopic point of view—tangential to the walls. (This application is analogous to the possibility examined by Lifshitz[5] of generating second sound by the oscillations of a small sphere, which, however, was shown by him to be inefficient.)

The properties of such surfaces (briefly semi-impervious surfaces) can be conveniently described in terms of an acoustic impedance, which in the complex hydrodynamics of helium II will have the form of a matrix.

Section II will contain the matrix formulation of the general wave propagation in helium II whereby an intrinsic matrix impedance will be defined. Sections III, IV, and V will contain the discussion of transmission and reflection of various surfaces characterized by different impedances.

II. WAVE PROPAGATION IN HELIUM II. INTRINSIC IMPEDANCE

The hydrodynamic equations and in particular the equations of wave propagation in helium II can be described in two different sets of coordinates. In the first set (called briefly the x-scheme), one considers the displacement vectors of the two fluids x_n, x_s and the corresponding velocities $\dot{x}_n$, $\dot{x}_s$. In the second set (briefly the ξ-scheme) one considers "normal coordinates" ξ_1, ξ_2 introduced by Tisza[2] corresponding to the two modes of sound propagation. (Also x_n, x_s are identical to ξ_n, ξ_s in Tisza's notation.) The transformation connecting these schemes is

$$\begin{aligned} x_n &= \xi_1+\xi_2, & \xi_1 &= (\rho_n x_n+\rho_s x_s)/\rho, \\ x_s &= \xi_1-(\rho_n/\rho_s)\xi_2, & \xi_2 &= (\rho_s/\rho)(x_n-x_s). \end{aligned} \tag{2}$$

Obviously ξ_1, refers to a "center of mass" motion (first sound) and ξ_2 to a "relative motion" with vanishing net flow (second sound). The general wave motion in the interior of the liquid can most conveniently be described in the ξ-scheme. On the other hand, the boundary conditions, particularly at semi-impervious surfaces can be expressed rather in the x-scheme. Hence the transformations between the two schemes are of interest. These can be represented best in a matrix form. We consider only plane waves traveling in one direction and define the following two component "vectors."

$$\mathbf{X}=\begin{pmatrix} x_n \\ x \end{pmatrix}, \quad \xi=\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \tag{3}$$

and their adjoints

$$\mathbf{X}^* = \widehat{x_n, x_s}, \quad \xi^* = \widehat{\xi_1, \xi_2}. \tag{3a}$$

We write (2) in matrix form,

$$\mathbf{X}=\mathbf{S}\xi, \quad \xi=\mathbf{S}^{-1}\mathbf{X}, \tag{4}$$

with

$$\mathbf{S}=\begin{pmatrix} 1 & 1 \\ 1 & -\alpha \end{pmatrix}, \quad \mathbf{S}^{-1}=\frac{1}{1+\alpha}\begin{pmatrix} \alpha & 1 \\ 1 & -1 \end{pmatrix}, \tag{5}$$

where the abbreviation $\alpha=\rho_n/\rho_s$ is used. $\mathbf{S}$ is self-adjoint (does not change if rows and columns are interchanged); hence the adjoint relationships are

$$\mathbf{X}^*=\xi^*\mathbf{S}, \quad \xi^*=\mathbf{X}^*\mathbf{S}^{-1}. \tag{4a}$$

The density of the kinetic energy is

$$W=\tfrac{1}{2}\rho_n\dot{x}_n^2+\tfrac{1}{2}\rho_s\dot{x}_s^2=\tfrac{1}{2}\rho\dot{\xi}_1^2+\tfrac{1}{2}\rho\alpha\dot{\xi}_2^2, \tag{6}$$

where a dot above the symbol indicates the time derivative. The last two terms correspond to the two modes of sound propagations. The total energy flow, or intensity, is obtained by multiplying each energy density of Eq. (6) by its corresponding wave velocity, c_1 and c_2.

$$\gamma=\tfrac{1}{2}\rho c_1\dot{\xi}_1^2+\tfrac{1}{2}\rho\alpha c_2\dot{\xi}_2^2. \tag{7}$$

This expression suggests the definition of a generalized intrinsic impedance

$$\mathbf{Z}_{0\xi}=\begin{pmatrix} \rho c_1 & 0 \\ 0 & \alpha\rho c_2 \end{pmatrix}, \tag{8}$$

and the intensity appears then as

$$\gamma=\tfrac{1}{2}\dot{\xi}^*\mathbf{Z}_{0\xi}\dot{\xi}, \tag{9}$$

in close analogy with the usual acoustic case. $\mathbf{Z}_{0\xi}$ is a diagonal matrix because we neglect the coupling terms between the two sounds. This coupling term is proportional to the coefficient

[5] E. Lifshitz, J. Phys. U.S.S.R. 8, 110 (1944).

of thermal expansion and is extremely small (see Lifshitz and Tisza).

Equation (9) suggests the definition of a generalized pressure

$$\mathbf{P}_\xi=\mathbf{Z}_{0\xi}\dot{\boldsymbol{\xi}}=\begin{pmatrix}\rho c_1\dot{\xi}_1\\ \alpha\rho c_2\dot{\xi}_2\end{pmatrix}. \tag{10}$$

The intensity is then

$$\gamma=\tfrac{1}{2}\dot{\boldsymbol{\xi}}^*\mathbf{P}_\xi=\tfrac{1}{2}\mathbf{P}_\xi^*\dot{\boldsymbol{\xi}}. \tag{9a}$$

The physical meaning of $\mathbf{P}_\xi$ is more apparent from an alternative form which we are now going to derive.

One has for a plane wave of phase velocity c_1

$$\dot{\xi}_1=-c_1\nabla\cdot\xi_1 \tag{11}$$

and

$$-\nabla\cdot\xi_1=\Delta\rho/\rho. \tag{12}$$

Equation (12) is the equation of continuity (compare Tisza) and finally

$$c_1^2=\Delta P/\Delta\rho, \tag{13}$$

P and ρ are the pressure and density, respectively. From Eqs. (11), (12), and (13) one has

$$\rho c_1\dot{\xi}_1=\Delta P.$$

The analagous expressions for second sound are, according to Tisza,[2]

$$\dot{\xi}_2=-c_2\nabla\cdot\xi_2, \tag{11a}$$

$$-\nabla\cdot\xi_2=\Delta\rho_n/\rho_n, \tag{12a}$$

$$c_2^2=(\Delta P_n/\Delta\rho_n)(\rho_s/\rho) \tag{13a}$$

hence, $\alpha\rho c_2\dot{\xi}_2=\Delta P_n$ and

$$\mathbf{P}_\xi=\begin{pmatrix}\Delta P\\ \Delta P_n\end{pmatrix}. \tag{14}$$

Here ΔP_n is the excess of the "osmotic pressure," introduced by Tisza,[1,2] which plays the same role for second sound as the excess of ordinary pressure for first sound.

Although the impedance in the x-scheme is not necessary to the solution of boundary value problems, its formulation does give useful insight to the manner in which the boundary conditions for such problems may be introduced. Since the intensity is invariant, one has

$$\dot{\boldsymbol{\xi}}^*\mathbf{Z}_{0\xi}\dot{\boldsymbol{\xi}}=\dot{\mathbf{X}}^*\mathbf{S}^{-1}\mathbf{Z}_{0\xi}\mathbf{S}^{-1}\dot{\mathbf{X}}=\dot{\mathbf{X}}^*\mathbf{Z}_{0x}\dot{\mathbf{X}},$$

so that

$$\mathbf{Z}_{0x}=\mathbf{S}^{-1}\mathbf{Z}_{0\xi}\mathbf{S}^{-1},$$

and hence

$$\mathbf{Z}_{0x}=(\rho_n\rho_s/\rho)\begin{pmatrix}(\rho_n/\rho_s)c_1+c_2 & c_1-c_2\\ c_1-c_2 & (\rho_s/\rho_n)c_1+c_2\end{pmatrix}. \tag{16}$$

The four-element matrix of (16) bears a close analogy to a four-terminal electrical network in which one set of terminals is considered to correspond to the normal fluid component, the other to the superfluid component. We imagine a geometrical plane passed through a point of observation perpendicular to the direction of wave propagation. Then, for outgoing waves only, $(\rho_n\rho_s/\rho)[(\rho_n/\rho_s)c_1+c_2]$ is the impedance experienced by normal fluid when motion of superfluid across the plane is prohibited. To extend the analogy, the transfer impedance, $(\rho_n\rho_s/\rho)(c_1-c_2)$, is the pressure which must be exerted against the superfluid to suppress its motion for unit velocity amplitude of normal fluid (or the reverse for the other off-diagonal element). Similarly $(\rho_n\rho_s/\rho)[(\rho_s/\rho_n)c_1+c_2]$ is the superfluid input impedance for normal fluid "clamped."

As will become apparent in Section III, the boundary conditions imposed by the porous reflectors mentioned earlier may be introduced most logically in the x-scheme. For example, a thin layer, or region, exhibiting viscous properties serves effectively as a lumped resistance inserted in series between otherwise extended regions possessing intrinsic (or characteristic) impedance $\mathbf{Z}_{0x}$. After expressing in this manner the net resulting x-scheme impedance presented at the layer, conversion is made to the ξ-scheme for application of the boundary conditions derived in Section II.[6]

[6] It is of interest to express the generalized pressure in the x-scheme. One has

$$\dot{\boldsymbol{\xi}}^*\mathbf{P}_\xi=\dot{\mathbf{X}}^*\mathbf{P}_x,$$

and

$$\mathbf{P}_x=\mathbf{S}^{-1}\mathbf{P}_\xi=\begin{pmatrix}p_n\\ p_s\end{pmatrix}.$$

Here p_n and p_s are the excess sound pressures of normal fluid and superfluid considered by Landau.[3a] Accordingly

$$\begin{pmatrix}p_n\\ p_s\end{pmatrix}=\mathbf{P}_x=\frac{1}{1+\alpha}\begin{pmatrix}\alpha\Delta P+\Delta P_n\\ \Delta P-\Delta P_n\end{pmatrix}=\begin{pmatrix}\rho_n c_1\dot{\xi}_1+\rho_n c_2\dot{\xi}_2\\ \rho_s c_1\dot{\xi}_1-\rho_n c_2\dot{\xi}_2\end{pmatrix}.$$

The total pressure is the sum of the two components $=\Delta P$. Hence, only the first sound contributes to the total pressure! Ordinary transducers are accordingly incapable of generating or detecting second sound.

III. BOUNDARY CONDITIONS AT A PLANE-REFLECTING SURFACE

We proceed now to derive the boundary conditions which hold at a reflecting surface. In order to keep the problem purely mechanical for the time being, the conditions will be derived first for the case of a boundary which is a perfect heat insulator. That is, no interchange $\rho_s \rightleftarrows \rho_n$ will occur; and any unbalance introduced between the two modes of propagation will be due solely to unequal viscous forces on the two fluid components.

The boundary conditions of classical acoustics are that particle velocity and particle pressure must both be continuous across any interface. (This is equivalent to specifying continuity of velocity and energy flow.) The same is true for the case of He II, except that here both the velocity and the pressure are matrices. Having derived a relationship in the x-scheme in terms of true particle velocities and pressures, we may then transform to ξ-coordinates; the latter system is preferable for dealing with the distribution of energy flow between the two modes of propagation.

Let the incident sound energy of He II travel along the positive y axis and encounter a boundary surface defined by $y=0$ in Fig. 1. In general, there will be some energy reflected back in the negative y direction, and some will continue through the interface (where it may or may not involve two modes, depending upon whether liquid He II is involved for $y>0$).

Let $\dot{\mathbf{X}}_i$ represent the true particle velocity due to incident waves and $\dot{\mathbf{X}}_r$ that due to reflected waves; then the effective velocity experienced by the interface is just the sum of these, or $\dot{\mathbf{X}}_i+\dot{\mathbf{X}}_r$. In order to specify the requirements on pressure, a matrix impedance $\mathbf{Z}_x$ is assigned to the reflecting surface. This matrix determines the reflectivity characteristics. The effective pressure driving the surface then becomes the product of $\mathbf{Z}_x$ times the particle velocity, or $\mathbf{Z}_x(\dot{\mathbf{X}}_i+\dot{\mathbf{X}}_r)$. But the pressure supported by the standing-wave system must be identical with this value, and is given by $\mathbf{Z}_{0x}(\dot{\mathbf{X}}_i-\dot{\mathbf{X}}_r)$. The reversed direction of

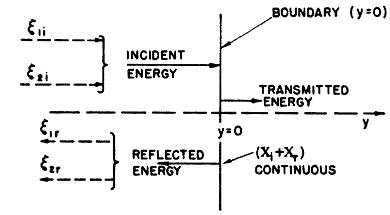

FIG. 1. Reflection of sound in helium II from plane boundary.

propagation for the reflected wave accounts for the minus sign. Combining, we have the boundary condition

$$\mathbf{Z}_{0x}(\dot{\mathbf{X}}_i-\dot{\mathbf{X}}_r)=\mathbf{Z}_x(\dot{\mathbf{X}}_i+\dot{\mathbf{X}}_r) \tag{17}$$

in the x-scheme. Since our concern is primarily with the relative amounts of first and second sound reflected, transformation is made to the ξ-scheme. Employing (4) and (4a), we have

$$\mathbf{Z}_{0\xi}(\dot{\xi}_i-\dot{\xi}_r)=\mathbf{Z}_\xi(\dot{\xi}_i+\dot{\xi}_r), \tag{18}$$

where $\mathbf{Z}_\xi$ is now the characteristic impedance for the surface in the ξ-scheme. The same relationship as used before holds for transforming reflector impedances from one scheme to the other, namely

$$\mathbf{Z}_\xi=\mathbf{S}\mathbf{Z}_x\mathbf{S}. \tag{19}$$

We may solve (18) for the amplitude ξ_r of the reflected waves to obtain the following matrix in terms of the incident waves ξ_i

$$\xi_r=(\mathbf{Z}_{0\xi}+\mathbf{Z}_\xi)^{-1}(\mathbf{Z}_{0\xi}-\mathbf{Z}_\xi)\xi_i=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix}\xi_i. \tag{20}$$

The diagonal elements A_{11}, A_{22} represent the fractional amounts of incident first- and second-sound amplitudes which are reflected unchanged. Conversely, the diagonal terms A_{12}, A_{21} represent transfer of amplitudes between modes. However, a more significant property of the reflector is the manner in which intensities leaving its surface are divided between first and second sound. For our purposes therefore we allow only one mode of sound at a time to strike the surface and build up a *reflectivity array*. This array, which must not be confused with a matrix, may be written:

$$\text{Reflectivity}=\left|\begin{array}{c|c} F_{11}=\gamma_{1r}/\gamma_{1i}=A_{11}^2 & F_{12}=\gamma_{1r}/\gamma_{2i}=(\rho_s c_1/\rho_n c_2)A_{12}^2 \\ \hline F_{21}=\gamma_{2r}/\gamma_{1i}=(\rho_n c_2/\rho_s c_1)A_{21}^2 & F_{22}=\gamma_{2r}/\gamma_{2i}=A_{22}^2 \end{array}\right|. \tag{21}$$

Thus, if unit intensity of first sound strikes the surface, then intensities $F_{11}=A_{11}^2$ of first sound and $F_{21}=(\rho_n c_2/\rho_s c_1)A_{21}^2$ of second sound will leave the surface. Similarly, for unit incident intensity of second sound, intensities F_{12} of first sound and F_{22} of second sound will be reflected. In this manner, the diagonal elements play the roles of ordinary reflection coefficients, while the off-diagonal elements act as transfer factors. Evaluation of the reflectivity array will be given for specific cases in the following section.

IV. SPECIAL REFLECTING SURFACES

We now turn to the examination of specific cases of reflecting boundaries, still restricting ourselves, however, to the purely mechanical case, i.e., no transition between fluid components. The types of reflectors in which we shall be interested may be divided logically into two main categories; (1) impervious surfaces and (2) semi-impervious surfaces (a third category comprising pervious boundaries[7] will not be considered here).

1. Impervious Surfaces

This category includes any boundary which is permeable to neither normal fluid nor superfluid. As a result the perpendicular component of the internal convection peculiar to second sound is prohibited at the surface (since we have specified infinitely poor heat conductivity). Therefore, ξ_2 is zero at $y=0$, so that all second sound is reflected. We have for the impedance and the reflectivity

$$\mathbf{Z}_\xi=\begin{pmatrix} Z_1 & 0 \\ 0 & 0 \end{pmatrix}; \quad \text{Reflectivity}=\begin{array}{|c|c|} \hline \gamma_{1r}/\gamma_{1i}=\left|\dfrac{\rho c_1-Z_1}{\rho c_1+Z_1}\right|^2 & \gamma_{1r}/\gamma_{2i}=0 \\ \hline \gamma_{2r}/\gamma_{1i}=0 & \gamma_{2r}/\gamma_{2i}=1 \\ \hline \end{array}, \tag{22}$$

where Z_1 is the usual mechanical impedance of the surface to an ordinary acoustic wave (and therefore the impedance experienced by normal sound). For the element γ_{1r}/γ_{1i} the absolute magnitude of the ratio is used, sinze Z_1 may have reactive components. Note the diagonal nature of this reflectivity array, which indicates no coupling between modes of propagation. In particular for a very thin membrane (thin compared to a first-sound quarter wave-length) with low heat conductivity and surrounded by liquid He II, the mechanical impedance will reduce simply to ρc_1. This results in complete transparency to first sound. The array becomes:

$$\underset{\text{(Membrane)}}{\text{Reflectivity}}=\begin{array}{|c|c|} \hline 0 & 0 \\ \hline 0 & 1 \\ \hline \end{array}. \tag{23}$$

Its significance is that ideally such a system would constitute a mode filter, transmitting all first sound, but still reflecting all second sound.

2. Semi-Impervious Surfaces

The other case we shall analyze involves reflections from regions of space which present different viscous drag to the two fluid components of He II. We shall consider only the most idealized situations. The unequal viscous drag will be supposed due to the presence of a porous or honeycomb structure through which superfluid may pass unhindered (zero viscosity is one of the properties of superfluid) but which presents ordinary viscous friction to the normal fluid component. Furthermore, the honeycomb will be considered so thin walled that negligible fluid is displaced by its presence, but sufficiently rigid not to participate in the mechanical vibrations. This highly artificial condition may then be represented mathematically by introducing a

[7] Pervious boundaries might constitute such a trite case as a change in cross-sectional area of a narrow duct containing He II and conducting sound. Then the effective impedance presented at the junction would differ from the intrinsic impedance of He II only by a linear scale factor f. The impedance and reflectivity would be, respectively,

$$\mathbf{Z}_\xi=f Z_{0\xi};$$

$$\text{Reflectivity}=\begin{array}{|c|c|} \hline [(1-f)/(1+f)]^2 & 0 \\ \hline 0 & [(1-f)/(1+f)]^2 \\ \hline \end{array}.$$

normal fluid pressure gradient[8] due to viscous drag throughout all regions occupied by the structure. We shall first examine the case where only a thin layer of space is thus occupied (i.e., a thin porous screen) following which the extension to a semi-infinite space will be given.

Semi-Impervious Rigid Screen

Let the position of the porous screen be represented by a thin layer of thickness Δl at the plane $y=0$. The influence of the layer will be manifest entirely through the viscous drag opposing flow of the normal fluid component. Treating the problem as analogous to a lumped electrical impedance inserted in a continuous transmission line, the effective impedance presented by the screen may be written as

$$\mathbf{Z}_x=\mathbf{Z}_{0x}+\mathbf{R}_x\Delta l. \tag{24}$$

Here the first term is the intrinsic impedance of He II, the second the added series resistance. This series term involves the matrix

$$\mathbf{R}_x=R_0\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \tag{25}$$

or in the ξ-scheme

$$\mathbf{R}_\xi=R_0\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}, \tag{26}$$

where the scalar coefficient R_0 is the *flow resistance* for the normal fluid component. Making the simplest assumption that flow resistance behaves in the classical manner with respect to normal fluid, its definition becomes

$$R_0\dot{x}_n=\mathrm{grad}\,p_n=\Delta p_n/\Delta l. \tag{27}$$

R_0 is a real, positive quantity determined entirely by the coefficient of viscosity of normal fluid (essentially equal to that of He I) and the detailed porous structure of the honeycomb. This p_n is the excess pressure of the normal fluid component, as defined in reference 6. Converted to the ξ-scheme, the resultant impedance encountered by incoming waves at the plane $y=0$, becomes

$$\mathbf{Z}_\xi=\mathbf{Z}_{0\xi}+\mathbf{R}_\xi\Delta l=\mathbf{Z}_{0\xi}R_0\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\Delta l. \tag{28}$$

Applied to (20) the matrix for the reflected waves is

$$\xi_r=\frac{1}{2/R_0\Delta l+(1/\rho c_1+1/\alpha\rho c_2)}\begin{pmatrix}1/\rho c_1 & 1/\rho c_1\\ 1/\alpha\rho c_2 & 1/\alpha\rho c_2\end{pmatrix}\xi_i, \tag{29}$$

all elements of which become maximum for very large values of R_0. The condition for one-half maximum effect is given by

$$R_0=\frac{2/\Delta l}{1/\rho c_1+1/\alpha\rho c_2}. \tag{30}$$

For R_0 greatly in excess of this value, the reflectivity would be

$$(\text{Reflectivity})_{\text{Screen}}=\frac{1}{(1+\rho_s c_1/\rho_n c_2)^2}\cdot\left|\begin{array}{c|c}1 & \rho_s c_1/\rho_n c_2\\ \hline \rho_s c_1/\rho_n c_2 & (\rho_s c_1/\rho_n c_2)^2\end{array}\right|. \tag{31}$$

Because of the extremely low viscosity of even the normal fluid component, porous material fulfilling this condition would be virtually impermeable for ordinary liquids.

Numerical values of the four intensity ratios corresponding to the reflectivity are shown in Fig. 2. Note that the off-diagonal elements represent transfer from one mode to the other upon reflection from the screen. Figure 2 illustrates the interesting fact that the two intensity transfer factors are equal ($F_{12}=F_{21}$), so that the conversion efficiency is the same for either mode of incident sound. This is a type of reciprocity.

[8] The semi-permeable membrane applicable to thermodynamic discussions (see references 1 and 2) represents the extreme case where motion of normal fluid is completely arrested.

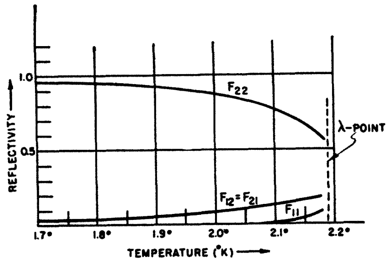

FIG. 2. Reflectivity factors *versus* temperature for thin porous screen.

The viscous properties of the thin porous screen result in a greater impedance mismatch for second sound than for first. Thus the factor F_{22} giving the reflectivity ratio for second sound (the fraction reflected without conversion) greatly exceeds F_{11}. This is a direct consequence of the relatively low wave velocity of second sound ($c_1/c_2 \geq 10$). Thus, from (16), the impedance $(\rho_n\rho_s/\rho)[(\rho_s/\rho_n)c_1+c_2]$ experienced by superfluid at the screen is determined primarily by c_1 and therefore provides a better match for first sound. Of course an actual reflector of this nature would displace an appreciable amount of liquid so that corrections would be necessary for deviation from the idealized situation assumed for deriving the curves of Fig. 2.

Extended Semi-Impervious Region

Perhaps a more practical reflector for converting first sound to second, or the reverse, would be provided by the plane face of an extended porous medium. Thus the interior honeycomb structure of, for example, a sintered material should provide differential viscous drag to the two fluids. For the case of pulsed energy, the structure could be considered infinite in extent (short enough pulses do not detect thickness of a reflector until the reflection process has been completed).

Here again the true situation can be approximated only crudely by visualizing all regions for $y>0$ as endowed with viscous properties (with respect to normal fluid) and determining the resulting characteristic impedance. Space does not permit complete analysis of this case, but it may be shown that in the extreme situation the flow resistance R will be sufficient to suppress virtually all normal fluid vibration. This occurs when $R \gg \omega\rho$.

Under such circumstances the mode corresponding to second sound reduces essentially to vibration of superfluid only, without attenuation, and with effective wave velocity

$$v_2 \rightarrow [(\rho_s c_1^2 + \rho_n c_2^2)/\rho]^{\frac{1}{2}}.$$

The mode corresponding to first sound, however, degenerates to an over-damped motion, for which the resistance to normal fluid vibration virtually precludes flow of energy.

Accordingly, this boundary should reflect a greater relative proportion of first sound than did the thin porous screen. Furthermore, the more drastic modifications of the boundary conditions should result in numerically greater coupling factors, F_{12} and F_{21}.

V. PHENOMENA INVOLVING HEAT TRANSFER

Thus far analysis has been simplified by the assumption of zero heat flow between the liquid He II and the reflector. We may now include the effects of such heat interchange. This process occurs for example when the boundary is formed by the liquid surface in equilibrium with its vapor, and is the case investigated experimentally by Lane.[4] Such a liquid-vapor interface provides coupling between the two types of sounds in the liquid and the classical sound in the vapor. For simplicity we consider only perpendicular incidence of sound waves against the free surface.

Coupling takes place due to periodic evaporation and condensation of helium at the surface. The situation for helium differs markedly from that of a classical liquid in equilibrium with its vapor. For ordinary liquids the temperature fluctuations accompanying the evaporation-condensation process occur only at the interface, being thus localized by the condition of adiabaticity. However, in He II an adiabatic means for heat transfer is provided by second sound. In this manner, temperature fluctuations occuring at the surface may be detected at (or, reciprocally, generated from) well submerged positions. We show, in fact, that the impedance encountered at the surface by incident waves of either first sound, second sound, or the vapor mode involves all three modes. This has a direct bearing on acoustical resonance methods such as employed by Lane.

To establish the boundary conditions existing at the interface it is necessary to consider both the temperature fluctuation and the heat flow inherent in second sound. Variations in temperature are produced by the varying relative concentration of normal fluid according to an empirical relationship deduced from experiment

$$\Delta\rho_n/\rho_n = r\Delta T/T. \qquad (32)$$

The factor r has been evaluated numerically as about 5.5. Equation (12a) relates this concentration to incident and reflected waves so that

$$\Delta\rho_n/\rho_n = -\nabla\cdot\xi_2 = 1/c_2(\dot{\xi}_{2i} - \dot{\xi}_{2r}) = r\Delta T/T \quad (33)$$

(where the sign has been reversed for the reflected wave velocity). For purposes of computation we now make the assumption that the vapor pressure fluctuations which occur at the surface are given directly in terms of $\Delta T/T$ by means of the Clausius-Clapyron equation. Although this is probably not the physical situation[9] the assumption suffices for specifying conditions of resonance. Hence the vapor pressure p_g is

$$p_g = \{\rho\rho_g/(\rho-\rho_g)\}L\Delta T/T$$
$$= (L/rc_2^2)\{\rho_g/(\rho-\rho_g)\}\rho c_2(\dot{\xi}_{2i} - \dot{\xi}_{2r}), \quad (34)$$

where L is the latent heat of vaporization at the ambient temperature, and ρ_g the vapor density. The boundary requirement that pressure be continuous across the interface results in

$$(L/rc_2^2)\{\rho_g/(\rho-\rho_g)\}\rho c_2(\dot{\xi}_{2i} - \dot{\xi}_{2r})$$
$$= \rho c_1(\dot{\xi}_{1i} - \dot{\xi}_{1r}) = -\rho_g c_g(\dot{x}_{gi} - \dot{x}_{gr}), \quad (35)$$

where $\dot{x}_g$ and c_g represent particle velocity and wave velocity, respectively, for the vapor. This states that the pressure associated with the interaction between second sound and the surface must support (and therefore equal) both first sound pressure in He II and classical sound pressure in the vapor. The minus sign preceding the last term accounts for the reversed sense of incidence for sound in the vapor.

The boundary condition for particle velocity at the surface is a statement of the equation of continuity. This must take into account the alternate changes in material volume due to the periodic interchange between liquid and vapor. The evaporation rate is fixed by the heat transfer characteristics of second sound in He II according to a relationship given by Tisza.[2] Therefore we have

$$\text{heat flow} = \rho_g L v_g = sT(\dot{\xi}_{2i} + \dot{\xi}_{2r}), \quad (36)$$

where s is now the specific entropy and v_g the vapor particle velocity. Note that since the wave velocity c_2 does not enter explicitly into (36) there is no change in sign for the reflected wave. We may now express our condition of continuity

$$(sT/\rho_g L)(\dot{\xi}_{2i} + \dot{\xi}_{2r}) + (\dot{\xi}_{1i} + \dot{\xi}_{1r})$$
$$- (\dot{x}_{gi} + \dot{x}_{gr}) = 0. \quad (37)$$

The first term represents source (or sink) of volume due to second sound; the latter two constitute ordinary particle flow due to He II first sound and classical sound in the vapor. Finally, by combining (35) and (37) and eliminating the time derivative, we obtain

$$(1/\beta\rho c_2)\left(\frac{\xi_{2i}+\xi_{2r}}{\xi_{2i}-\xi_{2r}}\right) + (1/\rho c_1)\left(\frac{\xi_{1i}+\xi_{1r}}{\xi_{1i}-\xi_{1r}}\right)$$
$$+ (1/\rho_g c_g)\left(\frac{x_{gi}+x_{gr}}{x_{gi}-x_{gr}}\right) = 0, \quad (38)$$

where the quantity β, occuring in the term for second sound, is given by

$$\beta = (\rho_g L)^2/(\rho-\rho_g) r c_2^2 sT.$$

Expression (38) establishes the relationship between the two types of sound in the liquid and the sound in the vapor. This is applicable either to a situation involving acoustical resonance (Yale experiment) or to the case of short pulses where the geometry of the equipment does not enter. Note that the above result (38) is completely analogous to an electric situation involving two different transmission lines in parallel with a third at a common junction; each line has a different characteristic impedance. In this respect $\beta\rho c_2$ enters as an effective characteristic impedance for second sound insofar as interactions with the other types are concerned.

[9] It has recently been learned from Dr. Onsager that a dissipative process occurs at the surface which could be taken into account by the insertion of a complex factor in (34). This would lead to expressions for the heights and widths of resonance peaks in the Lane experiment.

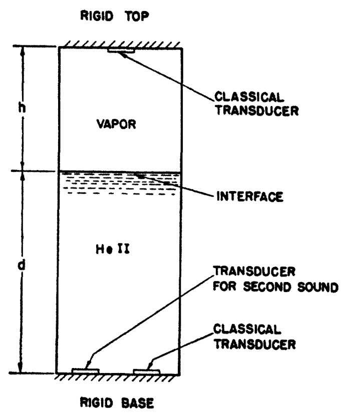

FIG. 3. Coupling due to liquid-vapor interface (idealized Yale resonance experiment).

The Experiment of Lane and Collaborators

Condition (38) is directly applicable to the resonance experiments conducted at Yale.[4] The physical situation is idealized in Fig. 3. Here is shown a vertical column of liquid He II of depth d, beneath a column of helium vapor of height h. Classical transducers, for ordinary sound in the vapor and first sound in the liquid, are provided both at the extreme top and extreme bottom of the containers. In addition, a second sound transducer is located at the bottom. Measurements consist, in general, of activating the second sound transducer and detecting a signal with one of the classical transducers, or the reverse. As either the frequency or the height of the liquid is altered, a succession of resonance peaks is observed.

The precise conditions of resonance depend upon a variety of factors, such as the top and bottom impedance of the container and the internal impedances of the transducers. Only the simplified situation of completely rigid container ends and infinite internal impedance for all transducers need be considered here. (That is, classical transducers of the pressure type; and second-second transducers of the temperature—i.e., low heat flow—type.) Also it is considered that for the steady state, the waves become one-dimensional and plane.

Under such conditions (not necessarily the experimental ones) maximum energy would be fed to the system for resonance conditions, i.e., matched to infinite impedance. This condition is specified by modifying (38) for infinite top and bottom impedances. Accordingly,

$$(1/\rho c_1)\tan(2\pi\nu d/c_1)+(1/\beta\rho c_2)\tan(2\pi\nu d/c_2) \\ +(1/\rho_g c_g)\tan(2\pi\nu h/c_g)=0 \quad (39)$$

gives the requirement for resonance, where ν is the frequency. This condition holds for any of the transmitter-receiver combinations.

Additional factors, such as dissipative effects occuring at the surface, would have to be introduced for computing heights and widths of resonance peaks. Furthermore, non-infinite impedances would alter the conditions of resonance (39), by modifying the effective depth d or height h. For example, a "low impedance" type of second sound generator (i.e., ratio of temperature fluctuation to heat flow, small) would result in the replacement of the *tan* of the second term by *cotan*. Similar alterations in (39) would occur for other modifications in the equipment.

Note that for this particular situation involving resonance, no recourse is made to the matrix method. For less specialized cases, however, such as reflection of short pulses from the surface, the previously derived matrix formulation would be used.

VI. CONCLUSIONS

The transmission and reflection of sound in He II is formulated on the basis of a matrix representation. A system of generalized coordinates (ξ-scheme) is used for expressing energy flow, whereas true coordinates (x-scheme) are used for setting up boundary conditions at the surface of a reflector. The reflectivity conditions for the case of normal incidence are expressed by means of transformations between these two systems. Distinction is made between reflectors (1) for which heat exchange with the liquid helium plays a basic role, and (2) those for which no heat transfer takes place. Concerning (2), it is shown that impervious (or impenetrable) surfaces reflect all incident second sound. Coupling between first and second sound occurs only for semi-impervious surfaces for which the superfluid component experiences less viscous retardation than does the normal fluid. Reflectivity curves are given for the case of a thin, semi-impervious screen, for which transfer of intensity between modes may reach 15 percent. That the coupling factor for such a surface is identical for either type of incident sound constitutes a type of reciprocity. Concerning (1), the boundary condition governing reflection of acoustic energy from a liquid-vapor interface is given. Special modifications for the case of resonance are applicable to the Yale type of experiment.

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The Electromagnetic Shift of Energy Levels

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The displacement between the $2s$ and $2p_{\frac{1}{2}}$ levels of hydrogen is calculated on the assumption that it is caused by interaction with the radiation field; the calculation is relativistic, but the spin of the electron is neglected. The theory gives a finite result, agreeing closely with the experimental value.

I. INTRODUCTION

BETHE[1] has proposed a method of calculating theoretically the observed[2] displacements of the hydrogen fine structure levels from the positions predicted by the Dirac theory. The displacements are attributed to interaction between the electron and the radiation field, and are to be calculated by the usual perturbation method of calculating electromagnetic self-energies. Since it is presumed that the measured mass of an electron already includes the electromagnetic self-energy, the level shifts are found by subtracting from the calculated self-energy of the bound electron the self-energy calculated for a "suitable" free electron. Bethe observed that the difference between these two divergent expressions should be finite and of the correct magnitude. Unfortunately, the non-relativistic approximation which he used in his preliminary calculation[1] is not good enough for an exact test of the theory, since in this approximation the formula for the shift is still divergent after the subtraction. An exact calculation, based on the Dirac electron theory, presents formidable, though not fundamental, difficulties.

[1] H. A. Bethe, Phys. Rev. **72**, 339 (1947).
[2] W. E. Lamb and R. C. Retherford, Phys. Rev. **72**, 241 (1947).

The present paper outlines a calculation undertaken as an interim program while the exact calculation is in progress; relativistic theory is used throughout, only the effects of spin are ignored. Specifically, the level shift is determined for an "atom" consisting of a proton and a particle of electronic mass and charge and zero spin. The light particle, which will be referred to as "scalar particle," satisfies the Klein-Gordon wave equation and obeys Bose statistics. Since the system thus defined is at least conceptually a physical system (which a non-relativistic system is not), it is satisfactory to find that the theory gives a convergent expression for the level shift. Further, the values obtained are very close to the non-relativistic approximations and to the observed shifts.

From the theoretical standpoint, the convergence of the present calculation is noteworthy and somewhat unexpected. The non-relativistic theory, in which the self-energy before subtraction diverges linearly, gives a logarithmically divergent level shift; and it was to be expected that the Dirac hole theory, in which the self-energy is only logarithmically divergent, would give a convergent level shift. In the case of the scalar particle the self-energy itself is quadratically divergent, and after the subtraction the