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Transients in Townsend Discharges

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An analysis is made of the transients in Townsend discharges for certain stepwise variations in the stimulating photoelectric current from the cathode, while the voltage across the tube is held steady. The method of analysis can be readily generalized to include any type of variation in the photoelectric current. The steady-state current, including the effects of metastable molecules produced in the gas, is also derived; from this result it is shown that the second Townsend coefficient is a function of the electrode spacing. The work also suggests experiments which can separate the radiation, ion, and metastable contributions to the second Townsend coefficient.

I. INTRODUCTION

We shall consider in this paper only gas discharges between infinite plane-parallel electrodes, operating below breakdown. Under these conditions, the current through the discharge is not self-sustaining, but must be stimulated by a primary current i_0. This primary current is usually obtained by illuminating the cathode with a source of light which is external to the discharge. It is well known that the steady-state current through the discharge, under these conditions, is given approximately by an expression of the form:

$$i = i_0 \exp[\alpha_i(X-x_0)] / [1-\gamma(\exp[\alpha_i(X-x_0)]-1)]. \quad (1)$$

In this, X is the separation of the electrodes, and α_i, γ, and x_0 are considered to be functions only of the gas pressure and the electric field and are independent of X.

In (1), α_i, called the first Townsend coefficient, is interpreted as the number of ionizing collisions which one electron makes while traveling one cm in the direction of the electric field, and x_0 is interpreted as the distance that an electron must travel before acquiring ionizing energy. γ is interpreted as representing the number of electrons emitted from the cathode under the influence of events which occur in the discharge, as we shall describe more fully below.

Our fundamental picture of a gas discharge will be as follows: An electron which leaves the cathode encounters a certain number of gas molecules while traveling to the anode. If one of these collisions is inelastic, there may be three possible results: (1) the molecule may be ionized, with the simultaneous release of an electron; (2) the molecule may be excited to a level from which it can decay by radiation; (3) the molecule may be excited to a metastable level. Subsequently, electrons may be emitted from the cathode (1) upon impact of a positive ion, (2) when radiation from an excited molecule strikes the cathode, and (3) when a metastable molecule strikes the cathode. (2) above is, of course, an example of photoelectric emission. Processes (1) and (3) have been studied experimentally by Oliphant,[1] and theoretically by Massey[2] and Cobas and Lamb.[3]

As we shall show formally in later sections, this mechanism leads to an expression for γ in Eq. (1), which is the sum of three terms, one each from the electron emission caused by radiation, ions, and metastables. It is of great interest to be able to separate these terms experimentally. This separation cannot be performed accurately by measurements of the steady-state current through a discharge but, as Engstrom and Huxford[4] have pointed out, it can be made by an analysis of the transient currents.

The basis for the separation lies in the different times required for radiation, ions, and metastables to produce effects at the cathode. Suppose that the discharge is initiated by an electron leaving the cathode. At a field of 50 volts/cm, an X of 1 cm, and a pressure of 1 mm, this electron and the ones released by ionization in the gas reach the anode in the order of 10^{-8} sec. The

[1] M. L. E. Oliphant, Proc. Roy. Soc. **A124**, 228 (1929).
[2] H. S. W. Massey, Proc. Camb. Phil. Soc. **26**, 386 (1930); Proc. Camb. Phil. Soc. **27**, 460 (1931).
[3] A. Cobas and W. E. Lamb, Phys. Rev. **65**, 327 (1944).
[4] R. W. Engstrom and W. S. Huxford, Phys. Rev. **58**, 67 (1940).

time required for an excited molecule to radiate, and for the radiation to reach the cathode and produce a photoelectron, is also of the order of 10^{-8} sec. An ion, drifting under the field, does not reach the cathode for about 10^{-6} sec., while a metastable, which must reach the cathode by diffusion, requires the order of 10^{-3} sec. Thus, transients should show three relatively distinct phases, governed by these differences in time required to produce new electrons at the cathode. These three phases are shown schematically in Fig. 1.*

The first phase, of duration about 10^{-8} sec., in which the first electrons are crossing the tube and new photoelectrons are being produced, is probably too short for observation by present techniques. Accordingly, we shall not analyze its form but shall show how to allow for its effect upon other phases.

The second phase, of duration about 10^{-6} sec. and characterized by the first arrival of ions at the cathode, has not previously been analyzed. We shall obtain its form and shall show how, under certain conditions, it contains discontinuities in the current.[5] Observation of these discontinuities may afford a convenient method of measuring α_i and the ion contribution to γ.

The third phase, of duration about 10^{-3} sec. and characterized by the first arrival of metastables at the cathode, has been studied by Engstrom and Huxford.[4] We shall analyze it by a different method and in somewhat greater detail, thus obtaining a different form for the results.

We shall perform the analysis by setting up boundary value problems to be satisfied by the densities of ions and metastables in the gas. Different types of transient are then obtained by solving these problems subject to different initial conditions and to different forms of i_0 as a function of time. Steady-state currents are also found by getting the time-independent solutions of the boundary value problems; from the steady-state currents, we can evaluate γ in (1) above. This process leads to no new results for the contribution of ions to γ, but, for the metastable contribution to γ, shows that γ is not in fact a constant but is dependent upon X.

Sections II through V will be devoted to the formal solution, and Section VI to a discussion of the results. A list of frequently used symbols is given below.

We shall assume that α_i and x_0 are constant from point to point within the discharge. For a discussion of these assumptions and a general survey of the field we refer the reader to Loeb[6] and to Druyvesteyn and Penning.[7] We shall neglect collisions of a molecule with more than one electron, and collisions of ions or excited molecules with each other.

List of Symbols

The following symbols are frequently used throughout this paper. We have not listed symbols which occur infrequently and in only a limited part of the paper.

α_i = number of ions formed per electron per cm; equals first Townsend coefficient.

α_m = number of metastables formed per electron per cm.

α_r = number of radiating molecules formed per electron per cm.

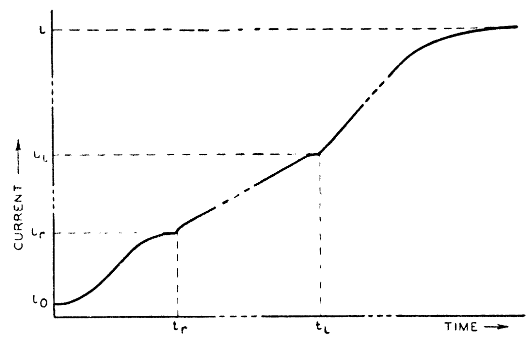

FIG. 1. Sketch of a discharge transient. i_0 = stimulating photo-current. i_r = radiation steady state. i_i = ion steady state. i = true steady state. t_r = duration of radiation transient. t_i = duration of ion transient. The scales are broken in order to display the entire transient.

* *Note added in proof:* In this paper, the phenomenon termed "imprisonment of resonance radiation" has been ignored. The significance of this phenomenon, which was first studied theoretically by K. T. Compton (Phys. Rev. **20**, 283 (1922)) was brought to our attention by T. Holstein (see Phys. Rev. **72**, 1212 (1947)). As a result of this phenomenon, the time required for radiation to reach the cathode, instead of being short, may equal or exceed the transit time of ions. The analysis of this paper is valid whenever the two times are significantly different; if the two times are approximately equal, a more complicated analysis will be necessary.

[5] This part of the present paper was presented before the American Physical Society on May 3, 1945. See Phys. Rev. **72**, 184 (1947).

[6] L. B. Loeb, *Fundamental Processes of Electrical Discharge in Gases* (John Wiley and Sons, Inc., New York, 1939).

[7] M. J. Druyvesteyn and F. M. Penning, Rev. Mod. Phys. **12**, 87 (1940).

$A_i=\alpha_i X$.
$A_m=\alpha_m X M_i M_r$.
γ = parameter occurring in steady state discharge current; very nearly equals second Townsend coefficient divided by α_i.
γ_i = number of electrons released from cathode per positive ion striking.
γ_m = number of electrons released from cathode per metastable striking.
γ_r = number of electrons released from cathode per photon striking.
$\eta=1-\xi_0$.
i = current density in electronic charges per cm².
i subscript: denotes a parameter characteristic of positive ions.
i_0 = stimulating photo-current from cathode, in electronic charges per cm².
i_+ = current density carried by ions.
i_- = current density carried by electrons.
κ = square root of diffusion coefficient for metastables.
λ = square root of reciprocal of time constants for metastable decay.
m subscript: denotes a parameter characteristic of metastables.
M_g = ratio of i_- to current of electrons leaving one cm² of cathode.
M_i = amplification of i_0 resulting from ion effects.
M_m = amplification of i_0 resulting from metastable effects.
M_r = amplification of i_0 resulting from radiation effects.
N, N_a, N_c = parameters occurring in boundary conditions for metastable diffusion.
ρ = density of ions or metastables, as shown by subscript.
P_i = density of ions when expressed in the moving coordinate system y.
r subscript: denotes a parameter characteristic of radiation or radiating molecules.
t = time in sec.
τ = dimensionless time used in ion flow $=vt/X$.
T = dimensionless time used in metastable flow $=\kappa^2 t/X^2$.
Θ = effective solid angle presented by cathode to radiation from the gas.
v = ion velocity.
x = distance from cathode in cm.
x_0 = distance in cm which an electron must travel to acquire ionizing energy.
X = electrode spacing.
ξ = dimensionless $x=x/X$.
ξ_0 = dimensionless $x_0=x_0/X$.
y = moving coordinate in ion flow $=x+vt$.

II. AMPLIFICATION BY IONIZATION AND RADIATION

We shall assume that electrons cross the tube in negligible time. As a result of ionizations in the gas, more electrons reach the anode than leave the cathode. Suppose n_0 electrons leave one sq. cm of the cathode at some instant, and let n be the number of electrons per cm² which reach a distance x from the cathode. Then:

$$dn/dx=\alpha_i n,$$
$$n=n_0\exp\alpha_i(x-x_0),\quad x\geq x_0. \tag{2}$$

For $x<x_0$, $n=n_0$.

Let α_r be the number of excited molecules created per electron per cm. (By excited, we shall mean that the molecules are raised to a level from which they can radiate photons of sufficient energy to eject photoelectrons from the cathode. Thus metastable molecules are not excited in this sense.) The number of excited molecules per cm² created in a distance dx is $\alpha_r n dx$, if x is greater than x_r, the distance an electron must travel to acquire excitation energy, and n is given by (2) above. For brevity, we shall neglect the difference between x_0 and x_r; the difference can readily be included if desired. If x is less than x_r, the number of excited molecules is 0.

Let $\Theta/4\pi$ be the probability that a photon radiated at the distance x will reach the cathode. $\Theta=2\pi$ if absorption in the gas is negligible. Let γ_r be the number of secondary electrons per photon. Then we have for the number of secondary electrons:

$$\alpha_r\gamma_r n_0\int_{x_0}^{X}(\Theta/4\pi)\exp[\alpha_i(x-x_0)]dx.$$

However, these secondary electrons can produce more secondary electrons, and so on. We assume that all these secondaries are emitted in negligible time, so that the effect of radiation is to increase the number of electrons leaving the cathode from n_0 to:

$$n_0\sum_{j=0}^{\infty}\left\{\alpha_r\gamma_r\int_{x_0}^{X}(\Theta/4\pi)\exp[\alpha_i(x-x_0)]dx\right\}^j$$
$$=n_0\Big/\left\{1-\alpha_r\gamma_r\int_{x_0}^{X}(\Theta/4\pi)\times\exp[\alpha_i(x-x_0)]dx\right\}, \tag{3}$$

when the geometric series converges. We shall call the factor by which n_0 is multiplied M_r, the multiplication factor for radiation.

III. EQUATIONS FOR THE ION AND METASTABLE DENSITIES

Let $\rho_i(x, t)$ be the density of ions and $\rho_m(x, t)$ the density of metastables at time t at a distance x from the cathode. We assume that the ions have a uniform velocity v, so that the current of ions per cm^2 at any point is $-v\rho_i(x, t)$. We assume that the metastables move by diffusion with a coefficient κ^2, so that the current of metastables per cm^2 is $-\kappa^2(\partial\rho_m/\partial x)$.[8] Thus, the flow equations for ρ_i and ρ_m are:

$$\frac{\partial\rho_i}{\partial t}-\frac{\partial}{\partial x}(v\rho_i)=S_i(x, t),$$

$$\frac{\partial\rho_m}{\partial t}-\frac{\partial}{\partial x}\left(\kappa^2\frac{\partial\rho_m}{\partial x}\right)=S_m(x, t),$$

where S_i and S_m are functions, now to be set up, which give the numbers of ions and metastables created per cm^3 per sec.

Let γ_i be the number of electrons liberated per positive ion striking the cathode, and γ_m the number of electrons per metastable. Then the total number of electrons leaving the cathode per cm^2 per second is:

$$M_r[i_0+\gamma_i v\rho_i(0, t)+\gamma_m\kappa^2\rho_m'(0, t)],$$

in which a prime denotes $\partial/\partial x$, and i_0 is the primary current density measured in electrons per cm^2 per sec. To get the number of electrons per cm^2 per sec. at a distance x from the cathode, we multiply by unity for $x\leq x_0$ and by $\exp[\alpha_i(x-x_0)]$ for $x\geq x_0$. If α_i is the number of ions formed per electron per cm, and α_m the number of metastables, we have:

$$\frac{\partial\rho_i}{\partial t}-v\frac{\partial\rho_i}{\partial x}=\begin{cases}0, & x\leq x_0,\\ \alpha_i M_r[i_0+\gamma_i v\rho_i(0, t) \\ \quad +\gamma_m\kappa^2\rho_m'(0, t)]\exp[\alpha_i(x-x_0)], & x\geq x_0.\end{cases} \tag{4}$$

$$\frac{\partial\rho_m}{\partial t}-\kappa^2\frac{\partial^2\rho_m}{\partial x^2}=\begin{cases}0, & x\leq x_0,\\ \alpha_m M_r[i_0+\gamma_i v\rho_i(0, t) \\ \quad +\gamma_m\kappa^2\rho_m'(0, t)]\exp[\alpha_i(x-x_0)], & x\geq x_0.\end{cases} \tag{5}$$

In these, we have taken the minimum distance at which metastables are formed to be the same as that at which ions are first formed, namely, x_0.

As auxiliary conditions to (4) and (5), we must know the forms of $\rho_i(x, t)$ and $\rho_m(x, t)$ when $t=0$. These depend upon the type of transient being studied. In addition, there are certain conditions imposed upon ρ_i and ρ_m at the electrodes. For ρ_i, there is only one condition, which can be derived from the flow conditions at the anode:

$$\rho_i(X, t)=0. \tag{4a}$$

The conditions on ρ_m are more complicated. We expect the boundary conditions to be homogeneous in ρ_m and ρ_m', and hence to be of the form:

$$\begin{aligned}\rho_m(0, t)-N_cX\rho_m'(0, t)&=0,\\ \rho_m(X, t)+N_aX\rho_m'(X, t)&=0,\end{aligned} \tag{5a}$$

which implies a relation between the density and the metastable molecular current at the boundaries. If both electrodes are of the same material, $N_c=N_a=N$. The sign is different in these two conditions because the normals to the two electrodes are oppositely directed.

To evaluate N, we proceed as follows: The number of particles crossing unit area per sec. at any point because of the random thermal motion is $\rho_m(kT/2\pi m)^{\frac{1}{2}}$. Considering also those which cross because of diffusion, the total number crossing unit area per sec. in the negative x direction is $\rho_m(kT/2\pi m)^{\frac{1}{2}}+\frac{1}{2}\kappa^2\rho_m'$. The number crossing in the positive x direction is

$$\rho_m(kT/2\pi m)^{\frac{1}{2}}-\tfrac{1}{2}\kappa^2\rho_m',$$

so that the net flow is $\kappa^2\rho_m'$ in the negative x direction. From continuity, $\kappa^2\rho_m'(0, t)$ must equal the number of metastables destroyed at the cathode per cm^2 per sec. The work of Oliphant[1] indicates that one electron is emitted for each metastable destroyed, and hence that γ_m is the probability of destruction for a metastable striking the cathode. Therefore:

$$\gamma_m[\rho_m(0, t)(kT/2\pi m)^{\frac{1}{2}}+\tfrac{1}{2}\kappa^2\rho_m'(0, t)]=\kappa^2\rho_m'(0, t).$$

Thus:

$$N=(\kappa^2/X)(2\pi m/kT)^{\frac{1}{2}}(\gamma_m^{-1}-\tfrac{1}{2}). \tag{5b}$$

Using the kinetic theory expression for κ^2, this is approximately

$$N=(4l/3X)(\gamma_m^{-1}-\tfrac{1}{2}),$$

[8] These assumptions require that X equal many mean free paths. If the assumption that α_i is constant is met, we may expect that this is met.

where l is the mean free path of a metastable in an atmosphere of normal molecules. This has two important limiting cases; if $\gamma_m=0$, N is infinite, and we must have $\rho_m'(0,t)=0$. If $l/X\ll\gamma_m$, we have very nearly that $\rho_m(0,t)=0$.

IV. ION TRANSIENT AND STEADY-STATE CURRENTS

In the first several microseconds, ions can cross the tube a number of times before the metastables have moved appreciably. To study the operation of the tube during this interval, we can set $\gamma_m=0$ in Eq. (4). Also, let $\epsilon(w)$ be the step function defined by:

$$\epsilon(w)=\begin{cases}1, & w\le 0,\\ 0, & w\ge 0.\end{cases}$$

The use of this step function avoids some trouble with limits. (4) now becomes:

$$(\partial\rho_i/\partial t)-v(\partial\rho_i/\partial x)=\alpha_i M_r[i_0+\gamma_i v\rho_i(0,t)]\times\exp[\alpha_i(x-x_0)]\epsilon(x-X)\epsilon(x_0-x). \quad (6)$$

General Solution

For brevity, let

$$\phi(x)=\alpha_i M_r\exp[\alpha_i(x-x_0)]\epsilon(x-X)\epsilon(x_0-x).$$

Also, transform the independent variables from x, t to t and $y=x+vt$, and let $\rho_i(x,t)=P_i(y,t)$. y is obviously a coordinate which moves with the stream of ions. Equation (6) now reads:

$$\partial P_i/\partial t=[i_0+\gamma_i vP_i(vt,t)]\phi(y-vt). \quad (6a)$$

We shall solve (6a) for an arbitrary form of $\phi(x)$, and for any variation of i_0 with t.

Solutions of (6a) which have the proper form at $t=0$ are also solutions of the integral equation:

$$P_i(y,t)=P_i(y,0)+\int_0^t\phi(y-vt')\times[i_0+\gamma_i vP_i(vt',t')]dt'. \quad (7)$$

The current can be calculated once the ion density is known. The current carried by positive ions is:

$$i_+(t)=(v/X)\int_0^X\rho_i(x,t)dx=(v/X)\int_{vt}^{vt+X}P_i(y,t)dy, \quad (8a)$$

in electronic charges per cm² per sec. The number of electrons leaving unit area of the cathode per second is

$$M_r[i_0+\gamma_i v\rho_i(0,t)]=M_r[i_0+\gamma_i vP_i(vt,t)].$$

In addition to these electrons, we have those which are formed in the gas and which do not traverse the entire distance X. Since $\phi(x)/M_r$ is the number of new electrons created per cm per electron leaving the cathode, we get the total current carried by electrons if we multiply the number of electrons leaving the cathode by:

$$M_g=1+\int_0^X[\phi(x)/M_r][(X-x)/X]dx.$$

This weights each electron according to the distance it travels in reaching the anode.[9] The current carried by electrons is therefore:

$$i_-(t)=M_rM_g[i_0+\gamma_i vP_i(vt,t)]. \quad (8b)$$

With the aid of this, we can replace (7) by a simpler equation. Substituting from (8b) for the quantity in square brackets in (7), we have for $y=vt$:

$$P_i(vt,t)=P_i(vt,0)+\int_0^t\phi(vt-vt')[i_-(t')/M_rM_g]dt'.$$

Substituting this back into (8b):

$$i_-(t)=M_rM_g[i_0+\gamma_i vP_i(vt,0)]+\gamma_i v\int_0^t\phi(vt-vt')i_-(t')dt'. \quad (9)$$

This equation can be solved for $i_-(t)$, then $P_i(vt,t)$ can be computed from (8b), and $P_i(y,t)$ and $i_+(t)$ can be computed from (7) and (8a), respectively.

Equation (9) is readily solved by Picard's method. Let:

$$i_-(t)=\sum_{n=0}^{\infty}i_-^{(n)}(t),$$

$$i_-^{(0)}(t)=M_rM_g[i_0+\gamma_i vP_i(vt,0)], \quad (10)$$

$$i_-^{(n+1)}(t)=\gamma_i v\int_0^t\phi(vt-vt')i_-^{(n)}(t')dt', \quad n=0,1,2,\cdots.$$

[9] W. Shockley, J. App. Phys. 9, 635 (1938).

If the sum exists, it is obviously the required solution.

Steady-State Current

Before applying (10) we derive the "steady-state" current. This has significance, in the first place, only if i_0 is constant and in the second place only if the ions can traverse the tube many times before metastables begin to reach the cathode in appreciable numbers. Under these circumstances, this steady-state current will be sensibly reached before the beginning of the metastable transient.

We find the steady state ion density $\rho_i(x, \infty)$ by setting $\partial/\partial t=0$ in Eq. (6) and integrating. The constant of integration is determined by the condition that $\rho_i(X, t)=0$. The result is:

$$\rho_i=(i_0/v)\left[\int_x^X \phi(x')dx'\right]\Big/\left[1-\gamma_i\int_0^X \phi(x')dx'\right].$$

The current is then found by applying formulas (8a) and (8b). If $\phi(x)$ is given by Eq. (6), the current has the well-known form:

$$i_i=i_0M_r\exp[\alpha_i(X-x_0)]/(1+\gamma_iM_r-\gamma_iM_r \times\exp[\alpha_i(X-x_0)]). \quad (11)$$

Pulse Transient

By a pulse transient we mean one resulting from a single pulse of stimulating current occurring at $t=0$. For this transient $i_0=0$ for $t>0$. We assume that n_0 primary electrons are released from the cathode in negligible time at $t=0$ and that these are instantly increased to M_rn_0, where M_r is defined below Eq. (3). Then:

$$P_i(y, 0)=n_0\phi(y).$$

We now find P_i and the current by substituting into Eqs. (10), (8a), and (8b).

When we return to our original assumption about the form of $\phi(x)$, that is, to the form of Eq. (6) with a constant α_i, we obtain the following results:

$$\rho_i(x, t)=\alpha_iM_rn_0\exp[A_i(\xi-\xi_0+\tau)] \times\sum_{n=0}^{\infty}(A_iM_r\gamma_ie^{-A_i\xi_0})^nI_n(\tau, \xi_0, \xi), \quad (12a)$$

$$\begin{aligned}i/\alpha_ivM_rn_0={}&(A_i)^{-1}\exp[A_i(\tau-\xi_0)]\\&\times\Big\{\exp[\max A_i(1-\tau), 0]\\&-\exp[\max A_i(\xi_0-\tau), 0]\\&+\sum_{n=1}^{\infty}(A_iM_r\gamma_ie^{-A_i\xi_0})^n\int_{\tau-1}^{\tau}I_{n-1}(\tau', \xi_0, 0)\\&\times(\exp[A_i(\tau'-\tau+1)]\\&-\exp[\max A_i(\tau'-\tau+\xi_0), 0])d\tau'\Big\}\\&+M_r\gamma_i(A_i)^{-1}\exp[A_i(\tau-\xi_0)]\\&\times(\exp[A_i(1-\xi_0)]+A_i\xi_0-1)\\&\times\sum_{n=0}^{\infty}(A_iM_r\gamma_ie^{-A_i\xi_0})^nI_n(\tau, \xi_0, 0). \quad (12b)\end{aligned}$$

These are expressed in terms of dimensionless variables $\tau=vt/X$, $\xi=x/X$, $\xi_0=x_0/X$, and $A_i=\alpha_iX$. The symbol $\max A, B$ means that we use the larger of A or B. Also:

$$I_n(\tau, \xi_0, \xi)=\int_{\tau+\xi-1}^{\min(\tau+\xi-\xi_0),\,\tau}I_{n-1}(\tau', \xi_0, 0)d\tau',$$

$$I_0(\tau, \xi_0, \xi)=\epsilon(\xi+\tau-1)\epsilon(\xi_0-\xi-\tau).$$

In order to express the current we need only the $I_n(\tau, \xi_0, 0)$, which are functions of only two variables. The first three of the $I_n(\tau, \xi_0, 0)$ are:

$$I_0(\tau, \xi_0, 0)=1, \qquad \xi_0\le\tau\le1.$$

$$I_1(\tau, \xi_0, 0)=\begin{cases}\tau-2\xi_0, & 2\xi_0\le\tau\le1+\xi_0,\\ 2-\tau & 1+\xi_0\le\tau\le2.\end{cases}$$

$$I_2(\tau, \xi_0, 0)=\begin{cases}\tfrac{1}{2}(\tau-3\xi_0)^2, & 3\xi_0\le\tau\le1+2\xi_0,\\ -\tfrac{1}{2}(\tau-1-2\xi_0)^2-\tfrac{1}{2}(2+\xi_0-\tau)^2\\ \quad+(1-\xi_0)^2, & 1+2\xi_0\le\tau\le2+\xi_0,\\ \tfrac{1}{2}(3-\tau)^2, & 2+\xi_0\le\tau\le3.\end{cases}$$

Others can readily be generated from the definition. The functions are zero outside the given ranges of τ.

The separate terms in (12a) have a simple physical interpretation. The first term, with $n=0$, is just the initial distribution due to the pulse of n_0 electrons. The term with $n=1$ is the distribution set up by the preceding term, and so on. Each successive term or "wave" begins a time x_0/v later than the preceding one, and lasts a time X/v longer, so that the waves overlap in a complicated fashion.

The integral

$$\int_0^\infty \exp(a\tau)I_n(\tau,\ \xi_0,\ 0)d\tau$$

will be useful. With $n=0$, this can be evaluated immediately. The integral with any other value of n can be expressed in terms of the integral with n decreased by unity by means of the definition of I_n. Thus, by induction, we find:

$$\int_0^\infty \exp(a\tau)I_n(\tau,\ \xi_0,\ 0)d\tau=(e^a-e^{a\xi_0})^{n+1}/a^{n+1}.$$

For $a=0$, this gives:

$$\int_0^\infty I_n(\tau,\ \xi_0,\ 0)d\tau=(1-\xi_0)^{n+1}.$$

With these results, we can readily compute the total number of electrons which leave the cathode and the total charge transferred through the tube. The first quantity is seen to be:

$$n_0M_r/\{1+\gamma_iM_r-\gamma_iM_r\exp[\alpha_i(X-x_0)]\}.$$

If we let:

$$M_i=\{1+\gamma_iM_r-\gamma_iM_r\exp[\alpha_i(X-x_0)]\}^{-1},\qquad(13)$$

the total number of electrons leaving the cathode is $n_0M_rM_i$ and the total charge transferred is

$$n_0M_rM_i\exp[\alpha_i(X-x_0)].$$

In this notation, Eq. (11) becomes

$$i_i=i_0M_rM_i\exp[\alpha_i(X-x_0)]$$

for the steady-state current, which is seen to be consistent with the value of the total charge transferred as the result of a pulse.

Finally, it should be noted that the product M_rM_i has the form:

$$M_rM_i=\left\{1-\alpha_r\gamma_r\int_{x_0}^X(\Theta/4\pi)\exp[\alpha_i(x-x_0)]dx+\gamma_i-\gamma_i\exp[\alpha_i(X-x_0)]\right\}^{-1}\qquad(13a)$$

Operating Transients

By this term, we mean those transients which occur when i_0 is changed abruptly from zero to a steady value (the rise) or when i_0 is changed abruptly from a steady value to zero (the fall). It is readily shown that the rise and fall have the same shape, so that we shall solve only for the rise.

We can find this transient by integrating the pulse transient over a suitable time variable or by usings Eqs. (10) with $P_i(vt,0)=0$. The currents carried by ions and by electrons are:

$$i_+(t)=i_0M_rA_i\int_\tau^{\tau+1}d\xi\int_{\xi-1}^{\xi-\xi_0}e^{A_i(\xi-\xi_0-\tau')}\times\left[\sum_0^\infty(\gamma_iM_r)^nJ_n(\tau',\ \xi_0)\right]d\tau',\qquad(14)$$

$$i_-(t)=i_0M_rM_0\sum_0^\infty(\gamma_iM_r)^nJ_n(\tau,\ \xi_0),$$

and the positive ion density can be found by application of Eqs. (7) and (8b). The $J_n(\tau,\ \xi_0)$ are given by:

$$J_0(\tau,\ \xi_0)=\begin{cases}0, & \tau\leq 0,\\ 1, & \tau\geq 0;\end{cases}$$

$$J_n(\tau,\ \xi_0)=A_i\int_{\tau-1}^{\tau-\xi_0}e^{A_i(\tau-\xi_0-\tau')}J_{n-1}(\tau',\ \xi_0)d\tau'.$$

As with the pulse transient, the individual terms have a simple interpretation. The first term, with $n=0$, represents the direct effects of the primary current. Ions formed by the primary current release new electrons from the cathode, whose direct effects are given by the second term, and so on. Unlike the terms in the pulse transient, these terms do not rise from zero and fall back to zero but instead rise from zero to a constant value. The nth term starts from zero at $\tau=n\xi_0$ and reaches a constant value at $\tau=n$. It is readily found that the constant value, which we denote by substituting $\tau=\infty$, is:

$$J_n(\infty,\ \xi_0)=[\exp A_i(1-\xi_0)-1]^n.$$

The form of Eqs. (14) is not convenient for large values of t or τ. We can find an approximate form for large τ, however, by making use of the property just mentioned. For τ equal to an integer k all the $J_n(\tau, \xi_0)$ for which $n \leq k$ have attained their limit while the others are still rising. Neglecting the terms which have not attained their limit, the electron current, for example, is:

$$i_-(\tau = k) \approx i_0 M_r M_0 \sum_{n=0}^{k} (\gamma_i M_r)^n \times[\exp\alpha_i(X - x_0) - 1]^n. \quad (15)$$

Letting $k \to \infty$, we see that this gives the steady-state electron current, provided that we are below breakdown.

Equation (15) can be used to estimate the rise time of the current, that is, the time required for the current to attain some specified fraction of its steady-state value. Since the current estimated by means of Eq. (15) is always too small, the resulting time is always too large.

V. METASTABLE TRANSIENT AND STEADY-STATE CURRENTS

After the metastables begin to move appreciably, we must solve Eqs. (4) and (5) simultaneously. To do this exactly would be a difficult task; we can get a reasonably accurate solution, however, if we assume that, at each instant, the ion density has the steady-state distribution corresponding to the instantaneous value of the electron current leaving the cathode. In other words, we can drop $\rho_i(0, t)$ from the right member of Eq. (5) provided that we multiply what remains by M_i, as defined by Eq. (13). Then, when we have solved for ρ_m the current is given by Eq. (11), with i_0 replaced by the instantaneous value of $i_0 + \gamma_m \kappa^2 \rho_m'(0, t)$.

We shall solve for the pulse transient and the falling operating transient, for which $i_0 = 0$ when $t > 0$. For these transients Eq. (5) now reads:

$$\frac{\partial \rho_m}{\partial T} - \frac{\partial^2 \rho_m}{\partial \xi^2} = \begin{cases} 0, & \xi \leq \xi_0, \\ \gamma_m A_m \exp[A_i(\xi - \xi_0)] \times (\partial \rho_m / \partial \xi)_{\xi=0}, & \xi \geq \xi_0, \end{cases} \quad (16)$$

in terms of the dimensionless variables

$$T = (\kappa^2 / X^2) t, \quad \xi = x / X,$$

with $\xi_0 = x_0 / X$, $A_i = \alpha_i X$, and $A_m = M_r M_i \alpha_m X$.

This is subject to the conditions:

$$\rho_m(0, T) - N(\partial \rho_m / \partial \xi)_{\xi=0} = \rho_m(1, T) + N(\partial \rho_m / \partial \xi)_{\xi=1} = 0.$$

For the pulse transient

$$\rho_m(\xi, 0) = n_0(A_m / X) \exp[A_i(\xi - \xi_0)], \quad \xi \geq \xi_0;$$

$\rho_m(\xi, 0) = 0$, $\xi \leq \xi_0$. For the falling operating transient $\rho_m(\xi, 0)$ has the steady-state value which we shall derive later. We are taking N to be the same at cathode and anode.

General Solution

We can solve Eq. (16) by separating the variables. We find that

$$\chi_n(\xi) \exp[-\lambda_n^2 T] \quad (17)$$

satisfies (16) and the conditions at $\xi = 0$, 1, provided that χ_n is a solution of

$$(d^2\chi / d\xi^2) + \lambda^2 \chi = \begin{cases} 0, & \xi \leq \xi_0, \\ -\gamma_m A_m \exp[A_i(\xi - \xi_0)](d\chi / d\xi)_{\xi=0}, & \xi \geq \xi_0, \end{cases} \quad (17a)$$

and that χ satisfies the conditions at $\xi = 0$, 1. As we shall see, (17a) has an infinite ordered set of eigenvalues λ_n, so that we can write a solution of Eq. (16) in the form:

$$\rho_m(\xi, T) = \sum a_n \chi_n(\xi) \exp[-\lambda_n^2 T]. \quad (18)$$

The coefficients a_n are then chosen so that this reduces to the given form of $\rho_m(\xi, 0)$ when $T = 0$.

Equation (17a) can be satisfied by:

$$\chi(\xi) = \begin{cases} c_1(\sin\lambda\xi + N\lambda \cos\lambda\xi), & \xi \leq \xi_0, \\ c_2 \sin\lambda(\xi - \xi_0) + c_3 \cos\lambda(\xi - \xi_0) + c_4 \exp A_i(\xi - \xi_0), & \xi \geq \xi_0. \end{cases} \quad (19)$$

This satisfies the condition on χ at $\xi = 0$. There are four other conditions which must be satisfied: χ and its first derivative must be continuous at $\xi = \xi_0$, χ must satisfy the boundary condition at $\xi = 1$, and χ must satisfy (17a). These conditions lead to the equations:

$$\begin{aligned} &c_1(\sin\lambda\xi_0 + N\lambda \cos\lambda\xi_0) - c_3 - c_4 = 0, \\ &c_1\lambda(\cos\lambda\xi_0 - N\lambda \sin\lambda\xi_0) - c_2\lambda - c_4 A_i = 0, \\ &c_2(\sin\eta\lambda + N\lambda \cos\eta\lambda) + c_3(\cos\eta\lambda - N\lambda \sin\eta\lambda) + c_4 e^{\eta A_i}(1 + NA_1) = 0, \\ &c_1 \gamma_m A_m \lambda + c_4(A_i^2 + \lambda^2) = 0, \end{aligned} \quad (20)$$

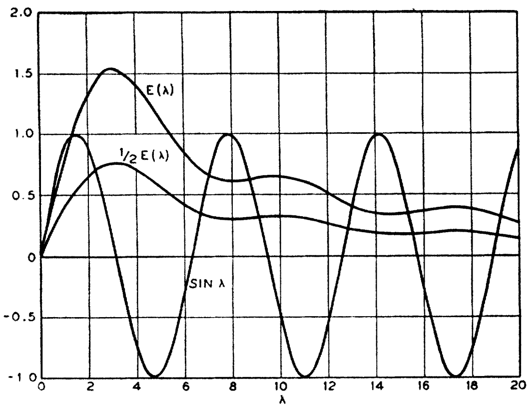

FIG. 2. Functions $\sin\lambda$ and $E(\lambda)$, for $N=0$, $A_i=2$, $\eta=0.9$.

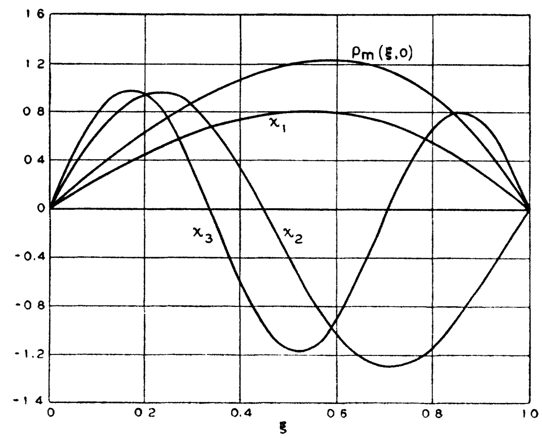

FIG. 3. The first three eigenfunctions, and the steady-state distribution, for $N=0$, $A_i=2$, $\eta=0.9$, $\gamma_m A_m=\frac{1}{2}$.

with $\eta=1-\xi_0$. If these are to have a non-trivial solution for c_1, c_2, c_3, c_4, the determinant of the coefficients must vanish. Multiplied out, this gives for the equation determining the eigenvalues λ:

$$(A_i^2+\lambda^2)[(1-N^2\lambda^2)\sin\lambda+2N\lambda\cos\lambda] = \gamma_m A_m[\lambda(1+NA_i)(e^{\eta A_i}-\cos\eta\lambda) - (A_i-N\lambda^2)\sin\eta\lambda]. \quad (21)$$

Without loss of generality we can take $R(\lambda)\geq 0$.

This has the roots $\lambda=0$, $\pm(-1)^{\frac{1}{2}}A_i$, but these are usually trivial since the values of the c's found for these values of λ give functions which are identically zero. If these roots should happen to be multiple, they become non-trivial.

If $A_m=0$, this reduces to the eigenvalue equation for flow without sources:

$$(1-N^2\lambda^2)\sin\lambda+2N\lambda\cos\lambda=0, \quad (21a)$$

whose roots are all real. The corresponding eigenfunctions are known to form a complete set on the interval $0\leq\xi\leq 1$.

To see the behavior of the eigenvalues when $A_m>0$, we can divide (21) through by $(\lambda^2+A_i^2)$. The behavior of the two members of the resulting equation is shown in Fig. 2, for the numerical values $N=0$, $A_i=2$, and $\eta=0.9$. For $N=0$, we can write (21) in the form:

$$\sin\lambda=\gamma_m A_m E(\lambda), \quad (21b)$$

introducing the function $E(\lambda)$ for brevity. If $N\neq 0$, (21) does not take so simple a form but the spirit of the following remarks about (21b) still holds.

$E(\lambda)$ is a function which rises to a maximum and falls slowly to zero as $\lambda\rightarrow\infty$. $E(\lambda)$ has small oscillations, but on the average goes as λ^{-1} for large λ. Thus, no matter how large $\gamma_m A_m$ may be, the right number of the above equation eventually becomes and remains less than unity and the eigenvalues approach the zeros of $\sin\lambda$.

Thus, we see that the eigenvalues of (21b) are continuous functions of $\gamma_m A_m$ and equal the integral multiples of π when $\gamma_m A_m=0$. As $\gamma_m A_m$ increases from zero, we see from Fig. 2 that the root originally at π moves toward the origin. The roots originally at 2π and 3π move toward each other, as do those originally at 4π and 5π, etc.

For some value of $\gamma_m A_m$, the root originally at π reaches the origin. This occurs when the two curves in Fig. 2 become tangent at the origin. $\lambda=0$ is then a double root and is no longer trivial. For still larger values of $\gamma_m A_m$ this eigenvalue becomes imaginary, so that $-\lambda^2$ is positive in Eq. (17) and the current increases without limit. Therefore, the breakdown condition is the condition that the slope of $\gamma_m A_m E(\lambda)$ at $\lambda=0$ be not less than unity. In the general case, with $N\neq 0$, similar remarks hold, and we find that the breakdown condition is that:

$$\gamma_m A_m \geq A_i^2(1+2N)[(1+NA_i) \times(e^{\eta A_i}-1)-\eta A_i]^{-1}. \quad (22)$$

We have seen that, for any value of $\gamma_m A_m$, the eigenvalues of (21) have a one-to-one correspondence to the known set defined by (21a). Further, the eigenvalues and eigenfunctions defined by (21) always approach those defined by (21a) at least as fast as λ^{-1}. Thus, we may plausibly assume that the set of functions $\chi_n(\xi)$ are complete and, hence, that the expansion of (18) is possible.

Expansion Coefficients and Initial Conditions

Having solved (21) for the eigenvalues, we can readily determine the c's from (20) and any convenient normalizing condition, and are ready to determine the a_n in (18). This determination is more difficult than for a Fourier expansion because the functions χ_n are not orthogonal. Evaluating the a_n is not necessary if we are only interested in the rise or fall time because this time is essentially determined by the smallest value of λ. This value is always less than π while the next value is equal to about 2π, and thus the first term in (18) rapidly becomes dominant.

We determine the coefficients a_n by substituting $T=0$ into Eq. (18), thus:

$$\rho_m(\xi,0)=\sum a_n\chi_n(\xi).$$

We proceed in the usual fashion to multiply through by $\chi_k(\xi)$ and to integrate over all ξ:

$$\sum a_n(\chi_k\cdot\chi_n)=(\chi_k\cdot\rho_m(\xi,0)). \tag{23}$$

If the χ_k were orthogonal, this would give a_k immediately. As it is, (23) represents a set of equations to be solved for the a_n.

For the pulse transient, we take:

$$\rho_m(\xi,0)=\begin{cases}0, & \xi\leq\xi_0,\\ n_0A_m\exp[A_i(\xi-\xi_0)], & \xi\geq\xi_0,\end{cases} \tag{24}$$

expressed in terms of number of metastables per cm^2 per unit ξ. The current is given by:

$$i=M_rM_ie^{\eta A_i}\gamma_m(\kappa^2/X^2)(\partial\rho_m/\partial\xi)_{\xi=0}. \tag{25}$$

For the falling transient we have everything the same as for the pulse transient, except for the form of $\rho_m(\xi,0)$. Since the rising transient is the same shape as the falling transient, we know both when we have determined one. For the falling transient we take $\rho_m(\xi,0)$ to be the steady-state distribution. As we shall show in a minute, this distribution is:

$$\rho_m=(i_0X^2/\kappa^2)M_rM_iM_m(\alpha_m/\alpha_i^2X)\times\begin{cases}(\xi+N)(1+2N)^{-1}[(1+NA_i)\\ \quad\times(e^{\eta A_i}-1)-\eta A_i], & \xi\leq\xi_0,\\ (\xi+N)(1+2N)^{-1}[(1+NA_i)\\ \quad\times(e^{\eta A_i}-1)-\eta A_i]-\exp[A_i(\xi-\xi_0)]\\ \quad+1+A_i(\xi-\xi_0), & \xi\geq\xi_0,\end{cases} \tag{26}$$

where ρ_m is in number of metastables per cm^2 per unit ξ, and:

$$M_m^{-1}=1-(\gamma_mM_rM_i\alpha_m/\alpha_i^2X)(1+2N)^{-1}\times[(1+NA_i)(e^{\eta A_i}-1)-\eta A_i]. \tag{26a}$$

If we have a problem in which the stimulating current i_0 is an arbitrary function of time, we can proceed by dividing the given function into a set of pulses of infinitesimal duration, finding the pulse transient resulting from each pulse and integrating the result over the set of pulses.

As an example, we have carried through the numerical work for the values $N=0$, $A_i=2$, $\eta=0.9$, and $\gamma_mA_m=\frac{1}{2}$. The eigenvalues are given by the intersections of the $\sin\lambda$ curve in Fig. 2 with the curve $\frac{1}{2}E(\lambda)$ (see (21b)). To four decimal places, the first five eigenvalues are:

$$\lambda_1=2.3392,\quad \lambda_2=6.6487,\quad \lambda_3=9.0955,\quad \lambda_4=12.7856,\quad \lambda_5=15.5246.$$

The first three eigenfunctions, normalized by setting $c_1=1$ in Eqs. (20), are shown in Fig. 3. They bear a general resemblance to the trigonometric functions $\sin\pi\xi$, $\sin2\pi\xi$, $\sin3\pi\xi$, but are somewhat distorted. In particular, χ_1 has its maximum at a larger value of ξ than the midpoint. The steady-state density function, indicated by $\rho_m(\xi,0)$, is also given in Fig. 3 on an arbitrary scale. From it we see that χ_1 is more suitably shaped to be an approximate representation than a symmetrical function would be.

The results of keeping one, two or three terms in Eq. (23) is shown in Fig. 4. We get the fol-

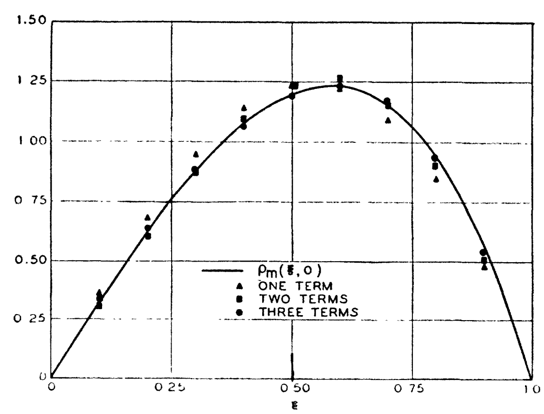

FIG. 4. Successive approximations to $\rho_m(\xi,0)$.

lowing values of the coefficients:

One term: $a_1=1.5311$.
Two terms: $a_1=1.5013$, $a_2=-0.0668$.
Three terms: $a_1=1.5060$, $a_2=-0.0761$, $a_3=0.0406$.

In this case at least we converge rapidly upon the accurate values of the coefficients. Thus, if we used only a one-term approximation in this case, our accuracy at large T (when only a_1 is important) would be about 2 percent.

We now find the current by substituting the expansion for ρ_m into Eq. (25). From (18) and (19), with the normalizing condition that $c_1=1$, we have:

$$(\partial\rho_m/\partial\xi)_{\xi=0}=\sum a_n\lambda_n \exp[-\lambda_n^2T],$$

and the current is proportional to this quantity. Since λ_2^2 is about eight times λ_1^2, and the other λ^2 are even larger, only the first term is important for T greater than about 0.1.

Steady-State Metastable and Ion Distribution, Steady-State Current

The steady-state densities are found readily by setting $(\partial/\partial t)=0$ in Eqs. (4) and (5), and integrating, subject to the same boundary conditions that we used in finding the time-dependent solutions. The metastable distribution has been given in Eq. (26). The ion distribution and the current are:

$$\rho_i=\frac{i_0X}{v}M_rM_iM_m\begin{cases}e^{\eta A_i}-1, & \xi\leq\xi_0,\\ e^{\eta A_i}-e^{A_i(\xi-\xi_0)}, & \xi\geq\xi_0;\end{cases}\tag{27}$$

$$i=i_0M_rM_iM_me^{A_i(1-\xi_0)},\tag{28}$$

where ρ_i is in number of ions per cm² per unit ξ, and M_m is defined in (26a).

The product $M_rM_iM_m$ is a fraction whose numerator is unity and whose denominator is unity minus a sum of terms, each of which is characteristic of one of the three processes of secondary emission: photo-emission, emission under ion bombardment, and emission under metastable bombardment. Thus, if $\Theta=2\pi$:

$$\begin{aligned}(M_rM_iM_m)^{-1}&=1-(\alpha_r\gamma_r/2\alpha_i)\\&\times[\exp(A_i(1-\xi_0))-1]\\&-\gamma_i[\exp(A_i(1-\xi_0))-1]\\&-(\gamma_m\alpha_m/\alpha_i^2X)(1+2N)^{-1}[(1+NA_i)\\&\times[\exp(A_i(1-\xi_0))-1]-A_i(1-\xi_0)].\end{aligned}\tag{29}$$

In the expression for the steady-state current this quantity replaces the denominator in Eq. (1). Thus we see that Eq. (1) is correct, to the accuracy used in this work, only if γ_m is negligible.

Breakdown occurs when $(M_rM_iM_m)^{-1}$ becomes zero or negative. This condition is identical with the condition already given in Eq. (22), as one can see by substituting the values of M_r and M_i into Eq. (22).

Solution by an Integral Equation

Equation (16), like Eq. (6), can be transformed to an integral equation. To effect this transformation, consider first the following method for solving the diffusion equation with sources,

$$(\partial y/\partial T)-(\partial^2y/\partial\xi^2)=S(\xi,T),$$

with y subject to homogeneous boundary conditions, and with $y(\xi,0)$ a given function of ξ.

First, we find a function $y_1(\xi,T)$ which satisfies the homogeneous diffusion equation

$$(\partial y_1/\partial T)-(\partial^2y_1/\partial\xi^2)=0,$$

and which satisfies the same boundary conditions and initial condition as y, so that:

$$y_1(\xi,0)=y(\xi,0).$$

Then we find a function $y_2(\xi,T,T')$, where T' is an as yet undefined parameter, which also satisfies the *homogeneous* diffusion equation, which satisfies the same boundary conditions as y, but which satisfies the initial condition:

$$y_2(\xi,0,T')=S(\xi,T'),$$

which defines the parameter T'. Then one can verify by substitution that the desired solution $y(\xi,T)$ is:

$$y(\xi,T)=y_1(\xi,T)+\int_0^T y_2(\xi,T-T',T')dT',$$

provided that $y_2(\xi,T-T',T')$ can be differentiated under the integral sign.[10]

In other words, we first find the density function which results from the decay of the original distribution without sources. Then we consider that a density distribution equal to $S(\xi,T')dT'$

[10] G. Doetsch, *Laplace Transformation* (Dover Publications, New York, 1943), p. 358.

is set up at $T=T'$ and find the density which results as this decays without further sources. To find the actual density at time T we add up all the contributions from the initial distribution and from the sources which "occur" earlier than time T.

In the present problem, the source function $S(\xi, T)$ is, according to Eq. (16), the product of $(\partial\rho_m\partial\xi)_{\xi=0}$, which is a function only of time, by a function of ξ only. It is then convenient to take the function $y_2(\xi, T, T')$ to be:

$$y_2=(\partial\rho_m/\partial\xi)_{\substack{\xi=0\\T=T'}}\phi(\xi, T), \tag{30a}$$

where $\phi(\xi, T)$ satisfies the homogeneous diffusion equation, and

$$\phi(\xi, 0)=\begin{cases}0, & \xi\leq\xi_0,\\ \gamma_m A_m e^{A_i(\xi-\xi_0)} & \xi\geq\xi_0.\end{cases} \tag{30b}$$

Our method of solution then gives the result:

$$\rho_m(\xi, T)=y_1(\xi, T)+\int_0^T\phi(\xi, T-T')(\partial\rho_m/\partial\xi)_{\substack{\xi=0\\T=T'}}dT'. \tag{31}$$

Operating on both sides with $\partial/\partial\xi$ and evaluating at $\xi=0$, we get an integral equation for the time-dependent variable $(\partial\rho_m/\partial\xi)_{\xi=0}$:

$$(\partial\rho_m/\partial\xi)_{\xi=0}=(\partial y_1/\partial\xi)_{\xi=0}+\int_0^T(\partial/\partial\xi)_{\xi=0}\times\phi(\xi, T-T')(\partial\rho_m/\partial\xi)_{\substack{\xi=0\\T=T'}}dT'. \tag{32}$$

Since the current is proportional to $(\partial\rho_m/\partial\xi)_{\xi=0}$, this equation is analogous to Eq. (9). It is essentially the integral equation which Engstrom and Huxford[4] give for the current.

Equation (32) is identical in form with (9). In both cases, the first term on the right represents the contribution from the initial distribution (there is also an i_0 in (9) which we have taken to be zero in (32)). In both cases, the first factor in the integrand gives the number of electrons leaving the cathode at time T as a result of an electron which left the cathode at time T'.

Equation (32) can be solved by an infinite series, which we know from our study of Eq. (16) to have the form:

$$(\partial\rho_m/\partial\xi)_{\xi=0}=\sum_n a_n\exp(-\lambda_n^2T). \tag{33}$$

We also know y_1 and ϕ, and can write:

$$(\partial y_1/\partial\xi)_{\xi=0}=\sum_n b_n\exp[-p_n^2T],$$

$$(\partial\phi/\partial\xi)_{\xi=0}=\sum_n c_n\exp[-p_n^2T],$$

in which the p_n are the roots of (21a), and b_n and c_n are determined by Eqs. (30a) and (30b). Substituting these into (32), we have:

$$\sum_k c_k/(p_k^2-\lambda_n^2)=1, \tag{34a}$$

$$b_k-c_k\sum_n a_n/(p_k^2-\lambda_n^2)=0. \tag{34b}$$

We can determine the λ_n from (34a), and the a_n from (34b) and the values of λ_n.

The roots of (34a) must be identical with the roots of (21). Which of the two equations one uses will probably be determined by the type of information desired.

With either this method or the method presented earlier in this section one can achieve any desired accuracy by retaining a sufficient number of terms, either in series (18) or series (33). It should be noted that the use of, say, two terms of (18) does not give the same approximation as the use of two terms of (33). In the latter case, we approximate to both the eigenvalues (the λ's) and the expansion coefficients. In the former case, we always obtain the accurate eigenvalues from (21), and approximate only to the expansion coefficients.

VI. SOME EXPERIMENTAL APPLICATIONS

In this section, we shall point out some of the consequences of the preceding formal analysis which suggest themselves as bases for experimental measurement of the various parameters which characterize a discharge. We shall not attempt a discussion of the feasibility or accuracy of any experiments.

The parameters which characterize the discharge may be divided into two types: (1) those which determine the "steady-state" currents i_r, i_i, and i (see Fig. 1) and (2) those which deter-

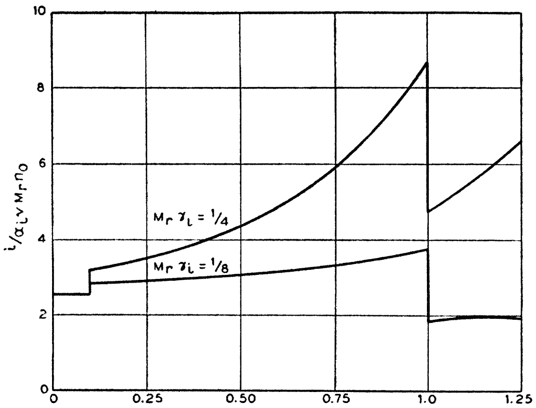

FIG. 5. Initial part of ion transient for $A_i=2$, $\xi_0=0.1$. The curves are only schematic beyond $\tau=1$.

mine the rise times t_r and t_i, and the time constants of the metastable transient.

The steady-state currents are given by:

$$i_r = i_0 M_r \exp[\alpha_i(X-x_0)],$$
$$i_i = i_0 M_r M_i \exp[\alpha_i(X-x_0)],$$
$$i = i_0 M_r M_i M_m \exp[\alpha_i(X-x_0)],$$

where M_r, M_i, and M_m are, respectively, defined below Eq. (3), by Eq. (13), and Eq. (26a). The products M_rM_i and $M_rM_iM_m$ are written out, respectively, in Eqs. (13a) and (29). By referring to these equations, we see that the steady-state currents depend upon the parameters $\alpha_r\gamma_r/\alpha_i$, γ_i, $\alpha_m\gamma_m/\alpha_i$ and α_i. (In addition to these parameters there occur N and ξ_0. We can usually estimate these with sufficient accuracy from other considerations.) Therefore, it is impossible to determine these four parameters from the three steady-state currents at one value of X.

We could overcome this difficulty by performing the classical Townsend experiment, in which the current is measured as a function of X, while keeping the electric field, and hence the α's, constant. The alternative procedure, which Engstrom and Huxford[4] have suggested, is to study the transients. This procedure brings in also the time parameters, namely, the speed v of the ions and the diffusion coefficient κ^2 of the metastables. If we had studied the radiation transient, we should also have a parameter (or parameters) for it.

An analysis of the transients is sufficient, as we shall show explicitly for the ion transient, to yield all the desired parameters. An analysis by this means has two great advantages over the classical method. One is that the parameters are all determined at one geometrical configuration, the second is that measurements can be made beyond the breakdown point, provided that the stimulating current is removed soon enough.

An experimental study of the metastable transient has been given by Engstrom and Huxford.[4] Their method involves measuring the time constants of the exponential decay terms which were discussed in Section V of this paper. Use of these time constants, in general, requires a knowledge of the diffusion coefficient κ^2. They show that, if one uses a one-term approximation in the integral equation method of solution (see Eq. (32) *et. seq.*), λ_1 is determined, to that accuracy, by the ratio i/i_i, without knowing κ^2.

For studying the positive ion transient, we make use of Eq. (12b) for the current following a pulse of stimulating electrons. Figure 5 shows the dimensionless current $(i/\alpha_i v M_r n_0)$ computed from (12b) as a function of the dimensionless time $\tau = vt/X$, for the particular values $A_i=2$, $\xi_0=0.1$, $(M_r\gamma_i)=\frac{1}{8}$ and $\frac{1}{4}$. The latter value is beyond the breakdown value.

The feature of greatest interest in Fig. 5 is that there are two discontinuities in the current, whose origin is not difficult to see. When the initial pulse of electrons passes through the tube, ions are created between $\xi=\xi_0$ and $\xi=1$. This mass of ions moves bodily toward the cathode. At $\tau=\xi_0$, the front wall of the ion distribution strikes the cathode and causes the emission of electrons. The current through the discharge therefore shows a sudden jump. The second discontinuity arises when the ions originally formed near the anode ($\xi=1$) reach the cathode. The initial density of ions at the anode is much greater than any density formed subsequently, so that after the ions originally at the anode strike the cathode there is a sudden decrease in the number of ions striking the cathode per sec., and therefore a sudden decrease in the current of electrons leaving the cathode.

Mathematically, the discontinuities arise in the first term of the last summation in Eq. (12b), that is, from the term:

$$M_r\gamma_i(A_i)^{-1}\exp[A_i(\tau-\xi_0)] \times\{\exp[A_i(1-\xi_0)]+A_i\xi_0-1\}I_0(\tau,\xi_0,0).$$

$I_0(\tau, \xi_0, 0)$ is a function which is unity inside the interval $\xi_0 \leq \tau \leq 1$, and zero outside that interval. Therefore, this term gives discontinuities in the current at $\tau = \xi_0$ and $\tau = 1$, which have the respective magnitudes:

$$M_r\gamma_i(A_i)^{-1}\{\exp[A_i(1-\xi_0)]+A_i\xi_0-1\}$$

and

$$M_r\gamma_i(A_i)^{-1}\exp[A_i(1-\xi_0)] \times\{\exp[A_i(1-\xi_0)]+A_i\xi_0-1\}.$$

Observation of these discontinuities thus allows us to compute $A_i(=\alpha_i X)$, and $M_r\gamma_i$. The combination of these observations with the steady-state currents in Fig. 4 gives one more than the number of relations needed to compute the parameters in the steady-state current.

On a time scale the discontinuities occur at $t = x_0/v$ and $t = X/v$. Therefore, from the times at which they occur, we can compute the ion velocity and hence the mobility.

It should be pointed out that, if metastable activity is significant in the discharge, the coefficient γ in Eq. (1), which is closely related to the second Townsend coefficient, is not independent of the electrode spacing X. Upon comparing (1) with (28) and (29), we see that γ is given by:

$$\gamma = (\alpha_r\gamma_r/2\alpha_i)+\gamma_i+[\gamma_m\alpha_m/\alpha_i(1+2N)] \times\{(1+N\alpha_i X)(\alpha_i X)^{-1} - X^{-1}(X-x_0)[\exp(\alpha_i(X-x_0))-1]^{-1}\}.$$

The radiation and ion contributions to this are constants, but the metastable contribution is a function of X, which varies as $N+(\alpha_i X)^{-1}$ for large X. In other words, if we try to determine a number γ to fit Eq. (1) by varying X, we should find that γ must decrease as X increases.

The reason that the ion and radiation contributions to γ are constant while the metastable contribution is not, is readily seen. Regardless of the value of X, essentially all ions formed in the discharge (if the diameter of the discharge is large compared with its length) reach the cathode. Likewise, one-half of the radiation emitted in the gas reaches the cathode, regardless of X. However, the fraction of the metastables formed which reaches the cathode is not independent of X, even if sidewise diffusion is negligible. As X increases, the distribution of metastables tends to become more concentrated near the anode. Since the metastables move by diffusion, more of the metastables reach the anode and fewer reach the cathode, under these conditions. Therefore, the metastable contribution to γ decreases with increasing X.

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