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Note on the Milne Problem for a Sphere

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A formal solution to the Milne problem for the sphere is presented which exhibits the shadow of the sphere and the corresponding discontinuity in the neutron distribution function. Two limiting cases leading to the Milne problem for the plane are briefly discussed.

STATEMENT OF THE PROBLEM

An infinite non-capturing medium which scatters neutrons isotropically without changing their velocity surrounds a sphere which completely absorbs all incident neutrons. A current density $1/r^2$ in the direction $-r$ is supposed to exist in the infinite medium. The problem is to determine the neutron distribution function in the medium.

The transport equation for the distribution function is:

$$\mu\frac{\partial\psi(r,\mu)}{\partial r}+\frac{(1-\mu^2)}{r}\frac{\partial\psi(r,\mu)}{\partial\mu}+\psi(r,\mu)=\tfrac{1}{2}\int_{1-}^{+1}\psi(r,\mu')d\mu'\equiv f(r). \quad (1)$$

The center of the sphere is taken as the coordinate origin and μ is the cosine of the angle between the direction of motion of the neutron and the radius vector r. $\psi(r,\mu)d\mu$ is the number of neutrons per unit volume at r with direction cosine between μ and $\mu+d\mu$. $2f(r)$ is the neutron density at r. The neutron mean free path is the unit of length.

The current density

$$j(r)=-\int_{-1}^{+1}\mu\psi(r,\mu)d\mu,$$

and on integrating Eq. (1) over μ it is seen that $4\pi r^2 j(r)$ is a constant. This constant is taken as 4π so that $j(r)=1/r^2$.

A solution of Eq. (1) is to be found such that $\psi(r,\mu)$ is finite everywhere in the medium and satisfies the boundary condition for a black sphere,

$$\psi(a,\mu)=0 \quad \text{for } 0<\mu\leqslant 1, \quad (2)$$

where a is the radius of the sphere.

The statement of the problem given here follows that of Marshak.[1] An exact solution appears to be difficult and for practical reasons approximate solutions have been obtained.[1] One would expect to see the shadow of the sphere and, as a result, a discontinuity in the neutron distribution function. The solutions previously given do not show this effect, and it is our purpose to present a formal solution which demonstrates the shadow and discontinuity.

THE FORMAL SOLUTION

Our method shall consist of the reduction of the transport Eq. (1) to a homogeneous integral equation for the neutron density. This is the usual treatment, for example, of the Milne problem for the plane.

If it is supposed that $f(r)$ is a known function, Eq. (1) is a first-order partial differential equation for ψ. A formal solution for ψ in terms of $f(r)$ can be obtained which exactly fulfills the boundary condition. This procedure demonstrates the discontinuity in ψ. Substitution of this ψ in the definition of $f(r)$ leads to a homogeneous integral equation for $f(r)$. The solution of the integral equation, normalized by the condition $j(r)=1/r^2$, yields the complete solution of the problem.

Supposing that $f(r)$ is known, we have for the subsidiary equations:

$$dr/\mu=rd\mu/(1-\mu^2)=d\psi/(-\psi+f(r)). \quad (3)$$

Integrating the relation between r and μ gives $r(1-\mu^2)^{\frac{1}{2}}=c$. The parameter c has the range $0<c<\infty$. For c fixed, we require $c\leqslant r<\infty$ in order that $|\mu|\leqslant 1$.

We also have the relation

$$d\psi/dr=-(1/\mu)\psi+(1/\mu)f(r).$$

[1] R. E. Marshak, Phys. Rev. 71, 443 (1947). References to previous work are given in this paper.

Expressing μ in terms of r and c, and holding c fixed, we find on integrating that

$$\psi(r,\mu)\exp[\pm(r^2-c^2)^{\frac{1}{2}}]\mp\int^r dyyf(y)(y^2-c^2)^{-\frac{1}{2}}\times\exp[\pm(y^2-c^2)^{\frac{1}{2}}]=h_\pm(c).\quad(4)$$

The signs $\pm$ refer to the sign of μ as we shall always take the positive square root.

The characteristics $c=$constant are sketched in Fig. 1. We can now discuss the determination of ψ.

$-1\leqslant\mu<0$

In this region $\mu=-(r^2-c^2)^{\frac{1}{2}}/r$. Let us denote the distribution function in this region by $\psi^-(r,\mu)$. Then, from (4),

$$\psi^-(r,\mu)\exp[-(r^2-c^2)^{\frac{1}{2}}]+\int^r dyyf(y)(y^2-c^2)^{-\frac{1}{2}}\times\exp[-(y^2-c^2)^{\frac{1}{2}}]=h_-(c).\quad(5)$$

Let the point (r,μ) move to infinity along $c=$constant. As $r\rightarrow\infty$, $\mu\rightarrow-1$. Since $\psi^-(\infty,-1)$ must be finite,

$$\int^\infty dyyf(y)(y^2-c^2)^{-\frac{1}{2}}\exp[-(y^2-c^2)^{\frac{1}{2}}]=h_-(c).\,(6)$$

Consequently,

$$\psi^-(r,\mu)=\exp[+(r^2-c^2)^{\frac{1}{2}}]\int_r^\infty dyyf(y)(y^2-c^2)^{-\frac{1}{2}}\times\exp[-(y^2-c^2)^{\frac{1}{2}}].\quad(7)$$

If we hold c fixed and let $r\rightarrow c$, then $\mu\rightarrow0^-$. Thus,

$$\psi^-(c,0^-)=\int_c^\infty dyyf(y)(y^2-c^2)^{-\frac{1}{2}}\times\exp[-(y^2-c^2)^{\frac{1}{2}}].\quad(8)$$

$0<\mu\leqslant+1$

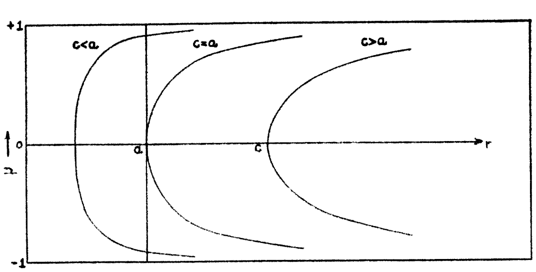

FIG. 1. The characteristics $r(1-\mu^2)^{\frac{1}{2}}=c$ for $0<c<\infty$.

In this region $\mu=+(r^2-c^2)^{\frac{1}{2}}/r$. We shall denote the distribution function in this region by $\psi^+(r,\mu)$. Then, from (4),

$$\psi^+(r,\mu)\exp[+(r^2-c^2)^{\frac{1}{2}}]-\int^r dyyf(y)(y^2-c^2)^{-\frac{1}{2}}\times\exp[+(y^2-c^2)^{\frac{1}{2}}]=h_+(c).\quad(9)$$

We see from Fig. 1 that in this region the sub-regions defined by $c\gtreqless a$ must be considered separately. When $c>a$, the characteristics do not cut the line $r=a$; whereas, when $c<a$ the curves cut the line $r=a$ and, therefore, allow the introduction of the boundary condition (2).

Sub-Region $c>a$

Let us denote the distribution function in this sub-region by $\psi^+(r,\mu;>)$. The distribution function must be continuous across the line $\mu=0$ when $r>a$, i.e., $\psi^+(r,0^+;>)=\psi^-(r,0^-)$. This condition enables $\psi^+(r,\mu;>)$ to be determined.

Holding c fixed and letting $r\rightarrow c$, then $\mu\rightarrow0^+$. From (9),

$$\psi^+(c,0^+;>)-\int^c dyyf(y)(y^2-c^2)^{-\frac{1}{2}}\times\exp[+(y^2-c^2)^{\frac{1}{2}}]=h_+(c;>).\quad(10)$$

Now, using the continuity condition and (8), we have

$$\psi^+(r,\mu;>)=\exp[-(r^2-c^2)^{\frac{1}{2}}]\times\int_c^\infty dyyf(y)(y^2-c^2)^{-\frac{1}{2}}\exp[-(y^2-c^2)^{\frac{1}{2}}]+\exp[-(r^2-c^2)^{\frac{1}{2}}]\int_c^r dyyf(y)(y^2-c^2)^{-\frac{1}{2}}\times\exp[+(y^2-c^2)^{\frac{1}{2}}].\quad(11)$$

Sub-Region $c<a$

The distribution function is denoted by $\psi^+(r,\mu;<)$. Since the curves for $c<a$ cut the line $r=a$, set $r=a$ in (9). This yields

$$\psi^+(a,\mu;<)\exp[+(a^2-c^2)^{\frac{1}{2}}]-\int^a dyyf(y)(y^2-c^2)^{-\frac{1}{2}}\exp[+(y^2-c^2)^{\frac{1}{2}}]=h_+(c;<).\quad(12)$$

μ and c are connected by

$$\mu=(a^2-c^2)^{\frac{1}{2}}/a,\quad 0<c<a. \tag{13}$$

Using the boundary condition (2), $\psi^+(a,\mu;<)=0$, we find $h_+(c;<)$ and, consequently,

$$\psi^+(r,\mu;<)=\exp[-(r^2-c^2)^{\frac{1}{2}}]\times\int_a^r dyyf(y)(y^2-c)^{2-\frac{1}{2}}\exp[+(y^2-c^2)^{\frac{1}{2}}]. \tag{14}$$

The boundary condition (2) is now exactly fulfilled and $\psi(r,\mu)$ is defined in the region $a<r<\infty$, $|\mu|\leqslant 1$ in terms of the function $f(r)$ by Eqs. (7), (11), and (14). ψ is continuous across $\mu=0$ when $r>a$. But we note that ψ is not continuous across $\mu=0$ when $0<r<a$, and $\psi^+(r,a;<)$ is negative when $r<a$. Although the region $r<a$ has no physical sense in that it is the inside of the black sphere, yet ψ is mathematically well defined in terms of $f(r)$.

We also see that the point $\mu=0$, $r=a$ $(c=a)$ is a singular point of ψ. The function ψ is essentially indeterminate at this point and this lack of determination is carried along the curve $c=a$, when $0<\mu<1$. Equations (11) and (14) immediately show that

$$\Delta\equiv\lim_{c=a}[\psi^+(r,\mu;>)-\psi^+(r,\mu;<)]=F(a)\exp[-(r^2-a^2)^{\frac{1}{2}}], \tag{15}$$

where

$$F(a)=\int_a^\infty dyyf(y)(y^2-a^2)^{-\frac{1}{2}}\exp[-(y^2-a^2)^{\frac{1}{2}}].$$

When $c=a$, the relation between μ and r is

$$\mu\equiv\mu_r=(r^2-a^2)^{\frac{1}{2}}/r. \tag{16}$$

The geometrical significance of relation (16) is shown in Fig. 2.

From the point at a distance r from the center of the sphere, the tangent to the sphere is drawn. Then, of course, $\mu_r=\cos\theta_r=(r^2-a^2)^{\frac{1}{2}}/r$. The arrow in Fig. 2 indicates the neutron direction of motion. The critical value of θ, θ_r, defines the shadow of the sphere. The shadow at the point r is within the cone of apex angle θ_r. With respect to Fig. 1, the shadow region, for the various r, is the region covered by the curves for which $c<a$ $(0<\mu<1)$, and it is these curves which cut the line $r=a$.

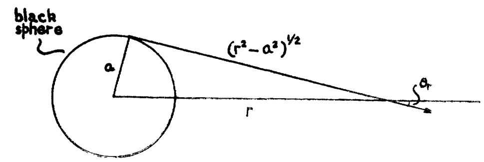

FIG. 2. The geometrical significance of $\mu_r=\cos\theta_r=(r^2-a^2)^{\frac{1}{2}}/r$.

Integral Equation for $f(r)$

The definition of $f(r)$ is given in (1). On substituting the ψ given by (7), (11), and (14) into this definition we are led, after some manipulation, to the following equation for $f(r)$ when $a<r<\infty$.

$$rf(r)=\int_a^\infty K(r,y)yf(y)dy, \tag{17}$$

where[2]

$$K(r,y)=\tfrac{1}{2}E_1(|r-y|)-\tfrac{1}{2}E_1[(r^2-a^2)^{\frac{1}{2}}+(y^2-a^2)^{\frac{1}{2}}]. \tag{18}$$

The function $f(r)$ is normalized by the requirement that

$$-\int_{-1}^{+1}\mu\psi(r,\mu)d\mu=1/r^2.$$

This completes the formal solution of the problem for the range of the variables $a<r<\infty$, $|\mu|\leqslant 1$.

DISCUSSION

An exact solution of the integral equation (17) is difficult to obtain. There are two limiting cases of physical interest, however, which lead to the same soluble specialization of (18). In one case $r\gg a$. Then the kernel $K(r,y)$ is essentially determined by the first integral in (18) as this integral is large for $y\simeq r$ and the second integral is small for all y. In the second case $a\to\infty$ and $r\to\infty$ so that $r-a$ is finite. The kernel $K(r,y)$ is again given by the first integral in (18). This limiting kernel is found in the Milne problem for the plane, and the exact solution of the integral equation is well known.[3] The first case has been discussed by Marshak[4] and Davison.[5] The

[2] $E_1(x)=\int_1^\infty[e^{-xt}]dt/t$.

[3] G. Placzek and W. Seidel, Phys. Rev. **72**, 550 (1947). These authors have discussed the solution for the plane in a form adapted to problems in neutron diffusion.

[4] R. E. Marshak, Phys. Rev. **71**, 688 (1947).

[5] B. Davison, Phys. Rev. **71**, 694 (1947).

shadow of the sphere plays no significant role in this case as the points of interest are far from the sphere. In the second case the shadow of the sphere is important as the points of interest are close to the boundary. In the limit, the region $c>a$ for $\mu>0$ disappears. The distribution function at points near the sphere may be discussed from the point of view of this approximation.

Because of the existence of the shadow and the discontinuity in ψ, measurements of the critical angle θ_r and Δ can, in principle, determine the radius a of the sphere and the mean free path λ of the neutrons. If it is supposed that μ_r is determined at two positions along a radius so that $\delta r=r_2-r_1$ is known, then, from (16),

$$a^2=r_1^2(1-\mu_{r1}^2)=r_2^2(1-\mu_{r2}^2).$$

There are three equations for the three unknowns a, r_1, and r_2. In (15), r is expressed in terms of λ as the unit of length. We then write, from (15),

$$\ln\Delta=-(1/\lambda)r\mu_r+\ln F.$$

The slope of the line obtained from a plot of $\ln\Delta$ against $r\mu_r$ determines λ. The quantity Δ may be measured in an arbitrary unit and the quantity $r\mu_r$ is provided by the measurements of μ_r and δr.

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Angular Correlation of Scattered Annihilation Radiation*

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If the two photons emitted in an annihilation process are scattered, their initial cross-polarization leads to an angular correlation of the scattered radiation. This correlation effect is calculated, and yields a substantial azimuthal asymmetry. It is shown that one may regard the scattering of one photon as performing a partial analysis of the polarization of the other photon.

1. INTRODUCTION

According to pair theory[1] the dominant type of annihilation is one in which the positron-electron pair has zero relative angular momentum. Associated with this is the cross-polarization of the two quanta emitted in the annihilation process. If one photon is linearly polarized in one plane, the other photon, which goes off in the opposite direction, is linearly polarized in the perpendicular plane. A similar relation exists for any state of polarization of one photon.

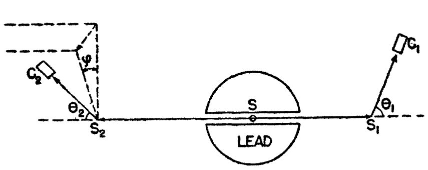

FIG. 1. Schematic diagram of experimental arrangement.

Wheeler[2] has suggested an experiment to test this prediction, involving coincidence measurements of the scattering of both of the annihilation photons. The arrangement is represented schematically in Fig. 1.

A source S of annihilation radiation (a radioactive source of slow positrons covered with a foil) is placed at the center of a lead sphere with a narrow channel drilled through it. The photons, each of energy mc^2, passing through the channel are scattered by scatterers S_1 and S_2 and recorded by gamma-ray counters C_1 and C_2. Coincidences between the two counters are recorded when the azimuths of the two counters are identical ($\varphi=0$) and when the azimuths differ by a right

* Research carried out at the Brookhaven National Laboratory under the auspices of the Atomic Energy Commission.

[1] P. A. M. Dirac, Proc. Camb. Phil. Soc. 26, 361 (1930).

[2] J. A. Wheeler, Ann. N. Y. Acad. Sci. 48, 219 (1946).