Higher Order Field Equations

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January 16, 1948

CONSIDER the quadratic Lagrangian

$$L=(1/2a^2)(c_0\varphi_\alpha{}^2+c_1a^2\varphi_{\alpha,\beta_1\beta_1}{}^2+c_2a^4\varphi_{\alpha,\beta_1\beta_2}{}^2+c_3a^6\varphi_{\alpha,\beta_1\beta_2\beta_3}{}^2+c_4a^8\varphi_{\alpha,\beta_1\beta_2\beta_3\beta_4}{}^2), \quad (1)$$

where a is an arbitrary length and the c's are dimensionless natural constants. Higher derivatives, if admissible, may be treated by induction. Applying Hamilton's principle[1] we obtain the set of linear eighth-order field equations

$$(c_0-c_1a^2\Box+c_2a^4\Box^2-c_3a^6\Box^3+c_4a^8\Box^4)\varphi_\alpha(\mathbf{r},t)=0 \quad (2)$$

which may be written as

$$[\prod_{\sigma=1}^{4}(a^2\Box-s_\sigma{}^2)]\varphi_\alpha(\mathbf{r},t)=0, \quad (3)$$

where $s_\sigma{}^2$ represents the four roots of the auxiliary algebraic equation obtained by replacing $a^2\Box$ by s^2 in (2).

A solution of (3) expressed in Fourier integrals is

$$\varphi_\alpha(\mathbf{r},t)=(1/2\pi)^{\frac{3}{2}}\sum_\sigma\int[\varphi_{\sigma\alpha}(\mathbf{k})\exp(ix_\beta k_{\sigma\beta})+\varphi_{\sigma\alpha}{}^*(\mathbf{k})\exp(-ix_\beta k_{\sigma\beta})]d\mathbf{k}, \quad (4)$$

where

$$k_{\sigma\beta}=(\mathbf{k},ik_\sigma)\quad\text{and}\quad k_\sigma{}^2=\mathbf{k}\cdot\mathbf{k}+s_\sigma{}^2/a^2. \quad (5)$$

Corresponding to our Lagrangian we can find an energy momentum density tensor $t_{\mu\nu}$, for which energy and momentum is conserved, i.e., $t_{\mu\nu,\nu}=0$. Expressing $t_{\mu\nu}$ in terms of Fourier amplitudes (we may discard all time-dependent terms) we get the Hamiltonian

$$H=-\int ict_{44}dV=\sum_{\sigma\alpha}2\gamma_\sigma\int k_\sigma 2\varphi_{\sigma\alpha}{}^*\varphi_{\sigma\alpha}d\mathbf{k}=\sum_{\sigma\alpha}\int\hbar ck_\sigma\eta_{\sigma\alpha}{}^*\eta_{\sigma\alpha}d\mathbf{k}, \quad (6)$$

where

$$\gamma_\sigma=(-c_1+2c_2s_\sigma{}^2-3c_3s_\sigma{}^4+4c_4s_\sigma{}^6), \quad (7)$$

and

$$\eta_{\sigma\alpha}(\mathbf{k})=(2\gamma_\sigma k_\sigma/c\hbar)^{\frac{1}{2}}\varphi_{\sigma\alpha}(\mathbf{k}). \quad (8)$$

Going over to quantum-field theory the commutation rules are

$$[\varphi_{\sigma\alpha}{}^*(\mathbf{k}),\varphi_{\tau\beta}(\mathbf{k}')]=-\delta_{\sigma\tau}\delta_{\alpha\beta}\delta(\mathbf{k}-\mathbf{k}')c\hbar/2k_\sigma\gamma_\sigma. \quad (9)$$

Suitable supplementary conditions which permit only transverse states of positive energy to exist in the "empty" field are

$$[(1-is_\sigma/ak)\mathbf{k}\cdot\mathbf{A}_\sigma(\mathbf{k})-k_\sigma\phi_\sigma(\mathbf{k})]\Omega=0. \quad (10)$$

The use of Dirac's expansors[2] is indicated here.

When nucleons are present in the field we find the interaction energy by a fourfold application of the formalism of Fock,[3] modified for bosons with finite masses ($m_\sigma=s_\sigma\hbar/ac$). From symmetry and superposition considerations we obtain the interaction energy[4]

$$U=\sum_\sigma(1/\gamma_\sigma)(\Sigma_{uv}{}'g_ug_v/8\pi)[(1-\tfrac{1}{2}\boldsymbol{\alpha}_u\cdot\boldsymbol{\alpha}_v)\times(1/R)\exp(-s_\sigma R/a)-\tfrac{1}{2}(\boldsymbol{\alpha}_u\cdot\mathbf{R}/R)(\boldsymbol{\alpha}_v\cdot\mathbf{R}/R)\times(s_\sigma/a+1/R)\exp(-s_\sigma R/a)]. \quad (11)$$

The $1/R$ singularity at short ranges, which is present in second-order field theory (i.e., when $c_2=c_3=c_4=0$), vanishes in fourth-[4] ($c_3=c_4=0$), sixth- ($c_4=0$), and eighth-order theories because $\Sigma(1/\gamma_\sigma)=0$, independent of the magnitudes of the natural constants. The self-energy is thus finite in the higher order theories. It is given by

$$U_s=-(\Sigma s_\sigma/\gamma_\sigma)(1-\tfrac{1}{2}\boldsymbol{\alpha}\cdot\boldsymbol{\alpha})g^2/8\pi a. \quad (12)$$

If we admit only field particles with real, positive, or zero masses then the c's must all have the same sign and the factor $(\Sigma s_\sigma/\gamma_\sigma)$ is positive or negative according to whether the c's are positive or negative.

Exploratory attempts to apply the results of fourth-order theory to the deuteron indicate that agreement with experiment in energy level and scattering questions can be achieved, with a reasonable choice of constants, if we introduce the isotopic spin formalism. Following Kemmer[5] for the charge-independent hypotheses one treats each component of $\varphi_\alpha(\mathbf{r},t)$ as a three vector in isotopic spin space. The modified formalism leads finally to the appearance of the isotopic spin operator T_{ab} as a factor in the interaction function. This approach is a radical one, however, if the more complicated interaction function of sixth- or eighth-order theories is used, because of the large number of field particles implied. In these cases the close similarity between Kemmer's Eq. (21) and Eq. (6) above invites consideration of the possibility of associating the sigma-quanta with the isotopic spin quanta themselves.

[1] Compare B. Podolsky and C. Kikuchi, Phys. Rev. **65**, 228 (1944).
[2] P. A. M. Dirac, Proc. Roy. Soc. **A183**, 284 (1945).
[3] V. Fock, Physik. Zeits. Sowjetunion **6**, 425 (1934).
[4] A. E. S. Green, Phys. Rev. **73**, 26 (1948).
[5] N. Kemmer, Proc. Camb. Phil. Soc. **34**, 354 (1938).

Spectroscopic Evidence of Methane in the Earth's Atmosphere*

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January 23, 1948

WHEN mapping the solar spectrum, under high resolving power, in the 3.3-μ region, fourteen regularly spaced and intense lines have been found between 3.33μ and 3.47μ. The wave numbers of these lines agree, within a maximum difference of ±0.7 cm^{-1}, with the measurements of A. H. Nielsen and H. H. Nielsen[1] on the 3.3-μ fundamental band of CH_4. The intense central line (Q branch) of the CH_4 band falls in a spectral region absorbed by water vapor. For shorter wave-lengths than 3.3μ, many intense lines of water vapor prevent also the possibility of finding most of the CH_4 lines of the R branch.

The new lines have been found on spectrograms taken on January 10, 1948, with a new prism-grating recording spectrometer design by Dr. Robert Noble, under the direction of Professor H. H. Nielsen. This instrument is of the Pfund type, has an aperture of f:5, and parabolic mirrors with a focal length of 100 cm. An echelette grating with 7200 lines per inch was used for the observations.

The other fundamental band of CH_4 is situated at 7.7μ and is more intense than the $3.3\text{-}\mu$ band. Consequently, a band of CH_4 is to be expected in the solar spectrum at 7.7μ. In fact, A. Adel[2] has found at low dispersion a band in the solar spectrum at 7.6μ and has proposed N_2O and perhaps N_2O_5, as possible identifications.[3] The $7.7\text{-}\mu$ region of the solar spectrum was also mapped by A. Adel[4] at high dispersion by using a grating with 2400 lines per inch, and this author[5] has attributed part of the observed fine structure to N_2O. Adel's measurements have been compared with the data obtained by A. H. Nielsen and H. H. Nielsen[1] for the $7.7\text{-}\mu$ band of CH_4. It is possible to find many corresponding lines, within the errors of observations. However, the solar spectrum in this region is complicated by the presence of many water vapor lines and probably also by the fine structure of N_2O.

With the aim of finding the CH_4 lines in the $7.7\text{-}\mu$ region of the solar spectrum, an echelette grating with 3600 lines per inch has been placed in the spectrometer. A comparison between the expected spectrograms and the observations from Adel may also show if the detected methane is not due to a local impurity of the atmosphere.

Details concerning the identification of the $3.3\text{-}\mu$ band of methane in the solar spectrum will be published, in the near future in the Astrophysical Journal.

* Supported by Air Materiel Command, Wright Field, Dayton, Ohio.
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1 A. H. Nielsen and H. H. Nielsen, Phys. Rev. **48**, 864 (1935).
2 A. Adel, Ap. J. **87**, 198 (1938).
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Fluctuations in the Refractivity of Water

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December 26, 1947

At the Chicago meeting of the American Physical Society Antonoff, Yakimac, and Randall report finding that the density of water fluctuates with time in a random manner. The density fluctuations are said to be in the fourth decimal place. Since the refractivity of water $(n-1)$ is roughly proportional to the density, a similar fluctuation in the index should exist.

We are able to measure changes of index with temperature, by means of a Jamin interferometer, to a precision of 1×10^{-6} index units per degree Centigrade. Light is supplied from a sodium arc. Fluctuations in density of 10^{-4} g/cc as reported by Antonoff would result in fluctuations of 3×10^{-5} index unit. This, in turn, would produce a fringe shift of about 10 fringes in our instrument.

We find that when the temperature of the water is held constant to 0.005°C, we observe fringe fluctuations no greater than $\frac{1}{2}$ fringe. This fluctuation is believed to be due to small temperature variations arising from the operation of the thermostat. This $\frac{1}{2}$-fringe fluctuation corresponds to an index change of 1.5×10^{-6} and a density fluctuation of no greater than 5×10^{-6} g/cc. Continuous observation of the fringe pattern has been made over periods of several hours. The fringe fluctuations in our apparatus have a period of about 30 seconds, which is of the same order as the operating period of the thermostat.

The Neutron-Proton Scattering Formula

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January 12, 1948

The neutron-proton scattering cross section can be described easily by the often used Bethe formula[1]

$$\sigma=\frac{3}{4}\sigma_T+\frac{1}{4}\sigma_S=\frac{3}{4}\frac{4\pi}{k^2+\kappa^2}+\frac{1}{4}\frac{4\pi}{k^2+\kappa'^2},$$

which is derived for a square well of radius a and a *real* singlet state of the deuteron. In this equation, k^2 is the kinetic energy of the incident neutron, measured in the center of gravity frame, in units of $\hbar^2/M$. The first term belongs to the triplet interaction with κ^2 being the binding energy of the deuteron in the same units. In the second term, κ'^2 is the binding energy of the real singlet state. If κa or $\kappa' a$ is not much less than unity, a correction has to be made by a factor $1+2\kappa a$ or $1+2\kappa' a$, respectively, not $1+\kappa a$ as suggested by Bethe.

We should like to point out in this letter that the second term of this expression does not hold for a *virtual* singlet state of the deuteron as commonly supposed. An easy calculation used instead gives

$$\sigma_S=\frac{4\pi}{k^2+(a^2/4)(\kappa'^2-k^2)^2},$$

where $\kappa'^2>0$ is now the energy of the virtual level in the same units. This expression is of the same form as Breit and Wigner's well-known resonance dispersion formula with a line breadth which is proportional to k and of the same order of magnitude as the resonance energy. In consequence of this large line breadth we do not get a maximum for $k^2=\kappa'^2$ but a cross section monotonously falling with growing energy in the same manner as Bethe's formula suggests and experiments[2] show.

The limit for $k=0$ comes out to be $\sigma_S=16\pi/(a^2\kappa'^4)$ instead of Bethe's $\sigma_S=4\pi/\kappa'^2$, i.e., larger by a factor $4/(\kappa' a)^2$. With the usual values for the virtual level energy of about 100 kev, this factor is much larger than unity. In order to get the right observed slow neutron scattering cross section of about 20.10^{-24} cm^2, it is necessary to take a much larger virtual level energy, about 1.5 Mev.

The change in order has many consequences, e.g., for the capture cross section of slow neutrons by protons, and for the field theory of nuclear forces. Details of the deduction of the new formula and of these consequences will be published elsewhere.

1 H. A. Bethe and R. F. Bacher, Rev. Mod. Phys. **8**, 114 (1936); J. Schwinger and E. Teller, Phys. Rev. **52**, 286 (1937); C. Kittel and G. Breit, *ibid.* **56**, 744 (1939).
2 Compare Bailey, Bennett, Bergstrahl, Nuckolls, Richards, and Williams, Phys. Rev. **70**, 583 (1946); D. H. Frisch, *ibid.* **70**, 589 (1946).