## Higher Order Field Equations

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CONSIDER the quadratic Lagrangian

$$L=(1/2a^2)(c_0\varphi_\alpha^2+c_1a^2\varphi_{\alpha,\beta_1}{}^2+c_2a^4\varphi_{\alpha,\beta_1\beta_2}{}^2+c_3a^6\varphi_{\alpha,\beta_1\beta_2\beta_3}{}^2+c_4a^8\varphi_{\alpha,\beta_1\beta_2\beta_3\beta_4}{}^2),\quad(1)$$

where $a$ is an arbitrary length and the $c$'s are dimensionless natural constants. Higher derivatives, if admissible, may be treated by induction. Applying Hamilton's principle[1] we obtain the set of linear eighth-order field equations

$$(c_0-c_1a^2\Box+c_2a^4\Box^2-c_3a^6\Box^3+c_4a^8\Box^4)\varphi_\alpha(\mathbf{r},t)=0\quad(2)$$

which may be written as

$$\left[\prod_{\sigma=1}^{4}(a^2\Box-s_\sigma{}^2)\right]\varphi_\alpha(\mathbf{r},t)=0,\quad(3)$$

where $s_\sigma{}^2$ represents the four roots of the auxiliary algebraic equation obtained by replacing $a^2\Box$ by $s^2$ in (2).

A solution of (3) expressed in Fourier integrals is

$$\varphi_\alpha(\mathbf{r},t)=(1/2\pi)^{\frac{3}{2}}\sum_\sigma\int[\varphi_{\sigma\alpha}(\mathbf{k})\exp(ix_\beta k_{\sigma\beta})+\varphi_{\sigma\alpha}{}^*(\mathbf{k})\exp(-ix_\beta k_{\sigma\beta})]d\mathbf{k},\quad(4)$$

where

$$k_{\sigma\beta}=(\mathbf{k},ik_\sigma)\quad\text{and}\quad k_\sigma{}^2=\mathbf{k}\cdot\mathbf{k}+s_\sigma{}^2/a^2.\quad(5)$$

Corresponding to our Lagrangian we can find an energy momentum density tensor $t_{\mu\nu}$, for which energy and momentum is conserved, i.e., $t_{\mu\nu,\,\nu}=0$. Expressing $t_{\mu\nu}$ in terms of Fourier amplitudes (we may discard all time-dependent terms) we get the Hamiltonian

$$H=-\int ict_{44}dV=\sum_{\sigma\alpha}2\gamma_\sigma\int k_\sigma 2\varphi_{\sigma\alpha}{}^*\varphi_{\sigma\alpha}d\mathbf{k}=\sum_{\sigma\alpha}\int \hbar ck_\sigma\eta_{\sigma\alpha}{}^*\eta_{\sigma\alpha}d\mathbf{k},\quad(6)$$

where

$$\gamma_\sigma=(-c_1+2c_2s_\sigma{}^2-3c_3s_\sigma{}^4+4c_4s_\sigma{}^6),\quad(7)$$

and

$$\eta_{\sigma\alpha}(\mathbf{k})=(2\gamma_\sigma k_\sigma/c\hbar)^{\frac{1}{2}}\varphi_{\sigma\alpha}(\mathbf{k}).\quad(8)$$

Going over to quantum-field theory the commutation rules are

$$[\varphi_{\sigma\alpha}{}^*(\mathbf{k}),\varphi_{\tau\beta}(\mathbf{k}')]=-\delta_{\sigma\tau}\delta_{\alpha\beta}\delta(\mathbf{k}-\mathbf{k}')c\hbar/2k_\sigma\gamma_\sigma.\quad(9)$$

Suitable supplementary conditions which permit only transverse states of positive energy to exist in the "empty" field are

$$[(1-is_\sigma/ak)\mathbf{k}\cdot\mathbf{A}_\sigma(\mathbf{k})-k_\sigma\phi_\sigma(\mathbf{k})]\Omega=0.\quad(10)$$

The use of Dirac's expansors[2] is indicated here.

When nucleons are present in the field we find the interaction energy by a fourfold application of the formalism of Fock,[3] modified for bosons with finite masses ($m_\sigma=s_\sigma\hbar/ac$). From symmetry and superposition considerations we obtain the interaction energy[4]

$$U=\sum_\sigma(1/\gamma_\sigma)(\Sigma_{uv}{}'g_ug_v/8\pi)[(1-\tfrac{1}{2}\boldsymbol{\alpha}_u\cdot\boldsymbol{\alpha}_v)\times(1/R)\exp(-s_\sigma R/a)-\tfrac{1}{2}(\boldsymbol{\alpha}_u\cdot\mathbf{R}/R)(\boldsymbol{\alpha}_v\cdot\mathbf{R}/R)\times(s_\sigma/a+1/R)\exp(-s_\sigma R/a)].\quad(11)$$

The $1/R$ singularity at short ranges, which is present in second-order field theory (i.e., when $c_2=c_3=c_4=0$), vanishes in fourth-[4] ($c_3=c_4=0$), sixth- ($c_4=0$), and eighth-order theories because $\Sigma(1/\gamma_\sigma)=0$, independent of the magnitudes of the natural constants. The self-energy is thus finite in the higher order theories. It is given by

$$U_s=-(\Sigma s_\sigma/\gamma_\sigma)(1-\tfrac{1}{2}\boldsymbol{\alpha}\cdot\boldsymbol{\alpha})g^2/8\pi a.\quad(12)$$

If we admit only field particles with real, positive, or zero masses then the $c$'s must all have the same sign and the factor $(\Sigma s_\sigma/\gamma_\sigma)$ is positive or negative according to whether the $c$'s are positive or negative.

Exploratory attempts to apply the results of fourth-order theory to the deuteron indicate that agreement with experiment in energy level and scattering questions can be achieved, with a reasonable choice of constants, if we introduce the isotopic spin formalism. Following Kemmer[5] for the charge-independent hypotheses one treats each component of $\varphi_\alpha(\mathbf{r},t)$ as a three vector in isotopic spin space. The modified formalism leads finally to the appearance of the isotopic spin operator $T_{ab}$ as a factor in the interaction function. This approach is a radical one, however, if the more complicated interaction function of sixth- or eighth-order theories is used, because of the large number of field particles implied. In these cases the close similarity between Kemmer's Eq. (21) and Eq. (6) above invites consideration of the possibility of associating the sigma-quanta with the isotopic spin quanta themselves.

[1] Compare B. Podolsky and C. Kikuchi, Phys. Rev. **65**, 228 (1944).
[2] P. A. M. Dirac, Proc. Roy. Soc. **A183**, 284 (1945).
[3] V. Fock, Physik. Zeits. Sowjetunion **6**, 425 (1934).
[4] A. E. S. Green, Phys. Rev. **73**, 26 (1948).
[5] N. Kemmer, Proc. Camb. Phil. Soc. **34**, 354 (1938).

## Spectroscopic Evidence of Methane in the Earth's Atmosphere*

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WHEN mapping the solar spectrum, under high resolving power, in the 3.3-$\mu$ region, fourteen regularly spaced and intense lines have been found between $3.33\mu$ and $3.47\mu$. The wave numbers of these lines agree, within a maximum difference of $\pm0.7$ cm$^{-1}$, with the measurements of A. H. Nielsen and H. H. Nielsen[1] on the 3.3-$\mu$ fundamental band of $CH_4$. The intense central line ($Q$ branch) of the $CH_4$ band falls in a spectral region absorbed by water vapor. For shorter wave-lengths than $3.3\mu$, many intense lines of water vapor prevent also the possibility of finding most of the $CH_4$ lines of the $R$ branch.

The new lines have been found on spectrograms taken on January 10, 1948, with a new prism-grating recording spectrometer design by Dr. Robert Noble, under the direction of Professor H. H. Nielsen. This instrument is of the Pfund type, has an aperture of $f$:5, and parabolic mirrors with a focal length of 100 cm. An echelette grating with 7200 lines per inch was used for the observations.